ON THE FORMALISM OF SHIMURA VARIETIES

JACK SEMPLINER AND RICHARD TAYLOR

1. Introduction

The formalism of Shimura varieties was laid out by Deligne [D1], [D2] and Langlands [L] 45 years ago. The formalism they suggested seems to us to have a number of possible shortcomings:

(1) Deligne's 'Shimura datum', a pair (G, X) of a connected reductive group over $\mathbb{Q}$ and a $G(\mathbb{R})$ conjugacy class of homormorphisms $h : \mathrm{RS}^{\mathbb{R}}_{\mathbb{C}}\mathbb{G}_m \to G$ over $\mathbb{R}$ satisfying certain axioms, paremtrizes not a (inverse system of) varieties $\mathrm{Sh}(G, X)$ over some number field $E(G, X)$, but the pair $(\mathrm{Sh}(G, X)/E(G, X), \rho^{\mathrm{can}} : E(G, X) \hookrightarrow \mathbb{C})$ of the Shimura variety together with an embedding of its field of definition into $\mathbb{C}$. Indeed the 'same (inverse system of) varieties' over E can be parametrized by different Shimura data depending on the choice of embedding $E \hookrightarrow \mathbb{C}$.

(2) The theory of conjugation of Shimura varieties conjectured by Langlands [L] and established by Milne [Mi1] depends for its formulation on some unmotivated, and somewhat non-canonical, choices of cocycles, which to the best of our knowledge are written down only in [L]. This makes it quite hard to work with, as does its reliance of choices of special points.

(3) In [D2], Deligne imposes an axiom that the group G^{ad} should have no simple factor over $\mathbb{Q}$, whose real points are compact. This allows him to use strong approximation to explicitly understand the connected components of his Shimura varieties, but it should be unnecessary for their existence and for the study of their conjugation properties.

The third of these points is unrelated to the other two and will be easily remedied in section 8.5. We will discuss it no further in this introduction.

As a simple illustration point (1), consider a non-Galois totally real cubic extension $F/\mathbb{Q}$. It has three different embeddings $\tau_i : F \hookrightarrow \mathbb{R}$ for $i = 1, 2, 3$. Write ∞_i for the infinite place of F corresponding to τ_i. Let D_i/F denote the quaternion algebra centre F ramified at exactly ∞_j for $j \neq i$. Denote by $G_i/\mathbb{Q}$ the reductive groups with $G_i(\mathbb{Q}) = D_i^{\times}$. These groups are not isomorphic over $\mathbb{Q}$. We have $G_i(\mathbb{R}) \cong GL_2(\mathbb{R}) \times \mathbb{H}^{\times} \times \mathbb{H}^{\times}$, where $\mathbb{H}$ denotes the Hamiltonian quaternions. Let X_i denote the $G_i(\mathbb{R})$-conjugacy class of the morphism $h_i : \mathrm{RS}^{\mathbb{C}}_{\mathbb{R}}\mathbb{G}_m \to G_i$ defined over $\mathbb{R}$ with

$$h_i(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \times 1 \times 1.$$

We have $E(G_i, X_i) = \tau_i F \subset \mathbb{C}$. Note that $G_i \times \mathbb{A}^\infty$ is independent of i. We will denote the group $G_i(\mathbb{A}^\infty)$, which does not depend on i, simply as Γ. Deligne's theory of Shimura varieties gives us for each i an inverse system $\{\mathrm{Sh}(G_i, X_i)_U\}$ of varieties over $\tau_i F \subset \mathbb{C}$ indexed by neat open compact subgroups of Γ and with an action of Γ. However there is one such system $\{S_U\}$ of varieties over F indexed by neat open compact subgroups of Γ and with an action of Γ, such that $\{\tau_i S_U\}$ with its Γ-action is identified with $\{\mathrm{Sh}(G_i, X_i)_U\}$ with its Γ-action. It seems to us unnecessarily cumbersome and confusing to index the one system $\{S_U\}$ over F by three different Shimura data, depending on how one wants to view F as a subfield of $\mathbb{C}$. It would seem to be preferable to index $\{S_U\}$ by some other data $\mathcal{D}$ over F and then to give a recipe that to $\mathcal{D}$ and any embedding $\tau : F \hookrightarrow \mathbb{C}$ attaches a Shimura datum $(G_{\mathcal{D},\tau}, X_{\mathcal{D},\tau})$ so that

$$(\tau S_U)(\mathbb{C}) = G_{\mathcal{D},\tau}\backslash(G_{\mathcal{D},\tau}(\mathbb{A}^\infty)/U \times X_{\mathcal{D},\tau}).$$

It turns out that points (1) and (2) above are closely related. Indeed the second only became apparent to us as we tried to understand the first, and once we felt we understood the second, the first was easily remedied.

To us the key to understanding possible shortcomings (1) and (2) is, perhaps not surprisingly, to make use of Kottiwitz's cohomology groups $B(G)$. However it will be essential for us to work with 1-cocycles, not only 1-cohomology classes. This causes us extensive problems because in Kottwitz's original theory only the cohomology groups are canonically defined. In fact more than two thirds of this paper is devoted to understanding how to work with such cocyles.

In the rest of this introduction we will first recall Kottwitz's theory in a way that emphasizes cocycles not only cohomology classes. We will then explain our hopefully more canonical reformulation of the theory of conjugation of Deligne's Shimura varieties. Finally we will state an alternative formulation which avoids the shortcoming (1).

1.1. **Algebraic cohomology.** Kottwitz's groups $B(G)$ are defined by what we will call algebraic cohomology. If E/F is a finite Galois extension of local or global fields Kottwitz considers certain extensions $\mathcal{E}(E/F)$ of $\mathrm{Gal}\,(E/F)$ by certain abelian groups $\mathcal{E}(E/F)^0$. The most familiar examples are the local and global Weil groups $W_{E/F}$: extensions of $\mathrm{Gal}\,(E/F)$ by $E^\times$ in the local case and by $\mathbb{A}_E^\times/E^\times$ in the global case. These extensions are defined in terms of a canonical class $[\alpha_{E/F}] \in H^2(\mathrm{Gal}\,(E/F), \mathcal{E}(E/F)^0)$, but not by a canonical cocycle. It turns out (because $H^1(\mathrm{Gal}\,(E/F), \mathcal{E}(E/F)^0) = (0)$) that $\mathcal{E}(E/F)$ is unique up to an isomorphism, which is only unique up to composition with conjugation by an element of $\mathcal{E}(E/F)^0$.

Having defined these extensions $\mathcal{E}(E/F)$ we will, following Kottwitz, consider what we call the *algebraic* (non-abelian) cohomology $H^1_{\mathrm{alg}}(\mathcal{E}(E/F), G)$, where G is a group (often the E or $\mathbb{A}_E$ points of an algebraic group) with an action of $\mathrm{Gal}\,(E/F)$. To define this cohomology we consider only the set $Z^1_{\mathrm{alg}}(\mathcal{E}(E/F), G) \subset Z^1(\mathcal{E}(E/F), G)$ of cocycles whose restriction to $\mathcal{E}(E/F)^0$ lie in some chosen class of homomorphisms, usually a class of homomorphisms coming from certain morphisms of algebraic groups. These 'algebraic' cocycles will be preserved by the usual equivalence relation and hence gives rise to a cohomology group $H^1_{\mathrm{alg}}(\mathcal{E}(E/F), G)$. It is easy to verify that despite the ambiguity in the definition of $\mathcal{E}(E/F)$, the pointed set $H^1_{\mathrm{alg}}(\mathcal{E}(E/F), G)$ is well defined up to unique isomorphism. However $Z^1_{\mathrm{alg}}(\mathcal{E}(E/F), G)$ is not.

If F is a local field there will be only one such extension of interest to us: $\mathcal{E}(E/F) = W_{E/F}$ - the Weil group defined by the usual canonical class $[\alpha^W_{E/F}] \in H^2(\mathrm{Gal}\,(E/F), E^\times)$. In this case, if G/F is an algebraic group with centre $Z(G)$, then $Z^1_{\mathrm{alg}}(W_{E/F}, G(E))_{\mathrm{basic}}$ will denote those cocycles which are given on $E^\times$ by an algebraic character $\nu : \mathbb{G}_m \to Z(G)$. These are sometimes called 'basic algebraic cocycles'. (There is also a non-basic version where one allows all characters $\nu : \mathbb{G}_m \to G$, but this more general definition will play little role in our story.)

However, when F is a global field, will need to consider several examples of these groups $\mathcal{E}(E/F)$, which we will now describe.

(1) We define $\mathcal{E}^{\mathrm{loc}}(E/F)^0 = \prod_{w \in V_E} E_w^\times$, where V_E denotes the set of all places of E. There is a unique class $[\alpha^{\mathrm{loc}}_{E/F}] \in H^2(\mathrm{Gal}\,(E/F), \prod_{w\in V_E} E_w)$ whose image in $H^2(\mathrm{Gal}\,(E_w/F_w), E_w^\times)$ equals $[\alpha^W_{E_w/F_w}]$ for all $w \in V_E$. We let $\mathcal{E}^{\mathrm{loc}}(E/F)$ denote the corresponding extension of $\mathrm{Gal}\,(E/F)$ by $\prod_{w\in V_E} E_w^\times$.

For an algebraic group G/F, basic algebraic cohomology of $G(\mathbb{A}_E)$ will be defined in terms of those cocycles whose restriction to $\mathcal{E}^{\mathrm{loc}}(E/F)^0$ are of the form $\prod_w \nu_w$, where $\nu_w : \mathbb{G}_m \to Z(G)_{/F_w}$ is an algebraic character, non-trivial for only finitely many w.

(2) We define $T_{2,E}/\mathbb{Q}$ to be the protorus with character group $\mathbb{Z}[V_E]$ with its natural action of $\mathrm{Gal}\,(E/F)$. Then we set $\mathcal{E}_2(E/F)^0 = T_{2,E}(\mathbb{A}_E)$ and define $\mathcal{E}_2(E/F)$ as the pushout of $\mathcal{E}^{\mathrm{loc}}(E/F)$ along the embedding $\prod_w E_w^\times \hookrightarrow \prod_w \mathbb{A}_E^\times \cong T_{2,E}(\mathbb{A}_E)$, where we identify $E_w^\times$ inside inside the copy of $\mathbb{A}_E^\times$ indexed by w.

In this case, for an algebraic group G/F, basic algebraic cohomology of $G(\mathbb{A}_E)$ will be defined in terms of those cocycles whose restriction to $\mathcal{E}_2(E/F)^0$ come from an algebraic character $\nu : T_{2,E} \to Z(G)_{/F}$. Thus there are natural restriction maps $Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F), G(\mathbb{A}_E))_{\mathrm{basic}} \to Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F), G(\mathbb{A}_E))_{\mathrm{basic}}$.

(3) $W_{E/F}$ will denote the global Weil group, i.e. the extension of $\mathrm{Gal}\,(E/F)$ by $\mathbb{A}_E^\times/E^\times$ coming from the usual canonical class $[\alpha^W_{E/F}] \in H^2(\mathrm{Gal}\,(E/F), \mathbb{A}_E^\times/E^\times)$.

(4) We will write $\mathcal{E}^{\mathrm{glob}}(E/F)^0$ for the subgroup of elements of $T_{2,E}(\mathbb{A}_E)$ whose image in $\mathbb{A}_E^\times/E^\times$ under any of the characters π_w corresponding to $w \in V_E$ is independent of w. It turns out (as observed by Nakayama and Tate), that there is a unique class $[\alpha_{E/F}^{\mathrm{glob}}] \in H^2(\mathrm{Gal}\,(E/F), \mathcal{E}^{\mathrm{glob}}(E/F)^0)$ which pushes forward to $[\alpha_{E/F}^W] \in H^2(\mathrm{Gal}\,(E/F), \mathbb{A}_E^\times/E^\times)$ and to $[\alpha_{E/F}^{\mathrm{loc}}] \in H^2(\mathrm{Gal}\,(E/F), T_{2,E}(\mathbb{A}_E))$. We write $\mathcal{E}^{\mathrm{glob}}(E/F)$ for the corresponding extension of $\mathrm{Gal}\,(E/F)$ by $\mathcal{E}^{\mathrm{glob}}(E/F)^0$.

In this case, for an algebraic group G/F, basic algebraic cohomology of $G(\mathbb{A}_E)$ will be defined in terms of those cocycles whose restriction to $\mathcal{E}_2(E/F)^0$ come from an algebraic character $\nu : T_{2,E} \to Z(G)_{/F}$.

There are embeddings of extensions $\mathrm{loc}_{\mathfrak{a}} : \mathcal{E}^{\mathrm{glob}}(E/F) \hookrightarrow \mathcal{E}_2(E/F)$ giving rise to isomorphisms

$$\mathrm{loc}_{\mathfrak{a}} = (\mathrm{loc}_{\mathfrak{a}}^*)^{-1} : Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F), G(\mathbb{A}_E))_{\mathrm{basic}} \xrightarrow{\sim} Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F), G(\mathbb{A}_E))_{\mathrm{basic}}.$$

The map of extensions is only defined up to composition with conjugation by an element of $T_{2,E}(\mathbb{A}_E)$; and the map of cocycles is canonically defined only up to composition with the map from $Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F), G(\mathbb{A}_E))_{\mathrm{basic}}$ to itself given by $\phi \mapsto {}^{\phi(t)}\phi$ for some $t \in T_{2,E}(\mathbb{A}_E)$.

(5) Finally we will write $T_{3,E}$ for the protorus over $\mathbb{Q}$ with character group $\mathbb{Z}[V_E]_0$, the subabelian group of $\mathbb{Z}[V_E]$ consisting of elements $\sum m_w w$ for which $\sum m_w = 0$. We will write $\mathcal{E}_3(E/F)$ for the pushout of $\mathcal{E}^{\mathrm{glob}}(E/F)$ along $\mathcal{E}^{\mathrm{glob}}(E/F)^0 \to T_{3,E}(E)$.

In this case, for an algebraic group G/F, basic algebraic cohomology of $G(E)$ will be defined in terms of those cocycles whose restriction to $T_{3,E}(E)$ come from an algebraic character $\nu : T_{2,E} \to Z(G)_{/F}$. Thus there is a natural morphism $Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F), G(E))_{\mathrm{basic}} \to Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F), G(\mathbb{A}_E))_{\mathrm{basic}}$.

We have a diagram of morphisms of extensions:

$$\begin{array}{ccccccc} & & \mathcal{E}_3(E/F) & \twoheadleftarrow & \mathcal{E}^{\mathrm{glob}}(E/F) & \twoheadrightarrow & W_{E/F} \\ & & & & \mathrm{loc}_{\mathfrak{a}} \downarrow & & \\ & & \mathcal{E}^{\mathrm{loc}}(E/F) & \hookrightarrow & \mathcal{E}_2(E/F) & & \\ & & \bigcup & & & & \\ W_{E_w/F_v} & \twoheadleftarrow & \mathcal{E}^{\mathrm{loc}}(E/F)|_{\mathrm{Gal}\,(E/F)_w} & & & & \end{array}$$

for any places $w|v$ of E and F. A key observation is that, although individually the extensions $\mathcal{E}(E/F)$ we consider here have automorphisms, the diagram as a whole does not. Thus if we fix such a diagram it makes sense to consider algebraic cocycles and not just algebraic cohomology classes.

However we have not specified such a diagram uniquely. There are many choices for the localization map $\mathrm{loc}_{\mathfrak{a}}$. To the best of our knowledge there is no preferred choice. The various choices form a set which we will denote $\mathcal{H}(E/F)$, which comes with a transitive action of $T_{2,E}(\mathbb{A}_E)$. It seems to us that choosing one element of $\mathcal{H}(E/F)$ is a bit like choosing one place of $\overline{E}$ above a given place of E: there are many choices, but for most purposes the choice is irrelevant. We will decorate the various extensions $\mathcal{E}^?(E/F)$ with a subscript $\mathfrak{a}$ to indicate it is the one uniquely determined up to unique isomorphism by $\mathfrak{a}$. If $\mathfrak{a}, \mathfrak{a}' \in \mathcal{H}(E/F)$ and $\mathfrak{a}' = {}^t\mathfrak{a}$ for some $t \in T_{2,E}(\mathbb{A}_E)$, then we get isomorphisms

$$z_t : Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/F)_{\mathfrak{a}}, G(A_E)) \xrightarrow{\sim} Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/F)_{\mathfrak{a}'}, G(A_E)),$$

where A_E denotes E or $\mathbb{A}_E$. There may not be a unique choice of t and the isomorphism may depend on the t chosen. However, after one passes to cohomology groups, it will no longer depend on the choice of t.

Given a choice $\mathfrak{a} \in \mathcal{H}(E/F)$ we have maps

$$\mathrm{loc}_{\mathfrak{a}} : Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_{\mathfrak{a}}, G(E))_{\mathrm{basic}} \longrightarrow Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{\mathfrak{a}}, G(\mathbb{A}_E))_{\mathrm{basic}}$$

and

$$\mathrm{res} : Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{\mathfrak{a}}, G(\mathbb{A}_E))_{\mathrm{basic}} \longrightarrow \prod_{w \in V_E} Z^1_{\mathrm{alg}}(W_{E_w/F_w,\mathfrak{a}}, G(E_w))_{\mathrm{basic}}.$$

We will decorate res with an super (resp. sub) script S to denote projection to only those w not in (resp. in) S. These induce maps in cohomology

$$\mathrm{loc} : H^1_{\mathrm{alg}}(\mathcal{E}_3(E/F), G(E))_{\mathrm{basic}} \longrightarrow H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F), G(\mathbb{A}_E))_{\mathrm{basic}}$$

and

$$\mathrm{res} : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F), G(\mathbb{A}_E))_{\mathrm{basic}} \longrightarrow \prod_{w \in V_E} H^1_{\mathrm{alg}}(W_{E_w/F_w}, G(E_w))_{\mathrm{basic}},$$

which are canonically independent of $\mathfrak{a}$. Moreover if E'/F' is an extension of local fields isomorphic (but not canonically so) to E_w/F_w, then there is a well defined map

$$\mathrm{res}_{E'/F'} : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F), G(\mathbb{A}_E))_{\mathrm{basic}} \longrightarrow H^1_{\mathrm{alg}}(W_{E'/F'}, G(E'))_{\mathrm{basic}}.$$

If F is local and $\phi \in Z^1_{\mathrm{alg}}(W_{E/F,\mathfrak{a}}, G(E))_{\mathrm{basic}}$ or if F is global and $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$, then $\mathrm{ad}\,\phi \in Z^1(\mathrm{Gal}\,(E/F), (G/Z(G))(E))$ and we get an inner form ${}^{\phi}G$ of G over F. If $Z(G)$ is a torus then Kottwitz showed that

$$H^1_{\mathrm{alg}}(W_{E/F}, G(E))_{\mathrm{basic}} \twoheadrightarrow H^1(\mathrm{Gal}\,(E/F), (G/Z(G))(E))$$

and

$$H^1_{\mathrm{alg}}(\mathcal{E}_3(E/F), G(E))_{\mathrm{basic}} \twoheadrightarrow H^1(\mathrm{Gal}\,(E/F), (G/Z(G))(E))$$

are surjective.

We recall that when G is connected reductive, Kottwitz defined important maps

$$\kappa_G : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F), G(\mathbb{A}_E))_{\mathrm{basic}} \longrightarrow (\mathbb{Z}[V_E] \otimes \Lambda_G)_{\mathrm{Gal}\,(E/F)}$$

and

$$\kappa_G : H^1_{\mathrm{alg}}(\mathcal{E}_3(E/F), G(E))_{\mathrm{basic}} \longrightarrow (\mathbb{Z}[V_E]_0 \otimes \Lambda_G)_{\mathrm{Gal}\,(E/F)},$$

where Λ_G denotes the algebraic fundamental group of G. They are compatible in that $\kappa_G \circ \mathrm{loc}$ equals κ_G composed with the obvious map $(\mathbb{Z}[V_E]_0 \otimes \Lambda_G)_{\mathrm{Gal}\,(E/F)} \to (\mathbb{Z}[V_E] \otimes \Lambda_G)_{\mathrm{Gal}\,(E/F)}$. We will also write

$$\overline{\kappa}_G : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F), G(\mathbb{A}_E)) \longrightarrow \Lambda_{G,\mathrm{Gal}\,(E/F)}$$

for the composition of κ_G with the map

$$\begin{array}{rcl} \mathbb{Z}[V_E] \otimes \Lambda_G & \longrightarrow & \Lambda_G \\ \sum_w x_w w & \longmapsto & \sum_w x_w. \end{array}$$

This theory is explained in sections 3, 4 and 5.

It turns out we need to fix slightly more than $\mathfrak{a} \in \mathcal{H}(E/F)$. Although the Weil group $W_{E/F}$ is only determined up to inner automorphism by an element of $\mathbb{A}_E^\times/E^\times$, the absolute Weil group $W_{\overline{F}/F}$ is much more rigid. It is determined up to conjugation by an element of $\ker(W_{\overline{F}/F} \to \mathrm{Gal}\,(\overline{F}/F))$. The extra data we will add is roughly speaking a collection of isomorphisms between $W_{\overline{F}/F}/\overline{[W_{\overline{F}/E}, W_{\overline{F}/E}]}$ and $W_{E/F}$. One might wonder why one works with $W_{E/F}$ at all, and not $W_{\overline{F}/F}/\overline{[W_{\overline{F}/E}, W_{\overline{F}/E}]}$ directly. The answer seems to be that, when $D \supset E \supset F$, the way we compare $Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/F), G(A_E))_{\mathrm{basic}}$ and $Z^1_{\mathrm{alg}}(\mathcal{E}^?(D/F), G(A_D))_{\mathrm{basic}}$ is *not* compatible with the natural map

$$W_{\overline{F}/F}/\overline{[W_{\overline{F}/D}, W_{\overline{F}/D}]} \twoheadrightarrow W_{\overline{F}/F}/\overline{[W_{\overline{F}/E}, W_{\overline{F}/E}]}.$$

More precisely by *complete rigidification data* for $\mathfrak{a} \in \mathcal{H}(E/F)$ we will mean the choice for each place v of F and each F-linear embedding $\rho : E^{\mathrm{ab}} \hookrightarrow \overline{F_v}$ (giving rise to a place $w(\rho)$ of E) a $E^\times_{w(\rho)}$-conjugacy class $[\Gamma_{v,\rho}]$ of isomorphisms of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathrm{Gal}\,(E^{\mathrm{ab}}/E) & \longrightarrow & \mathrm{Gal}\,(E^{\mathrm{ab}}/F) & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & (0) \\ & & \mathrm{Art}_E \uparrow \wr & & \widetilde{\Gamma}_{v,\rho} \downarrow \wr & & || & & \\ (0) & \longrightarrow & \mathbb{A}_E^\times/\overline{(E_\infty^\times)^0 E^\times} & \longrightarrow & W_{E/F,\mathfrak{a}}/\overline{(E_\infty^\times)^0 E^\times} & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & (0) \end{array}$$

such that

(1) $\Gamma_{v,\rho}$ lifts to an isomorphism of extensions

$$\widetilde{\Gamma}_{v,\rho} : W_{\overline{F}/F}/\overline{[W_{\overline{F}/E}, W_{\overline{F}/E}]} \xrightarrow{\sim} W_{E/F,\mathfrak{a}}$$

whose composition with the natural map

$$\theta_\rho : W_{\overline{F}_v/F_v}/\overline{[W_{\overline{F_v}/\rho(E)F_v}, W_{\overline{F_v}/\rho(E)F_v}]} \longrightarrow W_{\overline{F}/F}/\overline{[W_{\overline{F}/E}, W_{\overline{F}/E}]}$$

is equal to the composition of a canonical map

$$\iota^{\mathfrak{a}}_{w(\rho)} : W_{E_{w(\rho)}/F_v,\mathfrak{a}} \longrightarrow W_{E/F,\mathfrak{a}}$$

with some isomorphism of extensions

$$\widetilde{\Theta} : W_{\overline{F}_v/F_v}/\overline{[W_{\overline{F_v}/\rho(E)F_v}, W_{\overline{F_v}/\rho(E)F_v}]} \xrightarrow{\sim} W_{E_{w(\rho)}/F_v,\mathfrak{a}};$$

(2) and if $\sigma \in \mathrm{Gal}\,(E^{\mathrm{ab}}/F)$ then $[\Gamma_{v,\rho\sigma}]$ is determined in an explicit way by $[\Gamma_{v,\rho}]$ and σ.

We will denote by $\mathcal{H}(E/F)^+$ the set pairs $(\mathfrak{a}, \{[\Gamma_{v,\rho}]\})$, where $\{[\Gamma_{v,\rho}]\}$ is complete rigidification data for $\mathfrak{a}$. The action of $T_{2,E}(\mathbb{A}_E)$ on $\mathcal{H}(E/F)$ lifts to a transitive action on $\mathcal{H}(E/F)^+$. One consequence of the choice of $\mathfrak{a}^+ \in \mathcal{H}(E/F)^+$ is that if T/F is a torus split by E, if $\mu \in X_*(T)(\overline{F_v})$ and if $\tau \in \mathrm{Aut}\,(\overline{F}_v/F)$, then we can associate an important element

$$\bar{b}_{\mathfrak{a}^+,v,\mu,\tau} \in T(\mathbb{A}_E)/\overline{T(F)T(F_\infty)^0}T(E)T(E_v).$$

We have

$$\bar{b}_{t\mathfrak{a}^+,v,\mu,\tau} = \bar{b}_{\mathfrak{a}^+,v,\mu,\tau} \prod_\rho ({}^{\rho^{-1}}\mu)(t_{w(\rho)}/t_{w(\tau\rho)}),$$

where ρ runs over F-linear embeddings $E \hookrightarrow \overline{F_v}$. This is all discussed in section 6.

Finally when $D \supset E \supset F$ and $\mathfrak{a}_E^+ \in \mathcal{H}(E/F)^+$ and $\mathfrak{a}_D^+ \in \mathcal{H}(D/F)^+$ we need to compare the sets $Z^1_{\mathrm{alg}}(\mathcal{E}(E/F)_{\mathfrak{a}_E}, G(A_E))_{\mathrm{basic}}$ with the sets $Z^1_{\mathrm{alg}}(\mathcal{E}(D/F)_{\mathfrak{a}_D}, G(A_D))_{\mathrm{basic}}$; and the elements $\bar{b}_{\mathfrak{a}_E^+,v,\mu,\tau}$ and $\bar{b}_{\mathfrak{a}_D^+,v,\mu,\tau}$. It turns out that $\mathfrak{a}_D^+$ and $\mathfrak{a}_E^+$ can be related by certain elements $t \in T_{2,E}(\mathbb{A}_D)$ and the choice of such an element both gives rise to maps

$$\overset{?}{\underset{D/E,t}{\inf}} : Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/F)_{\mathfrak{a}_E}, G(A_E)) \longrightarrow Z^1_{\mathrm{alg}}(\mathcal{E}^?(D/F)_{\mathfrak{a}_D}, G(A_D))$$

and to equalities

$$\bar{b}_{\mathfrak{a}_D^+,v,\mu,\tau} = \bar{b}_{\mathfrak{a}_E^+,v,\mu,\tau} \prod_\rho ({}^{\rho^{-1}}\mu)(t_{w(\rho)}/t_{w(\tau\rho)}),$$

where again ρ runs over F-linear embeddings $E \hookrightarrow \overline{F_v}$.

1.2. **Some algebraic cohomology classes.** We must introduce some algebraic cohomology classes needed for our discussion of Shimura varieties. First we consider the algebraic cohomology of $W_{\mathbb{C}/\mathbb{R}}$ and its relation to cocharacters. For details of this see section 3.4.

Recall that

$$W_{\mathbb{C}/\mathbb{R}} = \langle \mathbb{C}^\times, j : \; jzj^{-1} = {}^c z \text{ and } j^2 = -1 \rangle.$$

To a character $\mu : \mathbb{G}_m \to G_{/\mathbb{C}}$ such that $\mu^c\mu$ is central we can associate a cocycle $\widehat{\lambda}_G(\mu) \in Z^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R}}, G(\mathbb{C}))_{\mathrm{basic}}$ defined by $\widehat{\lambda}_G(\mu)(z) = (\mu^c\mu)(z)$ and $\widehat{\lambda}_G(\mu)(j) = \mu(-1)$. (This depends on a choice of $j \in W_{\mathbb{C}/\mathbb{R}}$.) The class $\widehat{\boldsymbol{\lambda}}_G(\mu) \in H^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R}}, G(\mathbb{C}))_{\mathrm{basic}}$ of $\widehat{\lambda}_G(\mu)$ only depends on the $G(\mathbb{R})$ conjugacy class of μ (and is independent of the choice of j). We will denote it $\widehat{\boldsymbol{\lambda}}_G([\mu]_{G(\mathbb{R})})$. If G is connected reductive then $\kappa_G\widehat{\boldsymbol{\lambda}}_G(\mu)$ equals the image of μ in $\Lambda_{G,\mathrm{Gal}\,(\mathbb{C}/\mathbb{R})}$.

If $G/\mathbb{R}$ is connected reductive, we will call a $G(\mathbb{R})$-conjugacy class Y of cocharacaters of G *compactifying* if

- if $\mu \in Y$, then $\mu^c\mu$ is central;

- and $\mathrm{ad}\,\mu(-1)$ is a Cartan involution.

In this case $\widehat{\boldsymbol{\lambda}}(Y)$ determines Y.

If G is connected reductive and $G^{\mathrm{ad}}(\mathbb{R})$ is compact, then any $G(\mathbb{C})$-conjugacy class C of morphisms $\mu : \mathbb{G}_m \to G_{/\mathbb{C}}$ contains a unique $G(\mathbb{R})$-conjugacy class of characters μ with $\mu^c\mu$ central. In this case we can define $\widetilde{\boldsymbol{\lambda}}_G(C) = [\widehat{\boldsymbol{\lambda}}_G(\mu^{-1})]$ for any such $\mu \in C$. (Note the possibly confusing choice of sign.) The group ${}^{\widetilde{\boldsymbol{\lambda}}_G(C)}G$, which is well defined up to conjugation by elements of ${}^{\widetilde{\boldsymbol{\lambda}}_G(C)}G(\mathbb{R})$, comes equipped with a canonical compactifying ${}^{\widetilde{\boldsymbol{\lambda}}_G(C)}G(\mathbb{R})$-conjugacy class of cocharacters $Y(C)_{{}^{\widetilde{\boldsymbol{\lambda}}_G(C)}G}$ contained in C.

Now suppose that Y is a compactifying $G(\mathbb{R})$-conjugacy class of cocharacters of G and that C is a $G(\mathbb{C})$-conjugacy class of cocharacters of G. We set

$$\widehat{\boldsymbol{\lambda}}_G(Y - C) = \widetilde{\boldsymbol{\lambda}}_{{}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G}(C)[\widehat{\boldsymbol{\lambda}}_G(Y)] \in H^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R}}, G)_{\mathrm{basic}}.$$

It comes equipped with a canonical compactifying ${}^{\widehat{\boldsymbol{\lambda}}_G(Y-C)}G(\mathbb{R})$-conjugacy class of cocharacters $Y(C)_{{}^{\widehat{\boldsymbol{\lambda}}_G(Y-C)}G}$.

Now suppose that $G/\mathbb{Q}$ is a connected reductive group, that $E/\mathbb{Q}$ is a sufficiently large finite Galois extension, and that Y is a compactifying $G(\mathbb{R})$-conjugacy class of cocharacters of G defined over $\mathbb{C}$. If $\tau \in \mathrm{Aut}\,(\mathbb{C})$, then by an important theorem of Kottwitz there is a unique class $\boldsymbol{\phi}_{G,Y,\tau} \in H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(K))_{\mathrm{basic}}$ such that

- $\kappa_G(\boldsymbol{\phi}_{G,Y,\tau}) = (v(\rho) - v(\tau\rho)) \otimes \lambda_G(Y)$, where $\rho : E \hookrightarrow \mathbb{C}$ and $v(\rho)$ denotes the corresponding infinite place of E (this is independent of the choice of ρ);
- and $\mathrm{res}_{\mathbb{C}/\mathbb{R}}\mathrm{loc}\boldsymbol{\phi}_{G,Y,\tau} = \widehat{\boldsymbol{\lambda}}_G(Y - {}^{\tau}[Y]_{G(\mathbb{C})})$.

In this case $\mathrm{res}^{\infty}\mathrm{loc}\boldsymbol{\phi}_{G,Y,\tau} = 1$. If $\phi \in \boldsymbol{\phi}_{G,Y,\tau}$, then ${}^{\phi}G$ comes equipped with a canonical compactifying ${}^{\phi}G(\mathbb{R})$-conjugacy class $Y({}^{\tau}[Y]_{G(\mathbb{C})})_{{}^{\phi}G}$ of cocharacters, which we will simply denote ${}^{\tau,\phi}Y$.

1.3. **Conjugation of Deligne's Shimura varieties.** One can define a Shimura datum (in the sense of Deligne) to be a pair (G, Y), where $G/\mathbb{Q}$ is a connected reductive group and Y is a compactifying $G(\mathbb{R})$-conjugacy class of *miniscule* cocharacters $\mu : \mathbb{G}_m \to G_{/\mathbb{C}}$. It is more common to consider instead of Y a $G(\mathbb{R})$-conjugacy class of morphisms $h : \mathrm{RS}^{\mathbb{C}}_{\mathbb{R}}\mathbb{G}_m \to G_{/\mathbb{R}}$ satisfying certain properties, but these two notions are easily seen to be equivalent. (To a μ as above we associate h_μ which is the descent from $\mathbb{C}$ to $\mathbb{R}$ of $(\mu, {}^c\mu)$.) Also note that Deligne assumes that G^{ad} has no simple factor over $\mathbb{Q}$ whose real points are compact. However, as we will see, everything (that we will be discussing) remains true without this assumption.

To the Shimura datum (G, Y) and a neat open compact subgroup $U \subset G(\mathbb{A}^\infty)$, Deligne associates a smooth quasi-projective variety $\mathrm{Sh}(G, Y)_U/\mathbb{C}$ (called a Shimura variety) together with an identification of complex manifolds

$$G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)/U \times Y) \xrightarrow{\sim} \mathrm{Sh}(G, Y)_U(\mathbb{C}).$$

The system of these Shimura varieties as U varies has an action of $G(\mathbb{A}^\infty)$ (by right translation). If $f : (G, Y) \to (G', Y')$ is a morphism of Shimura data (i.e. a morphism $f : G \to G'$ of algebraic groups over $\mathbb{Q}$ which carries Y to Y') then there is an induced maps of Shimura varieties. Deligne defines the reflex field $E(G, Y) \subset \mathbb{C}$ to be the number field which is the fixed field of all automorphsims of $\mathbb{C}$ which fix the $G(\mathbb{C})$ conjugacy class $[Y]_{G(\mathbb{C})}$ of cocharacters of G, which conatins Y. He conjectured that $\mathrm{Sh}(G, Y)_U$ has a model over $E(G, Y)$ satisfying certain additional properties, which determine it uniquely. He proved this in many cases and Milne proved it in all cases. Langlands conjectured a rather complicated and apparently ad hoc formula for the conjugate of $\mathrm{Sh}(G, Y)_U$ by any automorphism of $\mathbb{C}$. This was also proved by Milne.

Fix a sufficiently large finite Galois extension $E/\mathbb{Q}$ and $\mathfrak{a}^+ \in \mathcal{H}(E/\mathbb{Q})^+$. If (G, Y) is a Shimura datum and $\phi \in \boldsymbol{\phi}_{G,Y,\tau}$, then $({}^\phi G, {}^{\tau,\phi} Y)$ is another Shimura datum. If moreover $b \in G(\mathbb{A}_E^\infty)$ with $\mathrm{res}^\infty \mathrm{loc}_\mathfrak{a} \phi = {}^b 1$, then we will define an isomorphism

$$\Phi_{\mathfrak{a}^+}(\tau, \phi, b) : {}^\tau \mathrm{Sh}(G, Y)_U \xrightarrow{\sim} \mathrm{Sh}({}^\phi G, {}^{\tau,\phi} Y)_{bUb^{-1}}.$$

These maps commute with the action of $G(\mathbb{A}^\infty)$ (using the identification $\mathrm{conj}_b : G(\mathbb{A}^\infty) \overset{\sim}{\to} {}^\phi G(\mathbb{A}^\infty)$) and with the action of morphisms $f : (G, Y) \to (G', Y')$ of Shimura data. One has a cocycle relation

$$\Phi_{\mathfrak{a}^+}(\tau_1\tau_2, \phi_1\phi_2, b_1b_2) = \Phi_{\mathfrak{a}^+}(\tau_1, \phi_1, b_1) \circ {}^{\tau_1}\Phi_{\mathfrak{a}^+}(\tau_2, \phi_2, b_2).$$

In the case where $G = T$ is a torus there is an explicit formula for the $\Phi_{\mathfrak{a}^+}(\tau, \phi, b)$. These properties together completely (over) characterize the maps $\Phi_{\mathfrak{a}^+}(\tau, \phi, b)$. We also explain how the maps $\Phi_{\mathfrak{a}^+}(\tau, \phi, b)$ depend on E and $\mathfrak{a}^+$. (See theorem 8.5 for all this.)

In particular the maps $\Phi_{\mathfrak{a}^+}(\tau, 1, 1)$ for $\tau \in \mathrm{Aut}\,(\mathbb{C})$ fixing $E(G, Y)$ provide descent data for $\mathrm{Sh}(G, Y)_U$ from $\mathbb{C}$ to $E(G, Y)$, which yields the canonical model of $\mathrm{Sh}(G, Y)_U$ over $E(G, Y)$.

The conjugation morphisms, whose existence was conjectured by Langlands and proved by Milne, are special cases of our maps $\Phi_{\mathfrak{a}^+}(\tau, \phi, b)$ in which ϕ and b factor through a suitable maximal torus in G and take a very particular form. Indeed our theorem follows easily from Milne's theorem, once we were able to discover the correct formulation (and unravel Langlands definitions).

This is all discussed in section 8.

1.4. **Rational Shimura varieties.** Finally we propose an alternative formalism, which we feel is better suited to keeping track of the rationality properties of Shimura varieties.

Fix a sufficiently large Galois extension $E/\mathbb{Q}$ and $\mathfrak{a}^+ \in \mathcal{H}^+(E/\mathbb{Q})$. The theory we describe is independent of these choices, in a way that is described precisely in the body of the paper.

By a *rational Shimura datum* over a field L of characteristic 0 we mean a triple (G, ψ, C), where $G/\mathbb{Q}$ is a connected reductive group; $\psi \in Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_\mathfrak{a}, G(\mathbb{A}_E))_{\mathrm{basic}}$;

and C is a conjugacy class of miniscule cocharacters of G (considered as a variety) defined over L; such that

- ${}^{\mathrm{res}_{\mathbb{C}/\mathbb{R}}\psi}G^{\mathrm{ad}}(\mathbb{R})$ is compact
- and $\overline{\kappa}_G(\psi)$ equals the image of ${}^{\rho^{-1}}C$ in $\Lambda_{G,\mathrm{Gal}\,(E/\mathbb{Q})}$, where $\rho : L \hookrightarrow \mathbb{C}$. (This is independent with the choice of ρ.)

The group G plays very little role except as a basis point to identify the class of extended pure inner forms with which we are working. One gets a completely equivalent theory if one replaces G by ${}^{\phi}G$ for $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$. Thus, in the case that $Z(G)$ is connected, we may assume without loss of generality that G is quasi-split.

To a rational Shimura datum (G, ψ, C) over L and a neat open compact subgroup $U \subset {}^{\psi}G(\mathbb{A}^{\infty})$, where ${}^{\psi}G$ is the inner form of G over $\mathbb{A}$ defined by ψ, we associate a smooth quasi-projective variety $\mathrm{Sh}(G, \psi, C)_U/L$. As U varies the system of varieties has an action of ${}^{\psi}G(\mathbb{A}^{\infty})$. (Note that ${}^{\psi}G/\mathbb{A}$ may well not arise from a group over $\mathbb{Q}$, it is what one might call 'incoherent'.)

These rational Shimura varieties are not exactly equal to canonical models of Deligne's Shimura varieties, rather they are finite unions of isomorphic copies of a single such canonical model. Thus they carry the same information. Indeed when one describes Shimura varieties as moduli spaces over rings of mixed characteristics it is these rational Shimura varieties that arise, as has long been observed. (See for example [K2] and [HT].) An additional benefit is that these rational Shimura varieties actually have an action of a larger group than ${}^{\psi}G(\mathbb{A}^{\infty})$, a group that transitively permutes the constituent Deligne Shimura varieties. More precisely let Γ denote the abelian group

$$\{(\zeta, g) \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \times {}^{\psi}G(\mathbb{A}_E) : \ (\mathrm{loc}_{\mathfrak{a}}\zeta)^g\psi = \psi\}$$

with componentwise multiplication. There are embeddings

$$\begin{array}{rcl} {}^{\psi}G(\mathbb{A}) & \hookrightarrow & \Gamma \\ g & \longmapsto & (1, g) \end{array}$$

and

$$\begin{array}{rcl} Z(G)(E) & \hookrightarrow & \Gamma \\ \delta & \longmapsto & ((e \mapsto \delta/{}^e\delta), \delta^{-1}). \end{array}$$

We define

$$\widetilde{G}_{E,\psi}(\mathbb{A}^{\infty}) = \Gamma/Z(G)(E)\overline{Z(G)(\mathbb{Q})^{\psi}G(\mathbb{R})}.$$

(The notation is not meant to suggest that $\widetilde{G}_{E,\psi}(\mathbb{A}^{\infty})$ is the $\mathbb{A}^{\infty}$ points of any algebraic group.) Then we have an exact sequence

$$\begin{array}{l} (0) \longrightarrow {}^{\psi}G(\mathbb{A}^{\infty})/\overline{Z(G)(\mathbb{Q})} \longrightarrow \widetilde{G}_{E,\psi}(\mathbb{A}^{\infty}) \longrightarrow \\ \ker(H^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\psi}G(\mathbb{A}_E))) \longrightarrow (0). \end{array}$$

The action of ${}^{\psi}G(\mathbb{A}^\infty)$ on the system of the $\{\mathrm{Sh}(G,\psi,C)_U\}_U$ extends to an action of $\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$, which permutes transitively the constituent Deligne Shimura varieties.

The action of Galois on Shimura varieties for rational Shimura data becomes completely transparent. If $\tau : L \to L'$ then $\{{}^{\tau}\mathrm{Sh}(G,\psi,C)_U\}_U = \{\mathrm{Sh}(G,\psi,{}^{\tau}C)_U\}_U$ (with their $\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$-actions).

Shimura varieties for rational Shimura data are also functorial in the rational Shimura data in the following sense: By a *morphism* $(\phi,g,f) : (G_1,\psi_1,C_1) \to (G_2,\psi_2,C_2)$ of rational Shimura data over L, we mean

- a cocycle $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G_2(E))_{\mathrm{basic}}$,
- an element $g \in G_2(\mathbb{A}_E)$,
- and a morphism $f : G_1 \to {}^{\phi}G_2$ defined over $\mathbb{Q}$, such that $f \circ \psi_1 = {}^{g^{-1}}\psi_2 \mathrm{loc}_{\mathfrak{a}}\phi^{-1}$ and $f(C_1) \subset C_2$.

Given such a morphism we obtain a morphism a morphism of inverse systems of varieties over E:

$$\mathrm{Sh}(\phi,g,f) : \{\mathrm{Sh}(G_1,\psi_1,C_1)_U\}_U \longrightarrow \{\mathrm{Sh}(G_2,\psi_2,C_2)_V\}_V.$$

(The case $\phi = 1$ and $f = 1$ recovers the action of ${}^{\psi}G(\mathbb{A}^\infty)$.) We have

$$\mathrm{Sh}(\phi_1,g_1,f_1) \circ \mathrm{Sh}(\phi_1,g_2,f_2) = \mathrm{Sh}(f_1(\phi_2)\phi_1, g_1 f_1(g_2), f_1 \circ f_2).$$

If $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$, then $\mathrm{Sh}(\phi,1,1)$ gives a canonical isomorphism between the the tower $\{\mathrm{Sh}(G,\psi,C)_U\}_U$ with its $\widetilde{G}_{\psi}(\mathbb{A}^\infty)$-action and the alternative tower $\{\mathrm{Sh}({}^{\phi}G, \psi\mathrm{loc}_{\mathfrak{a}}\phi^{-1}, C)_U\}_U$ with its $\widetilde{{}^{\phi}G}_{\psi\mathrm{loc}_{\mathfrak{a}}\phi^{-1}}(\mathbb{A}^\infty) = \widetilde{G}_{G,\psi}(\mathbb{A}^\infty)$-action. Thus, as we have already mentioned, the exact choice of G amongst its class of inner forms is not so important.

For any $g \in G(\mathbb{A}_E)$ the map $\mathrm{Sh}(1,g,1)$ gives an isomorphism between the system $\{\mathrm{Sh}(G,\psi,C)_U\}_U$ with its $\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$-action and $\{\mathrm{Sh}(G,{}^{g}\psi,C)_V\}_V$ with its $\widetilde{G}_{E,{}^{g}\psi}(\mathbb{A}^\infty)$-action, where we use conjugation by g to identify $\widetilde{G}_{\psi}(\mathbb{A}^\infty)$ and $\widetilde{G}_{{}^{g}\psi}(\mathbb{A}^\infty)$. Thus in a sense $\{\mathrm{Sh}(G,\psi,C)_U\}_U$ only depends on $[\psi] \in H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q}), G(\mathbb{A}_E))_{\mathrm{basic}}$. However the identification is not canonical - it depends on the choice of g taking ψ to ${}^{g}\psi$. This is why we have to work with cocycles and not only cohomology classes.

There is of course a theory of complex uniformization for rational Shimura varieties. If $\rho : L \hookrightarrow \mathbb{C}$, then ${}^{\rho}\mathrm{Sh}(G,\psi,C)_U(\mathbb{C})$ admits a uniformization by an Hermitian symmetric space, but this depends on auxiliary choices. We must choose $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}}$ and $b \in G(\mathbb{A}_E^\infty)$ with $\mathrm{res}_{\mathbb{C}/\mathbb{R}}\mathrm{loc}[\phi] = \boldsymbol{\lambda}_{{}^{\psi}G}(C)\mathrm{res}_{\mathbb{C}/\mathbb{R}}[\psi]$ and $\mathrm{res}^\infty \mathrm{loc}_{\mathfrak{a}}\phi = {}^{b}\mathrm{res}^\infty\psi$. We will write ${}^{\phi}G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}}$ for the subgroup of elements of ${}^{\phi}G^{\mathrm{ad}}(\mathbb{Q})$ which can be lifted to both ${}^{\phi}G(\mathbb{R})$ and ${}^{\phi}G(E)$. Then ${}^{\phi}G^{\mathrm{ad}}(\mathbb{R})_{E,\mathbb{R}}$ acts on $Y({}^{\tau}C)$ and there is an embedding ${}^{\phi}G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}} \hookrightarrow \widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$. There is an isomorphism of complex manifolds

$$\pi_{\rho,(\phi,b)} : {}^{\phi}G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}} \backslash (\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)/U \times Y({}^{\tau}C)) \xrightarrow{\sim} {}^{\rho}\mathrm{Sh}(G,\psi,C)_U(\mathbb{C}).$$

Finally in the case that $G = T$ is a torus the action of Galois can be made explicit: if $\rho : L \hookrightarrow \mathbb{C}$ and $\tau \in \mathrm{Aut}\,(\mathbb{C})$ then

$$(\tau \circ \pi_{\rho,(\phi,b)})(\widetilde{g}, \mu) = \pi_{\tau\rho,(\phi_\tau \phi, b_\tau b)}(\widetilde{g}, {}^\tau\mu),$$

for any $\phi_\tau \in \boldsymbol{\phi}_{T,\{{}^\rho\mu\},\tau}$ and $b_\tau \in T(\mathbb{A}_E^\infty)/\overline{T(\mathbb{Q})}$ such that $\mathrm{res}^\infty \mathrm{loc}_{\mathfrak{a}} \phi_\tau = {}^{b_\tau}1$ and the image of b_τ in $T(\mathbb{A}_E^\infty)/\overline{T(\mathbb{Q})}T(E)$ is $\bar{b}_{\mathfrak{a}^+,\infty,\mu,\tau}$.

This is discussed in section 9. For a complete statement of the main theorem results mentioned here see theorem 9.1.

2. Algebraic background

2.1. **Notations.** *For simplicity we will assume all fields we consider in this paper will be assumed to be perfect unless we specifically say otherwise.*

If F is a field we will write $\overline{F}$ for an algebraic closure of F and F^{ab} for a maximal abelian Galois extension of F.

If F is a local field of characteristic 0 we will write $\mathrm{Art}_F : F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ for the Artin map. (Normalized to take uniformizers to geometric Frobenius elements.)

If F is an algebraic extension of $\mathbb{Q}$ we will write V_F for the set of places of F and $\mathbb{A}_F$ for the ring of adeles of F. (In the case that F is an infinite extension of $\mathbb{Q}$ then $\mathbb{A}_F = \lim_{\to E} \mathbb{A}_E$, where E runs over subfields of F finite over $\mathbb{Q}$.) If v is a place of F then by F_v we will mean $\lim_{\to E} E_v$ as E runs over subextensions of $F/\mathbb{Q}$ which are finite over $\mathbb{Q}$. (So F_v may not be complete, but it is algebraic over $\mathbb{Q}_v$.) If F is a number field will write $\mathrm{Art}_F : \mathbb{A}_F^\times/\overline{F^\times (F_\infty^\times)^0} \xrightarrow{\sim} \mathrm{Gal}(F^{\mathrm{ab}}/F)$ for the Artin map.

If E/F is an algebraic extension of fields with F a number field and if $S \subset V_F$ we will write S_E for the set of places of E above a place in S, and $\mathbb{A}_{E,S}$ for the ring of adeles of E supported at the primes in S. (If E is also a number field then $\mathbb{A}_{E,S}$ is the restricted product $\prod'_{w:\, w|_F \in S} E_w^\times$.) Moreover $\mathbb{A}_E^S = \mathbb{A}_{E,V_F - S}$.

We will write $\mathbb{Z}[V_{E,S}]$ for the free ablelian group on $V_{E,S}$ and $\mathbb{Z}[V_{E,S}]_0$ for the subabelian group consisting of elements $\sum_w m_w w$ with $\sum_w m_w = 0$. If E/F is Galois, both groups have a natural action of $\mathrm{Gal}(E/F)$ via $\sigma \sum_w m_w w = \sum_w m_w (\sigma w) = \sum_w m_{\sigma^{-1} w} w$.

If F is an algebraic extension of $\mathbb{Q}$ and K is a local field and $\rho : F \hookrightarrow \overline{K}$, then we will write $v(\rho)$ or $w(\rho)$ or $u(\rho)$ for the place of F induced by ρ. (We will tend to use $v(\rho)$ when the field is denoted F, $w(\rho)$ when it is denoted E and $u(\rho)$ otherwise.) If moreover $F/\mathbb{Q}$ is Galois and $\tau \in \mathrm{Aut}(\overline{K})$, then we will write $\tau^\rho = \rho^{-1}\tau\rho \in \mathrm{Gal}(F/\mathbb{Q})$.

If E/F is a Galois extension with F a number field, and if v is a real place of F we will write $[c_v]$ for the conjugacy class in $\mathrm{Gal}(E/F)$ consisting of complex conjugations at places above v. If $F = \mathbb{Q}$ and $v = \infty$ we will simply write $[c]$.

2.2. **Conjugation of schemes.** We recall some standard facts about the action of Galois groups on varieties. Suppose that $E \supset F$ are fields. We will write $\mathrm{Aut}(E)$ for the group of automorphisms of E and $\mathrm{Aut}(E/F)$ for the subgroup of elements fixing F. If E/F is Galois we shall usually write $\mathrm{Gal}(E/F)$ for $\mathrm{Aut}(E/F)$. If $X/\mathrm{Spec}\, E$ is a scheme and $\tau \in \mathrm{Aut}(E)$ then by ${}^\tau X$ we mean $X \times_{\mathrm{Spec}\, E, \mathrm{Spec}\, \tau} \mathrm{Spec}\, E$. Thus there is a natural identification ${}^{\tau_1\tau_2}X = {}^{\tau_1}({}^{\tau_2}X)$. Note that the isomorphism $\tau \times 1 : E[T_1, .., T_n] \otimes_{E,\tau} E \xrightarrow{\sim} E[T_1, ..., T_n]$ gives an identification $\mathrm{Aff}^n \xrightarrow{\sim} {}^\tau\mathrm{Aff}^n$. If $X \subset \mathrm{Aff}^n$ is the affine subscheme cut out by polynomials $f_1, ..., f_r$ then this identifies ${}^\tau X$ with the variety cut out by the polynomials ${}^\tau f_1, ..., {}^\tau f_r$.

If X is again any variety over E then there is a natural bijection

$$\begin{array}{rcl} \tau : X(E) & \xrightarrow{\sim} & ({}^\tau X)(E) \\ P & \longmapsto & P \times \mathrm{Spec}\,(\tau^{-1}). \end{array}$$

In the special case that X is an affine variety cut out by equations $f_1, ..., f_r$ in n variables, the map τ is just given by applying τ to the coordinates.

If $\psi : X \to Y$ over $\mathrm{Spec}\, E$ then ${}^\tau\psi = \psi \times_{\mathrm{Spec}\, E, \mathrm{Spec}\, \tau} \mathrm{Spec}\, E : {}^\tau X \to {}^\tau Y$. We have $({}^\tau\psi)({}^\tau P) = {}^\tau(\psi(P))$.

Suppose that X_0/F is a variety, and let $X = X_0 \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, E$. If $\tau \in \mathrm{Aut}\,(E/F)$ we get a natural identification

$$\phi_\tau^{X_0} : {}^\tau X = (X_0 \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, E) \times_{\mathrm{Spec}\, E, \mathrm{Spec}\, \tau} \mathrm{Spec}\, E \xrightarrow{\sim} X_0 \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, E = X.$$

Under the natural identification ${}^{\tau_1}({}^{\tau_2}X) = {}^{\tau_1\tau_2}X$ we see that $\phi_{\tau_1}^{X_0}{}^{\tau_1}\phi_{\tau_2}^{X_0} = \phi_{\tau_1\tau_2}^{X_0}$. We also have a natural identification $X(E) \cong X_0(E)$. Under these identifications the map $\phi_\tau^{X_0} \circ \tau : X(E) \xrightarrow{\sim} X(E)$ becomes identified with the map $\tau : X_0(E) \xrightarrow{\sim} X_0(E)$ which sends P to $P \circ \mathrm{Spec}\, \tau^{-1}$. If $f_0 : X_0 \to Y_0$ over F and $f = f_0 \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, E$ then $f \circ \phi_\tau^{X_0} = \phi_\tau^{Y_0} \circ f$.

Suppose either that E/F is algebraic and Galois, or that E is algebraically closed of infinite transcendence degree over F. If $X/\mathrm{Spec}\, E$ is a variety, then by *descent data for X over F* we mean the choice for all $\tau \in \mathrm{Aut}\,(E/F)$ of an isomorphism

$$\phi_\tau : {}^\tau X \xrightarrow{\sim} X$$

such that $\phi_{\tau_1}{}^{\tau_1}\phi_{\tau_2} = \phi_{\tau_1\tau_2}$ for all $\tau_1, \tau_2 \in \mathrm{Aut}\,(E/F)$. We call a point $x \in X(E)$ with respect to $\{\phi_\tau\}$ *formally defined over* an intermediate field E' if $\phi_\tau(\tau x) = x$ for all $\tau \in \mathrm{Aut}\,(E/E')$. We call the descent data $\{\phi_\tau\}$ *effective* if there is a variety $X_0/\mathrm{Spec}\, F$ together with an isomorphism $\psi : X_0 \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, E \xrightarrow{\sim} X$ such that $\phi_\tau \circ {}^\tau\psi = \psi \circ \phi_\tau^{X_0}$ for all $\tau \in \mathrm{Aut}\,(E/F)$. In this case we also have $\psi \circ \tau \circ \psi^{-1} = (\tau \circ \phi_\tau) : X(E) \to X(E)$. Moreover if X_0' and ψ' also have this property, then there is a unique isomorphism $f : X_0 \xrightarrow{\sim} X_0'$ over F such that $\psi' \circ (f \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, E) = \psi$. If $\{\phi_\tau'\}$ is effective descent data for Y over F, which gives rise to a pair (Y_0, ψ'), and if $f : X \to Y$ satisfies $\phi_\tau'{}^\tau f = f\phi_\tau$ for all $\tau \in \mathrm{Aut}\,(E/F)$ then there is a unique $f_0 : X_0 \to Y_0$ such that $\psi' \circ (f_0 \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, E) = f \circ \psi$. If $\sigma \in \mathrm{Aut}\,(E)$, then ${}^\sigma X_0/\mathrm{Spec}\, {}^\sigma F$ corresponds to ${}^\sigma X$ and the descent data over ${}^\sigma F$ given by

$$\tau \longmapsto {}^\sigma\phi_{\sigma^{-1}\tau\sigma}.$$

If X and Y are two varieties over E and if $\{\phi_\tau\}$ is effective descent data for X over F and if $\{\phi_\tau'\}$ is effective descent data for Y over ${}^\sigma F$, then giving a map

$$f_0 : {}^\sigma X_0 \longrightarrow Y_0$$

over ${}^\sigma F$ is the same as giving a map $f : {}^\sigma X \to Y$ such that

$$\phi_\tau' \circ {}^\tau f = f \circ {}^\sigma\phi_{\sigma^{-1}\tau\sigma}$$

for all $\tau \in \mathrm{Aut}\,(E/{}^\sigma F)$.

It is important to know when descent data is effective. We recall four criteria each of which guarantees that $\{\phi_\tau\}$ is effective:

(1) X is quasi-projective and there is an intermediate field $E \supset E' \supset F$ with E' finitely generated over F such that $\{\phi_\tau\}_{\tau \in \mathrm{Aut}\,(E/E')}$ is effective.
(2) X is quasi-projective and E/F is finite Galois.
(3) X is quasi-projective and there are points $x_i \in X(E)$ and an intermediate field $E \supset E' \supset F$ with E' finitely generated over F such that
   (a) the identity is the only automorphism of X fixing each x_i,
   (b) and each x_i is formally defined over E'.
(4) X is quasi-projective, $\mathrm{Aut}\,(X)$ is finite and the set of points of $X(E)$ formally defined over a finitely generated subfield of E is Zariski dense in X.

For the first criterion follows from theorems 3 and 6 of [W2] as in the proof of theorem 1.1 of [Mi2]. (In the case E/F is Galois theorem 6 of [W2] is not required.) The second criterion clearly follows from the first (or directly from theorem 3 of [W2]). The third criterion also follows from the first. Indeed we may assume (after enlarging E') that X has some model X_1 over E', that each x_i is formally defined over E' and that each $x_i \in X_1(E')$. Then, for $\tau \in \mathrm{Aut}\,(E/E')$ we have $\phi_\tau = \alpha_\tau \circ \phi_\tau^{X_1}$ with $\alpha_\tau \in \mathrm{Aut}\,(X)$. We see that each α_τ fixes each x_i and so must be the identity. Thus $\{\phi_\tau\}_{\tau \in \mathrm{Aut}\,(E/E')}$ is effective. The fourth criterion follows from the third.

2.3. **Algebraic groups.** We will first recall some facts about algebraic groups over fields of characteristic 0.

If G is an (algebraic) group then $Z(G)$ will denote its centre and G^{ad} will denote $G/Z(G)$. Moreover G^{der} will denote its commutator subgroup and $C(G) = G^{\mathrm{ab}}$ will denote it co-center/abelianization G/G^{der}. If $H \subset G$ is a subgroup we will write $N_G(H)$ for its normalizer and $Z_G(H)$ its centralizer. If H has finite index, we will also write $\mathrm{tr}_{G/H} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ for the transfer map. If G acts on X we will write $[x]_G$ for the G orbit of $x \in X$ and $Z_G(x)$ for the centralizer of x in G. If G is an algebraic group acting on a variety X over a field F and $x \in X(F)$, then $[x]_G$ is a variety, and $[x]_G(F) \supset [x]_{G(F)}$, but these two sets may not be equal.

If F is a field, if E_1,...,E_r are fields containing F, and if G/F is an algebraic group; then we will write $G^{\mathrm{ad}}(F)_{E_1,\ldots,E_r}$ for the subgroup of $G(F)$ consisting of elements which admit lifts to each $G(E_i)$.

If G is an affine algebraic group over F then there a scheme $X_*(G)$, smooth and separated over F, and a homomorphism $\mu^{\mathrm{univ}} : \mathbb{G}_m \times_F X_*(G) \to G \times_F X_*(G)$, such that if S is any F-scheme and $\mu : \mathbb{G}_{m,S} \to G_S$ is a homomorphism, then there is a unique morphism $S \to X_*(G)$ under which μ^{univ} pulls back to μ. Moreover

$$\begin{array}{ccc} G \times X_*(G) & \longrightarrow & X_*(G) \times X_*(G) \\ (g, \mu) & \longmapsto & (\mathrm{conj}_g \circ \mu, \mu) \end{array}$$

is smooth; and

$$X_*(G)_{\overline{F}} = \coprod_{[\mu] \in G(\overline{F}) \backslash X_*(G)(\overline{F})} G/Z_G(\mu).$$

(See sections 4 and 5 of exposé XI in [SGA3].) If G is geometrically connected, then the $G/Z_G(\mu)$ are the connected components of $X_*(G)$. Moreover if $\mu \in X_*(G)(\overline{F})$ and if $F([\mu])$ denotes the fixed field of $Z_{\mathrm{Gal}\,(\overline{F}/F)}([\mu]_{G(\overline{F})})$, then $X_*(G)_{F([\mu])}$ has a (unique) connected component $[\mu]$ such that $[\mu](\overline{F}) = [\mu]_{G(\overline{F})}$. (Use lemma 33.7.18 of [Stacks].)

We will require all our reductive groups to be geometrically connected, i.e. by the term 'reductive group' we will mean what is often referred to as 'connected reductive group'. If G/F is a reductive group then $G(F)$ is Zariski dense in G. (See theorem 2.2 of [PR].) We will write G^{SC} for the simply connected semi-simple cover of G^{der}. If T is a maximal torus of G we will write T^{ad} for the image of T in G^{ad} (a maximal torus in G^{ad}) and $T^{\mathrm{der}} = (G^{\mathrm{der}} \cap T)$ (which is a maximal torus in G^{der}, see remark 3.5 of [Co]) and T^{SC} for the preimage of T in G^{SC} (which is a maximal torus in G^{SC}, see for instance proposition 4.1 of [Co]). We have $T = Z_G(T)$. We will also write W_T for the Weyl group $N_G(T)/T$, which we think of a a finite algebraic group. It acts faithfully on T. We will also write $W_{T,F}$ for $N_G(T)(F)/T(F) \subset W_T(F)$. If T is split over F, then we have equalities $W_T(\overline{F}) = W_T(F) = W_{T,F}$. Moreover any two split maximal tori in G over F are conjugate by $G(F)$. (See section 2.1.14 of [PR].)

We remark that if $T \subset G$ is a maximal torus and $\mu_1, \mu_2 \in X_*(T)$ are conjugate under $G(\overline{F})$ then they are conjugate under $W_T(\overline{F})$. (This is probably well known, but as we don't know a reference we will sketch the proof. Let H denote the connected component of the identity of the centralizer of $\mu_1(\mathbb{G}_m)$ in G. It is reductive. (See theorem 2.1 of [Co].) Suppose that $\mu_1 = g\mu_2 g^{-1}$. Then $\mu_1(\mathbb{G}_m) \subset gTg^{-1}$ so that T and gTg^{-1} are both maximal tori in H. Hence we have $gTg^{-1} = hTh^{-1}$ for some $h \in H$. Then $h^{-1}g \in N_G(T)$ and $\mu_1 = h^{-1}g\mu_2 g^{-1}h$, as desired.)

We will let Λ_G denote the algebraic fundamental group of G, i.e. $X_*(T)/X_*(T^{\mathrm{SC}})$ for any maximal torus T of G. Note that the Weyl group W_T acts trivially on $X_*(T)/X_*(T^{\mathrm{SC}})$. Any two maximal tori T and T' defined over F are conjugate over the separable closure $\overline{F}$ of F by $g \in G(\overline{F})$ with $gN_G(T)$ uniquely defined. Then conj_g induces an isomorphism $X_*(T)/X_*(T^{\mathrm{SC}}) \xrightarrow{\sim} X_*(T')/X_*(T'^{\mathrm{SC}})$. If we alter g by an element $h \in N_G(T)(\overline{F})$ then this isomorphism changes by an element of $W_T(\overline{F})$, i.e. is in fact unchanged. Thus Λ_G is canonically defined independent of the choice of T. In particular it has a canonical action of $\mathrm{Gal}\,(\overline{F}/F)$. (If $T' = \mathrm{conj}_g T$ and $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$, then $\sigma \circ \mathrm{conj}_g = \mathrm{conj}_g \circ \sigma \circ \mathrm{conj}_{w_\sigma}$ on $X_*(T)$ for some $w_\sigma \in W_T(\overline{F})$, and so $\sigma \circ \mathrm{conj}_g = \mathrm{conj}_g \circ \sigma$ on Λ_T.) If $[\mu]$ is a conjugacy class of cocharacters $\mu : \mathbb{G}_m \to G$, then $[\mu]$ gives rise to well defined element $\lambda_G([\mu]) \in \Lambda_G$. If $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$ then $\lambda_G({}^\sigma[\mu]) = {}^\sigma\lambda_G([\mu])$.

Now suppose that F is a number field and G/F is a connected algebraic group. Then $G(F)$ is dense in $\prod_{v \in V_{F,\infty}} G(F_v)$. (See theorem 7.7 of [PR].) Suppose further that S is any finite set of places of F and that $T_v \subset G \times \mathrm{Spec}\, F_v$ is a maximal torus for all $v \in S$. Then there is a maximal torus $T \subset G$ such that $T \times \mathrm{Spec}\, F_v$ is $G(F_v)$-conjugate to T_v for all $v \in S$. (See corollary 3 to proposition 7.3 of [PR].)

2.4. **Real groups.** Suppose that $G/\mathbb{R}$ is a reductive group. If either G is simply connected semi-simple or $G(\mathbb{R})$ compact, then $G(\mathbb{R})$ is connected. (See corollary 1 to theorem 3.6 and proposition 7.6 of [PR].) If $G/\mathbb{R}$ is a reductive group and H is a normal subgroup defined over $\mathbb{R}$ with $(G/H)(\mathbb{R})$ compact, then $G(\mathbb{R}) \twoheadrightarrow (G/H)(\mathbb{R})$. (The image is open by the open mapping theorem, but we have just seen that $(G/H)(\mathbb{R})$ is connected.) We will write $G(\mathbb{R})^+$ for the connected component of the identity in $G(\mathbb{R})$ in the archimedean topology. Because $G(\mathbb{R})$ is Zariski dense in G, we see that $Z_G(G) = Z_G(G(\mathbb{R}))$ and that $G(\mathbb{R})^{\mathrm{ad}}$ naturally embeds in $G^{\mathrm{ad}}(\mathbb{R})$ (and $G^{\mathrm{ad}}(\mathbb{R})^{\mathrm{ad}} \supset G^{\mathrm{ad}}(\mathbb{R})^+$.) If $F \subset \mathbb{R}$ is a subfield and G is defined over F we write $G(F)^+$ for $G(F) \cap G(\mathbb{R})^+$.

If $G/\mathbb{R}$ is reductive then a maximal torus $T \subset G_{/\mathbb{R}}$ is called *fundamental* if its split rank is minimal among the split ranks of all maximal tori. All fundamental maximal tori are conjugate by $G(\mathbb{R})$. (See [BW] section I.7.1.) If G' is an inner form of G, then fundamental maximal tori in G and G' are isomorphic. (See lemma 2.8 of [Sh].) If T is a fundamental torus and if $T^{\mathrm{ad}}(\mathbb{R})$ is compact (or equivalently if c acts by -1 on $X_*(T^{\mathrm{ad}})$) then $W_T(\mathbb{R}) = W_T(\mathbb{C})$. Moreover if T is a fundamental torus, then $T^{\mathrm{ad}}(\mathbb{R})$ is compact if and only if G has an inner form G' with $G'^{,\mathrm{ad}}(\mathbb{R})$ compact. (See proposition 3 of [LS].)

If $G^{\mathrm{ad}}(\mathbb{R})$ is compact then all maximal tori T are fundamental, and hence conjugate. Moreover, in this case, $W_{T,\mathbb{R}} = W_T(\mathbb{R}) = W_T(\mathbb{C})$, so that any two embeddings $i, i' : T \hookrightarrow G$ are conjugate under $G(\mathbb{R})$. (In the case that $G(\mathbb{R})$ is compact the equality $W_{T,\mathbb{R}} = W_T(\mathbb{R})$ is well known, see for instance theorem 11.36 of [H]. The more general case $G^{\mathrm{ad}}(\mathbb{R})$ compact reduces to this because $G(\mathbb{R}) \twoheadrightarrow G^{\mathrm{ad}}(\mathbb{R})$.)

If $\mu \in X_*(G)(\mathbb{C})$ then the image of μ commutes with that of ${}^c\mu$ if and only if μ factors through a maximal torus $T \subset G$ which is defined over $\mathbb{R}$. (To see the forward implication look at a maximal torus in $Z_G(\mu)$ containing the image of ${}^c\mu$.) We will call such cocharacters *commuting*. Being a commuting is preserved under $G(\mathbb{R})$-conjugacy.

If $\mu \in X_*(G)(\mathbb{C})$ then we will call μ *basic* if $\mu^c\mu$ factors through $Z(G)$. Being basic is preserved under $G(\mathbb{R})$-conjugacy; and basic cocharacters are commuting. If μ is basic, then μ factors through a fundamental maximal torus. (To see this work in G^{ad}. Then $(\mathrm{Im}\,\mu)(\mathbb{R})$ is compact and so contained in some maximal compact subgroup of $G^{\mathrm{ad}}(\mathbb{R})$. Hence it is contained in a maximal compact torus, and so in a fundamental torus. See section I.7.1 of [BW].) If $G^{\mathrm{ad}}(\mathbb{R})$ is compact then any commuting cocharacter $\mu \in X_*(G)(\mathbb{C})$ is basic (because if μ factors through a maximal torus T defined over $\mathbb{R}$, then c acts on $X_*(T^{\mathrm{ad}})$ by -1).

If $\mu \in X_*(G)(\mathbb{C})$ then we will call μ *compactifying* if μ is basic and $\mathrm{ad}\,\mu(-1) \in G^{\mathrm{ad}}(\mathbb{R})$ is a Cartan involution (i.e. $G^{\mathrm{ad}}(\mathbb{C})^{\mathrm{conj}_{\mu(-1)}\circ c=1}$ is compact). (See for instance section 2 of [BC] for basic facts about Cartan involutions.) Being basic is preserved under $G(\mathbb{R})$-conjugacy. If G admits a compactifying cocharacter, then G^{ad} has a compact inner form.

Lemma 2.1. *Suppose that $G/\mathbb{R}$ is a reductive group and that $G^{\mathrm{ad}}(\mathbb{R})$ is compact. Any G-conjugacy class $C \subset X_*(G)$ contains a unique $G(\mathbb{R})$-conjugacy class $Y(C)$ consisting of commuting cocharacters in $X_*(G)(\mathbb{C})$. The elements of $Y(C)$ are in fact basic.*

Proof: Any $\mu \in C(\mathbb{C})$ factors through some maximal torus and hence is conjugate to a cocharacter factoring through any other maximal torus, for instance one defined over $\mathbb{R}$. If $\mu, \mu' \in C(\mathbb{C})$ factor through maximal tori defined over $\mathbb{R}$, then replacing μ' by a $G(\mathbb{R})$-conjugate, we may assume it factors through the same maximal torus T (defined over $\mathbb{R}$) as μ. Then μ and μ' are conjugate by an element of $W_T(\mathbb{C}) = W_{T,\mathbb{R}}$, i.e. μ and μ' are $N_G(T)(\mathbb{R})$-conjugate. $\square$

2.5. **Cohomology.** If G/F is an algebraic group and E/F is a Galois extension then we will write $H^1(\mathrm{Gal}\,(E/F), G(E))$ for the first Galois cohomology. More precisely, by a 1 cocycle of $\mathrm{Gal}\,(E/F)$ we will mean a locally constant map $\phi : \mathrm{Gal}\,(E/F) \to G(E)$ such that $\phi(\sigma_1\sigma_2) = \phi(\sigma_1)^{\sigma_1}\phi(\sigma_2)$. We denote the set of 1 cocycles $Z^1(\mathrm{Gal}\,(E/F), G(E))$. If $\phi \in Z^1(\mathrm{Gal}\,(E/F), G(E))$ and $g \in G(E)$ we define ${}^g\phi \in Z^1(\mathrm{Gal}\,(E/F), G(E))$ by $({}^g\phi)(\sigma) = g\phi(\sigma)^\sigma g^{-1}$. We call two cocycles ϕ_1 and ϕ_2 equivalent if there is a $g \in G(E)$ such that $\phi_2 = {}^g\phi_1$. Then $H^1(\mathrm{Gal}\,(E/F), G(E))$ is the set of equivalence classes of cocycles. It is a pointed set with neutral element represented by the trivial cocycle (identically 1). If G is abelian then $H^1(\mathrm{Gal}\,(E/F), G(E))$ is an abelian group. (In the case $E = \overline{F}$, then we will also denote these sets $H^1(F, G)$ and $Z^1(F, G)$.)

If H is an algebraic subgroup of an algebraic group G, both defined over a field F, and if $g \in (H\backslash G)(F)$ (resp. $g \in (G/H)(F)$) one can define a class $o(g) \in H^1(F, H)$ such that g lifts to an element of $G(F)$ if and only if $o(g)$ is trivial. (If $\widetilde{g}$ is any lift of g to $G(\overline{F})$ then $o(g)$ is represented by the cocyle $\sigma \mapsto \widetilde{g}^\sigma \widetilde{g}^{-1}$.) In particular there is a bijection

$$G^{\mathrm{ad}}(F)/G(F)^{\mathrm{ad}} \xrightarrow{\sim} \ker(H^1(F, Z(G)) \to H^1(F, G)).$$

In this instance it is easily verified that both sides are groups and that the bijection is an isomorphism of abelian groups.

If F is a local field of characteristic 0, then there is a finite Galois extension E/F such that every element of $G^{\mathrm{ad}}(F)$ can be lifted to $G(E)$. (It is enough to see there is such an extension such that $H^1(\mathrm{Gal}\,(\overline{F}/F), Z(G)) \to H^1(\mathrm{Gal}\,(\overline{F}/E), Z(G))$ is the zero map. Choose E_1/F such that $Z(G)^0$ splits over E_1 and $\pi_0(Z(G))$ has a trivial $\mathrm{Gal}\,(\overline{F}/E_1)$ action, and then choose E/E_1 Galois over F such that the finitely many elements of $\mathrm{Hom}\,(\mathrm{Gal}\,(\overline{F}/E_1), \pi_0(Z(G)))$ vanish on $\mathrm{Gal}\,(\overline{F}/E)$.)

If G/F is an algebraic group and $\phi \in Z^1(\mathrm{Gal}\,(E/F), G^{\mathrm{ad}})$ then we can define ${}^\phi G$ to be the algebraic group over F obtained by descending $G \times_F E$ to F via the action

$$\sigma \longmapsto \mathrm{conj}_{\phi(\sigma)} \circ (1 \times \sigma).$$

Then $({}^{\phi}G) \times_F E = G \times_F E$. If $g \in G(E)$, then

$$\mathrm{conj}_g : {}^{\phi}G \xrightarrow{\sim} {}^{g\phi}G$$

over F. Thus ${}^{\phi}G$ only depends on $[\phi] \in H^1(\mathrm{Gal}\,(E/F), G^{\mathrm{ad}})$, but up to an isomorphism that is only unique up to composition with conjugation by an element of $G^{\mathrm{ad}}(F)$.

If G/F is an algebraic group, then by an *inner form* of G we mean a pair $(H, [i])$, where H/F is another algebraic groups and $[i]$ is a G-conjugacy class defined over F of isomorphisms $i : G \xrightarrow{\sim} H$. (Then $[i]$ has the structure of a variety over F, but may have no F-points.) By an isomorphism of inner forms $(H_1, [i_1])$ and $(H_2, [i_2])$ we mean an isomorphism of H_1 and H_2 over F which takes $[i_1]$ to $[i_2]$. Such an isomorphism is unique up to $H_1^{\mathrm{ad}}(F)$-conjugacy. The construction of the previous paragraph sets up a bijection between $H^1(F, G^{\mathrm{ad}})$ and (isomorphism classes of) inner forms of G. A pair $(H, [i])$ is sent to the class $\mathrm{inv}\,(H, [i])$ of the cocycle $\sigma \mapsto i^{-1} \circ {}^{\sigma}i$.

If $\phi \in Z^1(F, G)$ then there is a bijection

$$\begin{array}{ccc} Z^1(F, {}^{\phi}G) & \xrightarrow{\sim} & Z^1(F, G) \\ \psi & \longmapsto & \psi\phi \end{array}$$

which induces an isomorphism

$$H^1(F, {}^{\phi}G) \xrightarrow{\sim} H^1(F, G)$$

under which the trivial element maps to $[\phi]$. If $\phi_1, \phi_2 \in Z^1(F, G)$ and $[\phi_1] = [\phi_2] \in H^1(F, G^{\mathrm{ad}})$, then we can find $g \in G(\overline{F})$ and $\zeta \in Z^1(F, Z(G))$ such that

$$\phi_2 = \zeta^g\phi_1.$$

If $(H, [i])$ is an inner form of G, then it is elementary to see that all elements of $[i]$ induce the same isomorphism $Z(G) \xrightarrow{\sim} Z(H)$ which is thus defined over F. Hence we will identify $Z(G)$ and $Z(H)$ without further comment. If $[\mu]_G$ is a G-conjugacy class of cocharacters $\mu : \mathbb{G}_m \to G$, then one can associate an H-conjugacy class $[\mu]_{(H,[i])}$ of cocharacters $\mathbb{G}_m \to H$. Indeed

$$[\mu]_{(H,[i])} = \{i' \circ \mu : \ \mu \in [\mu]_G \text{ and } i \in [i]\}.$$

If $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$ then $({}^{\sigma}[\mu]_G)_{(H,[i])} = {}^{\sigma}([\mu]_{(H,[i])})$.

If G is a reductive group over F and $(H, [i])$ is an inner form of G, then the same argument that shows Λ_G with its $\mathrm{Gal}\,(\overline{F}/F)$-action is well defined shows that there is a canonical $\mathrm{Gal}\,(\overline{F}/F)$-invariant isomorphism $\Lambda_G \cong \Lambda_H$.

Suppose that G/F is a reductive group and that $T \subset G_{/F}$ is a maximal torus. Suppose that $\phi \in Z^1(F, G)$ and define $H^1(F, T)_\phi$ to be the preimage of $[\phi]$ in $H^1(F, T)$. Then there is a bijection between $H^1(F, T)_\phi$ and the set of ${}^{\phi}G(F)$-conjugacy classes $[j]$ of embeddings $j : T \hookrightarrow {}^{\phi}G$ over F conjugate over $\overline{F}$ to the given inclusion $T \hookrightarrow G_{/F}$. More precisely any embedding $T \hookrightarrow {}^{\phi}G$ is of the form $t \mapsto gtg^{-1}$ for some $g \in G(\overline{F})$

such that $g\phi(\sigma)^\sigma g^{-1} \in T(\overline{F})$ for all $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$. This sets up a bijection between the set of ${}^\phi G(F)$-conjugacy classes $[j]$

$$\{g \in T(\overline{F})\backslash G(\overline{F})/{}^\phi G(F) :\ g\phi(\sigma)^\sigma g^{-1} \in T(\overline{F})\ \ \forall \sigma \in \mathrm{Gal}\,(\overline{F}/F)\}.$$

This set is in turn in bijection with $H^1(F,T)_\phi$ via the map sending g to the cocyle $\sigma \mapsto g\phi(\sigma)^\sigma g^{-1}$.

If G is an algebraic group over a number field F, we set $H^1(\mathbb{A}_F, G) = H^1(\mathrm{Gal}\,(\overline{F}/F), G(\mathbb{A}_{\overline{F}}))$. We have that

$$H^1(\mathbb{A}_F, G) \cong {\prod_v}' H^1(F_v, G),$$

where the product is restricted with respect to $\{\mathrm{Im}\,(H^1(\mathrm{Gal}\,(F_v^{\mathrm{nr}}/F_v), G(\mathcal{O}_{F_v^{\mathrm{nr}}})) \to H^1(F_v, G))\}$. If G is connected then this becomes

$$H^1(\mathbb{A}_F, G) \cong \bigoplus_v H^1(F_v, G),$$

where the $\bigoplus_v$ of pointed sets means elements of the product that are at all but finitely many v equal to the neutral element. (See the corollary to theorem 6.8 of [PR].)

If E/F is Galois and $\phi \in Z^1(\mathrm{Gal}\,(E/F), G^{\mathrm{ad}}\,(\mathbb{A}_E))$ then we can define ${}^\phi G/\mathbb{A}_F$ to be the algebraic group defined by descending $G \times_F \mathbb{A}_E$ to $\mathbb{A}_F$ via the action

$$\sigma \longmapsto \mathrm{conj}_{\phi(\sigma)} \circ (1 \times \sigma).$$

If $g \in G(\mathbb{A}_E)$, then

$$\mathrm{conj}_g : {}^\phi G \xrightarrow{\sim} {}^{g\phi} G$$

over $\mathbb{A}_F$. Thus ${}^\phi G$ depends only on $[\phi] \in H^1(\mathrm{Gal}\,(E/F), G(\mathbb{A}_E))$, but up to an isomorphism that is only unique up to composition with conjugation by an element of $G^{\mathrm{ad}}\,(\mathbb{A}_F)$.

There is a natural map

$$H^1(F, G) \longrightarrow H^1(\mathbb{A}_F, G)$$

and we will denote the kernel $\ker^1(F,G)$. If G is reductive then $\ker^1(F,G)$ is finite. It vanishes if G is semi-simple and either adjoint or simply connected. (See for instance section 4 of [K1] and note that the 'no E_8-factors' restriction is no longer necessary because the Hasse principle is now known for all simply connected semi-simple groups by [Ch]. For finiteness see [BS] Theoreme 7.1.)

2.6. **Group extensions.** We recall some of the general theory of group extensions. Suppose that A is an abelian group and G is a finite group that acts on A. A 2-cocyle $\alpha \in Z^2(G, A)$ is a function $G \times G \to A$ satisfying the relation

$${}^{g_1}\alpha(g_2, g_3)\alpha(g_1, g_2 g_3) = \alpha(g_1 g_2, g_3)\alpha(g_1, g_2).$$

(We record that this implies that $\alpha(1,g)=\alpha(1,1)$ and $\alpha(g,1)={}^g\alpha(1,1)$.) If α is a 2-cocycle and $\beta: G\to A$ is any function then

$$^{\beta}\alpha(g_1,g_2)=\alpha(g_1,g_2)\beta(g_1g_2)\beta(g_1)^{-1}{}^{g_1}\beta(g_2)^{-1}$$

is another 2-cocycle. If α is a 2-cocycle we obtain an extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \mathcal{E}_\alpha & \longrightarrow & G & \longrightarrow & 0 \\ & & a & \longmapsto & a\alpha(1,1)^{-1}e(1) & & & & \\ & & & & ae(g) & \longmapsto & g, & & \end{array}$$

where $\mathcal{E}_\alpha$ is the group with elements $ae(g)$ with $a\in A$ and $g\in G$ with the multiplication rule

$$a_1e(g_1)a_2e(g_2)=a_1{}^{g_1}a_2\alpha(g_1,g_2)e(g_1g_2).$$

There is an isomorphism of extensions

$$\begin{array}{rcl} i_\beta:\mathcal{E}_\alpha & \xrightarrow{\sim} & \mathcal{E}_{^\beta\alpha} \\ ae_\alpha(g) & \longmapsto & a\beta(g)e_{^\beta\alpha}(g) \end{array}$$

for any map $\beta: G\to A$. Thus the isomorphism class of the extension $\mathcal{E}_\alpha$ only depends on $[\alpha]\in H^2(G,A)$, but *not canonically.* If $a\in A$ we set $({}^a\beta)(g)=\beta(g)a/{}^ga$, and we have

$$i_{^a\beta}=\mathrm{conj}_a\circ i_\beta.$$

Any element $\beta\in Z^1(G,A)$ gives rise to a map of extensions $i_\beta:\mathcal{E}_\alpha\to\mathcal{E}_\alpha$, and in fact this establishes an isomorphism between $Z^1(G,A)$ and the automorphisms of the extension $\mathcal{E}_\alpha$. The automorphism arises as conjugation by an element of A if and only if β is a coboundary. Thus, if $H^1(G,A)=(0)$, then every automorphism of the extension $\mathcal{E}_\alpha$ arises by conjugation by an element of A. Every extension of G by A arises from some $\alpha\in Z^2(G,A)$.

If $h:G\to G'$ and $f:A\to A'$ are morphisms such that

$$f({}^ga)={}^{h(g)}f(a)$$

and if $\alpha\in Z^2(G,A)$ and $\alpha'\in Z^2(G',A')$ satisfy

$$f(\alpha(g_1,g_2))=\alpha'(h(g_1),h(g_2))$$

then there is a morphism

$$\begin{array}{rcl} (f,h):\mathcal{E}_\alpha & \longrightarrow & \mathcal{E}_{\alpha'} \\ ae(g) & \longmapsto & f(a)e'(h(g)) \end{array}$$

such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \mathcal{E}_\alpha & \longrightarrow & G & \longrightarrow & 0 \\ & & f\downarrow & & (f,h)\downarrow & & h\downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & \mathcal{E}_{\alpha'} & \longrightarrow & G' & \longrightarrow & 0 \end{array}$$

commutes.

If $h : G' \to G$ and $\mathcal{E}$ is an extension of G by A, then we can form a pull-back extension

$$\mathcal{E}|_{G'} = \mathcal{E}|_{G',h} = \mathcal{E} \times_{G,h} G' = \{(e, g') : \ e \text{ and } g' \text{ have the same image in } G\} \subset G' \times \mathcal{E}$$

of G' by A. If $\mathcal{E}$ arises from $\alpha \in Z^2(G, A)$ then $\mathcal{E}|_{G',h}$ arises from $h^*\alpha = \alpha \circ h$. Similarly if $f : A \to A'$ is a map of G-modules we can form a push-out extension

$$f_*\mathcal{E} = (A' \rtimes \mathcal{E})/A$$

of A' by G. Here $\mathcal{E}$ acts on A' via its projection to G and we embed A as a normal subgroup of $A' \rtimes \mathcal{E}$ via $a \mapsto (f(a), a)$. If $\mathcal{E}$ arises from $\alpha \in Z^2(G, A)$ then $f_*\mathcal{E}$ arises from $f_*\alpha = f \circ \alpha$.

2.7. **Local Weil Groups.** We recall the theory of Weil groups for p-adic fields. See [T1].

First suppose that F is a p-adic field and that $\overline{F}$ is an algebraic closure. If k denotes the residue field of F, there is an exact sequence

$$(0) \longrightarrow I_F \longrightarrow \mathrm{Gal}\,(\overline{F}/F) \longrightarrow \mathrm{Gal}\,(\overline{k}/k) \longrightarrow (0).$$

We denote by $W_{\overline{F}/F}$ the preimage of $\mathrm{Frob}_k^{\mathbb{Z}} \subset \mathrm{Gal}\,(\overline{k}/k)$ and endow it with a topology decreeing that I_F should be an open subgroup with its usual topology. If $\sigma : \overline{F} \xrightarrow{\sim} \overline{F}'$ is a continuous automorphism with $\sigma F = F'$, then there is a canonical isomorphism

$$\begin{array}{rccc} \mathrm{conj}_\sigma : W_{\overline{F}/F} & \xrightarrow{\sim} & W_{\overline{F}'/F'} \\ \tau & \longmapsto & \sigma\tau\sigma^{-1}. \end{array}$$

Note that there is a canonical map $\varphi_{\overline{F}/F} : W_{\overline{F}/F} \to \mathrm{Gal}\,(\overline{F}/F)$ with dense image.

If E is an intermediate field between F and $\overline{F}$ we will write $W_{\overline{F}/E} = \varphi_{\overline{F}/F}^{-1}\mathrm{Gal}\,(\overline{F}/E)$. For E/F finite, there are canonical isomorphisms

$$r_E : E^\times \xrightarrow{\sim} W_{\overline{F}/E}^{\mathrm{ab}}$$

with the following properties:

- $\varphi_{\overline{F}/F} \circ r_E = \mathrm{Art}\,_E$.
- If $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$, then $\mathrm{conj}_\sigma \circ r_E = r_{\sigma E} \circ \sigma$.
- If $E' \subset E$ then $\mathrm{tr}\,_{E/E'} \circ r_{E'}$ equals r_E composed with the inclusion $(E')^\times \hookrightarrow E^\times$, where $\mathrm{tr}\,_{E/E'} : W_{\overline{F}/E'}^{\mathrm{ab}} \to W_{\overline{F}/E}$ is the transfer map.
- $W_{\overline{F}/F} \cong \lim_{\leftarrow E} W_{\overline{F}/F}/[W_{\overline{F}/E}, W_{\overline{F}/E}]$ as topological groups.

These properties imply that

- If $E' \subset E$ then $r_{E'} \circ \mathbf{N}_{E/E'}$ equals r_E followed by the map $W_{\overline{F}/E}^{\mathrm{ab}} \to W_{\overline{F}/E'}^{\mathrm{ab}}$ induced by the inclusion $W_{\overline{F}/E} \subset W_{\overline{F}/E'}$.

There are no non-identity automorphisms of $W_{\overline{F}/F}$ compatible with $\varphi_{\overline{F}/F}$.

If $F \cong \mathbb{C}$ we set $W_{\overline{F}/F} = F^\times$. If $F \cong \mathbb{R}$ and $\overline{F}$ is an algebraic closure we set $W_{\overline{F}/F} = \langle \overline{F}^\times, j : \ j^2 = -1, \ jzj^{-1} = {}^c z\rangle$. If $F = \mathbb{R}$ or $\mathbb{C}$ there is a natural map

$$\varphi_{\overline{F}/F} : W_{\overline{F}/F} \twoheadrightarrow \mathrm{Gal}\,(\overline{F}/F)$$

with kernel $\overline{F}^\times$. If E is an intermediate field between F and $\overline{F}$ we will write $W_{\overline{F}/E} = \varphi_{\overline{F}/F}^{-1}\mathrm{Gal}\,(\overline{F}/E)$. For E/F finite, there are canonical isomorphisms

$$r_E : E^\times \xrightarrow{\sim} W_{\overline{F}/E}^{\mathrm{ab}}$$

which are the identity if $E \cong \mathbb{C}$ and induced by $-1 \mapsto j$ and $x \mapsto \sqrt{x}$ if $x > 0$. These structures share the properties itemized above for p-adic fields. Again these constructions are functorial in the pair $\overline{F}/F$. In the case $F \cong \mathbb{C}$ the group $W_{\overline{F}/F}$ has no automorphisms compatible with r_F. On the other hand, if $F \cong \mathbb{R}$ then $W_{\overline{F}/F}$ does have automorphisms compatible with $\varphi_{\overline{F}/F}$ and r_F and $r_{\overline{F}}$, namely the inner automorphisms conj_z for $z \in \overline{F}^\times$. However the only ones compatible with the functoriality $W_F \to W_F$ induced by $c : F \to F$ are the identity and $\mathrm{conj}_{\sqrt{-1}}$.

If F is either a p-adic field or isomorphic to $\mathbb{R}$ or $\mathbb{C}$, and if $\overline{F}$ is an algebraic closure of F and if E is an intermediate field finite and Galois over F, then we write

$$W_{E^{\mathrm{ab}}/F} = W_{\overline{F}/F}/\overline{[W_{\overline{F}/E}, W_{\overline{F}/E}]}$$

so that there is a short exact sequence

$$(0) \longrightarrow E^\times \xrightarrow{r_E} W_{E^{\mathrm{ab}}/F} \longrightarrow \mathrm{Gal}\,(E/F) \longrightarrow (0)$$

which determines a class

$$[\alpha_{E/F}] \in H^2(\mathrm{Gal}\,(E/F), E^\times)$$

called the canonical class. It depends only on E/F, i.e. not on $\overline{F}$. If $D \supset E$ is another finite extension Galois over F then

- $\mathrm{inf}_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)}[\alpha_{E/F}] = [D : E][\alpha_{D/F}] \in H^2(\mathrm{Gal}\,(D/F), D^\times)$,
- $\mathrm{res}_{\mathrm{Gal}\,(D/E)}^{\mathrm{Gal}\,(D/F)}[\alpha_{D/F}] = [\alpha_{D/E}] \in H^2(\mathrm{Gal}\,(D/E), D^\times)$,
- and $\mathrm{cor}_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)}[\alpha_{D/E}] = [E : F][\alpha_{D/F}] \in H^2(\mathrm{Gal}\,(D/F), D^\times)$.

(See for instance section XI.3 of [Se]. The last two formulae do not require E/F to be Galois.)

If $D \supset E \supset F$ are finite Galois extensions of F in $\overline{F}$ then there is an obvious map

$$W_{D^{\mathrm{ab}}/F} \twoheadrightarrow W_{E^{\mathrm{ab}}/F}$$

which fits into a commutative diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & D^\times & \longrightarrow & W_{D^{\rm ab}/F} & \longrightarrow & {\rm Gal}\,(D/F) & \longrightarrow & (0) \\ & & N_{D/E} \downarrow & & \downarrow & & \downarrow & & \\ (0) & \longrightarrow & E^\times & \longrightarrow & W_{E^{\rm ab}/F} & \longrightarrow & {\rm Gal}\,(E/F) & \longrightarrow & (0). \end{array}$$

We will denote this map $\sigma \mapsto \sigma|_{E^{\rm ab}}$. However another method of comparing $W_{D^{\rm ab}/F}$ and $W_{E^{\rm ab}/F}$ will be important to us. We start with the following abstract lemma:

Lemma 2.2. *Suppose that $\pi : G \twoheadrightarrow H$ is a surjective group homomorphism, and that $A \lhd G$ and $B \lhd H$ are normal subgroups with $\pi A \subset B$. Then there is an isomorphism of groups*

$$\begin{array}{ccc} (B \rtimes G)/A & \xrightarrow{\sim} & H \times_{H/B} G/A \\ [(b, g)] & \longmapsto & (b\pi(g), gA). \end{array}$$

Here

$$\begin{array}{ccc} A & \longrightarrow & B \rtimes G \\ a & \longmapsto & (\pi(a)^{-1}, a). \end{array}$$

Proof: The map

$$\begin{array}{ccc} \phi : B \rtimes G & \xrightarrow{\sim} & H \times_{H/B} G/A \\ (b, g) & \longmapsto & (b\pi(g), gA) \end{array}$$

is easily checked to be a group homomorphism. We have $(b, g) \in \ker \phi$ if and only if $g \in A$ and $b = \pi(g)^{-1}$, i.e. (b, g) is in the image of A. It remains to check that ϕ is surjective. Suppose $(h, gA) \in H \times_{H/B} G/A$. Then $h\pi(g)^{-1} \in B$ and $\phi(h\pi(g)^{-1}, g) = (h, g)$, as desired.$\Box$

If we map

$$\begin{array}{ccc} D^\times & \longrightarrow & E^\times \rtimes W_{D^{\rm ab}/F} \\ a & \longmapsto & (N_{D/E}(a)^{-1}, r_D(a)), \end{array}$$

then we see that there is an isomorphism of extensions

$$\begin{array}{ccc} (E^\times \rtimes W_{D^{\rm ab}/F})/D^\times & \xrightarrow{\sim} & W_{E^{\rm ab}/F}|_{{\rm Gal}\,(D/F)} = W_{E^{\rm ab}/F} \times_{{\rm Gal}\,(E/F)} {\rm Gal}\,(D/F) \\ [(a, \tau)] & \longmapsto & (r_E(a)\tau|_{E^{\rm ab}}, \tau|_D). \end{array}$$

(Note that $r_D(b)|_{E^{\rm ab}} = r_E(N_{D/E}(b))$.) We see that we have maps of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & D^\times & \longrightarrow & W_{D^{\rm ab}/F} & \longrightarrow & {\rm Gal}\,(D/F) & \longrightarrow & (0) \\ & & N_{D/E} \downarrow & & \downarrow & & || & & \\ (0) & \longrightarrow & E^\times & \longrightarrow & W_{E^{\rm ab}/F}|_{{\rm Gal}\,(D/F)} & \longrightarrow & {\rm Gal}\,(D/F) & \longrightarrow & (0) \\ & & || & & \downarrow & & \downarrow & & \\ (0) & \longrightarrow & E^\times & \longrightarrow & W_{E^{\rm ab}/F} & \longrightarrow & {\rm Gal}\,(E/F) & \longrightarrow & (0), \end{array}$$

whose composite is the natural surjection $W_{D^{\rm ab}/F} \twoheadrightarrow W_{E^{\rm ab}/F}$, and where the middle row can either be obtained as a pushout from the top row or a pullback from the bottom row. We deduce that

$$N_{D/E,*}[\alpha_{D/F}] = \inf_{{\rm Gal}\,(E/F)}^{{\rm Gal}\,(D/F)} [\alpha_{E/F}] \in H^2({\rm Gal}\,(D/F), E^\times).$$

We define a pushout

$$W_{E^{\mathrm{ab}}/F,D} = (D^\times \rtimes W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)})/E^\times,$$

where

$$\begin{array}{rcl} E^\times & \longrightarrow & D^\times \rtimes W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)} \\ a & \longmapsto & (a^{-1}, (r_E(a), 1)). \end{array}$$

From the above discussion we see that this has a second description as $(D^\times \rtimes W_{D^{\mathrm{ab}}/F})/D^\times$, where

$$\begin{array}{rcl} D^\times & \hookrightarrow & D^\times \rtimes W_{D^{\mathrm{ab}}/F} \\ a & \longmapsto & ((N_{D/E}a)^{-1}, r_D(a)). \end{array}$$

It corresponds to the class

$$\inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)} [\alpha_{E/F}] = N_{D/E,*}[\alpha_{D/F}] \in H^2(\mathrm{Gal}\,(D/F), D^\times).$$

We have a commutative diagram of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & E^\times & \longrightarrow & W_{E^{\mathrm{ab}}/F} & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & (0) \\ & & \| & & \uparrow & & \uparrow & & \\ (0) & \longrightarrow & E^\times & \longrightarrow & W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \\ & & \bigcap & & \downarrow & & \| & & \\ (0) & \longrightarrow & D^\times & \longrightarrow & W_{E^{\mathrm{ab}}/F,D} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \\ & & N_{D/E}\uparrow & & \uparrow & & \| & & \\ (0) & \longrightarrow & D^\times & \longrightarrow & W_{D^{\mathrm{ab}}/F} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0). \end{array}$$

2.8. **Global Weil groups.** We now recall the theory of Weil groups for number fields fields. See [T1].

Now suppose that F is a number field and that $\overline{F}$ is an algebraic closure of F. One can associate to $\overline{F}/F$ a topological group $W_{\overline{F}/F}$ together with:

- A map $\varphi_{\overline{F}/F} : W_{\overline{F}/F} \twoheadrightarrow \mathrm{Gal}\,(\overline{F}/F)$. If E is an intermediate field we set $W_{\overline{F}/E} = \varphi_{\overline{F}/F}^{-1}\mathrm{Gal}\,(\overline{F}/E)$.
- For each intermediate field finite over F a map

$$r_E : \mathbb{A}_E^\times/E^\times \xrightarrow{\sim} W_{\overline{F}/E}^{\mathrm{ab}}$$

  such that $\varphi_{\overline{F}/F} \circ r_E = \mathrm{Art}_E$.

These maps also satisfy:

- If $w \in W_{\overline{F}/F}$, then $\mathrm{conj}_w \circ r_E = r_{\varphi_{\overline{F}/F}(w)_E} \circ \varphi_{\overline{F}/F}(w)$.
- If $E' \subset E$ then $\mathrm{tr}_{E/E'} \circ r_{E'}$ equals r_E composed with the inclusion $\mathbb{A}_{E'}^\times/(E')^\times \hookrightarrow \mathbb{A}_E^\times/E^\times$, where $\mathrm{tr}_{E/E'} = \mathrm{tr}_{W_{\overline{F}/E'}/W_{\overline{F}/E}}$ is the transfer map.
- $W_{\overline{F}/F} \cong \lim_{\leftarrow E} W_{\overline{F}/F}/[W_{\overline{F}/E}, W_{\overline{F}/E}]$ as topological groups. We will write $W_{E^{\mathrm{ab}}/F} = W_{\overline{F}/F}/\overline{[W_{\overline{F}/E}, W_{\overline{F}/E}]}$.

These properties imply that

- If $E' \subset E$ then $r_{E'} \circ \mathbf{N}_{E/E'}$ equals r_E followed by the map $W^{\rm ab}_{\overline{F}/E} \to W^{\rm ab}_{\overline{F}/E'}$ induced by the inclusion $W_{\overline{F}/E} \subset W_{\overline{F}/E'}$.

The only automorphisms of $W_{\overline{F}/F}$ compatible with $\varphi_{\overline{F}/F}$ and the r_E are the inner automorphisms conj_w for $w \in W_{\overline{F}/\overline{F}}$. The structure $(W_{\overline{F}/F}, \varphi_{\overline{F}/F}, \{r_E\})$ is unique up to isomorphism. However we do not know how to make the isomorphism canonical. (If it can be made canonical.)

The image of $W_{\overline{F}/\overline{F}}$ in $W_{E^{\rm ab}/F}$ is $\overline{E^\times (E_\infty^\times)^0}/E^\times = \ker \mathrm{Art}_E$. We will denote it Δ_E.

Lemma 2.3. *(1) For $i>0$ there is an isomorphism $H^i(\mathrm{Gal}(E/F), \prod_{v|\infty} E_v^\times) \xrightarrow{\sim} H^i(\mathrm{Gal}(E/F), \Delta_E)$. Moreover $H^i(\mathrm{Gal}(E/F), \prod_{v|\infty} E_v^\times) \cong \prod_w H^i(\mathrm{Gal}(E/F)_w, E_w^\times)$, where w runs over one place of E above each real place of F which does not split completely in E.*
(2) $N_{E/F}\Delta_E = \Delta_F$.
(3) If E is totally imaginary, then $\Delta_E^{\mathrm{Gal}(E/F)} = \overline{F^\times F_\infty^\times}/F^\times$.
(4) $\mathrm{Art}_E : \mathbb{A}_F^\times/F^\times \twoheadrightarrow \mathrm{Gal}(E^{\rm ab}/E)^{\mathrm{Gal}(E/F)}$.

Proof: The first three parts are proved in section III of [W1]. The fourth part follows because $H^1(\mathrm{Gal}(E/F), \Delta_E) = (0)$ (because in turn $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) = (0)$). □

If u is a place of $\overline{F}$ then there is a continuous homomorphism

$$\theta_u : W_{\overline{F}_u/F_u} \longrightarrow W_{\overline{F}/F}$$

such that

- $\varphi_{\overline{F}/F} \circ \theta_u$ equals the composite of $\varphi_{\overline{F}_u/F_u}$ with the canonical map $\mathrm{Gal}(\overline{F}_u/F_u) \xrightarrow{\sim} \mathrm{Gal}(\overline{F}/F)_u$;
- and, for E a finite intermediate field, $\theta_u \circ r_{E_u}$ equals the composite of r_E with the canonical map $E_u^\times \to \mathbb{A}_E^\times/E^\times$.

The map θ_u is determined up to conjugation by an element of $W_{\overline{F}/\overline{F}}$. The images of $\mathrm{conj}_a \circ \theta_u$, for any $a \in W_{\overline{F}/\overline{F}}$, are referred to as *decomposition groups* for u. The closure of the image under $\varphi_{\overline{F}/F}$ of any decomposition group for u is $\mathrm{Gal}(\overline{F}/F)_u$. If $\sigma \in W_{\overline{F}/F}$, then

$$\theta_{\varphi_{\overline{F}/F}(\sigma)u} \circ \mathrm{conj}_{\varphi_{\overline{F}/F}(\sigma)} = \mathrm{conj}_\sigma \circ \theta_u,$$

up to conjugation by an element of $W_{\overline{F}/\overline{F}}$.

If v is a place of F and if $\rho : \overline{F} \to \overline{F_v}$ is F-linear, then we write $u(\rho)$ for the place of $\overline{F}$ induced by ρ and define

$$\theta_\rho = \theta_{u(\rho)} \circ \rho^* : W_{\overline{F_v}/F_v} \longrightarrow W_{\overline{F}/F}$$

(where $\rho^* : W_{\overline{F_v}/F_v} \xrightarrow{\sim} W_{\overline{F}_{u(\rho)}/F_v}$).

(1) If $\sigma \in W_{\overline{F}/F}$, then $\theta_{\rho\sigma} = \mathrm{conj}_{\sigma^{-1}} \circ \theta_\rho$ up to conjugation by an element of $W_{\overline{F}/\overline{F}}$.
(2) If $\tau \in \mathrm{Gal}(\overline{F_v}/F_v)$, then $\theta_{\tau\rho} = \theta_\rho \circ \mathrm{conj}_{\tau^{-1}}$.

(3) If E is an intermediate field finite over F, then $\theta_\rho \circ r_{\rho(E)F_v}$ equals the composition of r_E with $(\rho(E)F_v)^\times \xrightarrow{\sim} E_{u(\rho)}^\times \to \mathbb{A}_E^\times/E^\times$, where the first map is the inverse of the continuous extension of ρ.

We get an induced map

$$\theta_\rho : W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v} \longrightarrow W_{E^{\mathrm{ab}}/E},$$

defined up to conjugation by an element of Δ_E. Up to this ambiguity, it only depends $\rho|_{E^{\mathrm{ab}}}$.

If E is an intermediate field, finite and Galois over F, the short exact sequence

$$(0) \longrightarrow \mathbb{A}_E^\times/E^\times \xrightarrow{r_E} W_{E^{\mathrm{ab}}/F} \longrightarrow \mathrm{Gal}\,(E/F) \longrightarrow (0)$$

which determines a class

$$[\alpha_{E/F}^W] \in H^2(\mathrm{Gal}\,(E/F), \mathbb{A}_E^\times/E^\times)$$

called the *canonical class*. This class depends only on E/F. If $\imath_w : E_w^\times \to \mathbb{A}_E^\times/E^\times$, then we have

$$\mathrm{res}[\alpha_{E/F}^W] = \imath_{w,*}[\alpha_{E_w/F_v}] \in H^2(\mathrm{Gal}\,(E/F)_w, \mathbb{A}_E^\times/E^\times).$$

(See formula (12) in [T2].) If $D \supset E$ is another finite Galois extension of F, then

$$\mathrm{res}^{\mathrm{Gal}\,(D/F)}_{\mathrm{Gal}\,(D/E)}[\alpha_{D/F}^W] = [\alpha_{D/E}^W] \in H^2(\mathrm{Gal}\,(D/E), \mathbb{A}_D^\times/D^\times)$$

and

$$\inf^{\mathrm{Gal}\,(D/F)}_{\mathrm{Gal}\,(E/F)}[\alpha_{E/F}^W] = [D:E][\alpha_{D/F}^W] \in H^2(\mathrm{Gal}\,(D/F), \mathbb{A}_D^\times/D^\times).$$

Now suppose that $D \supset E \supset F$ are finite Galois extensions of F in $\overline{F}$. We have a natural map

$$W_{D^{\mathrm{ab}}/F} \twoheadrightarrow W_{E^{\mathrm{ab}}/F}$$

which we will denote $\sigma \mapsto \sigma|_{E^{\mathrm{ab}}}$. Lemma 2.2 tells us there is an isomorphism

$$\begin{array}{rcl} (\mathbb{A}_E^\times/E^\times \rtimes W_{D^{\mathrm{ab}}/F})/(\mathbb{A}_D^\times/D^\times) & \xrightarrow{\sim} & W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)} = W_{E^{\mathrm{ab}}/F} \times_{\mathrm{Gal}\,(E/F)} \mathrm{Gal}\,(D/F) \\ {[(a,\tau)]} & \longmapsto & (r_E(a)\tau|_{E^{\mathrm{ab}}}, \tau|_D), \end{array}$$

where

$$\begin{array}{rcl} \mathbb{A}_D^\times/D^\times & \longrightarrow & \mathbb{A}_E^\times/E^\times \rtimes W_{D^{\mathrm{ab}}/F} \\ a & \longmapsto & (N_{D/E}(a)^{-1}, r_D(a)). \end{array}$$

We see that we have maps of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathbb{A}_D^\times/D^\times & \longrightarrow & W_{D^{\mathrm{ab}}/F} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \\ & & N_{D/E}\downarrow & & \downarrow & & || & & \\ (0) & \longrightarrow & \mathbb{A}_E^\times/E^\times & \longrightarrow & W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \\ & & || & & \downarrow & & \downarrow & & \\ (0) & \longrightarrow & \mathbb{A}_E^\times/E^\times & \longrightarrow & W_{E^{\mathrm{ab}}/F} & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & (0), \end{array}$$

whose composite is the natural surjection $W_{D^{\mathrm{ab}}/F} \twoheadrightarrow W_{E^{\mathrm{ab}}/F}$, and where the middle row can either be obtained as a push-out from the top row or a pullback from the bottom row. Thus

$$\inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)}[\alpha^W_{E/F}] = N_{D/E,*}[\alpha^W_{D/F}] \in H^2(\mathrm{Gal}\,(D/F), \mathbb{A}_E^\times/E^\times).$$

We define a pushout

$$W_{E^{\mathrm{ab}}/F,D} = (\mathbb{A}_D^\times/D^\times \rtimes W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)})/(\mathbb{A}_E^\times/E^\times),$$

where

$$\begin{array}{rcl} \mathbb{A}_E^\times/E^\times & \longrightarrow & \mathbb{A}_D^\times/D^\times \rtimes W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)} \\ a & \longmapsto & (a^{-1}, (r_E(a), 1)). \end{array}$$

From the above discussion we see that this has a second description as $(\mathbb{A}_D^\times/D^\times \rtimes W_{D^{\mathrm{ab}}/F})/\mathbb{A}_D^\times$, where

$$\begin{array}{rcl} \mathbb{A}_D^\times/D^\times & \hookrightarrow & \mathbb{A}_D^\times/D^\times \rtimes W_{D^{\mathrm{ab}}/F} \\ a & \longmapsto & ((N_{D/E}a)^{-1}, r_D(a)). \end{array}$$

It corresponds to the class

$$\inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)}[\alpha^W_{E/F}] = N_{D/E,*}[\alpha^W_{D/F}] \in H^2(\mathrm{Gal}\,(D/F), \mathbb{A}_D^\times/D^\times).$$

We have a commutative diagram of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathbb{A}_E^\times/E^\times & \longrightarrow & W_{E^{\mathrm{ab}}/F} & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & (0) \\ && \| && \uparrow && \uparrow && \\ (0) & \longrightarrow & \mathbb{A}_E^\times/E^\times & \longrightarrow & W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \\ && \bigcap && \downarrow && \| && \\ (0) & \longrightarrow & \mathbb{A}_D^\times/D^\times & \longrightarrow & W_{E^{\mathrm{ab}}/F,D} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \\ && N_{D/E}\uparrow && \uparrow && \| && \\ (0) & \longrightarrow & \mathbb{A}_D^\times/D^\times & \longrightarrow & W_{D^{\mathrm{ab}}/F} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0). \end{array}$$

The group $W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)}$ acts on $W_{E^{\mathrm{ab}}/F,D}$ by conjugation.

We will also write $\mathrm{Gal}\,(E^{\mathrm{ab}}/F)_D$ for

$$\begin{array}{rl} & (\mathrm{Gal}\,(D^{\mathrm{ab}}/D) \rtimes \mathrm{Gal}\,(D^{\mathrm{ab}}/F))/\mathrm{Gal}\,(D^{\mathrm{ab}}/D) \\ \cong & (\mathrm{Gal}\,(D^{\mathrm{ab}}/D) \rtimes (\mathrm{Gal}\,(E^{\mathrm{ab}}/F) \times_{\mathrm{Gal}\,(E/F)} \mathrm{Gal}\,(D/F)))/\mathrm{Gal}\,(E^{\mathrm{ab}}/E), \end{array}$$

where on the left hand side

$$\begin{array}{rcl} \mathrm{Gal}\,(D^{\mathrm{ab}}/D) & \longrightarrow & \mathrm{Gal}\,(D^{\mathrm{ab}}/D) \rtimes \mathrm{Gal}\,(D^{\mathrm{ab}}/F) \\ \tau & \longmapsto & (\mathrm{tr}_{E/F}(\tau|_{E^{\mathrm{ab}}})^{-1}, \tau), \end{array}$$

and on the right hand side

$$\begin{array}{rcl} \mathrm{Gal}\,(E^{\mathrm{ab}}/E) & \longrightarrow & \mathrm{Gal}\,(D^{\mathrm{ab}}/D) \rtimes (\mathrm{Gal}\,(E^{\mathrm{ab}}/F) \times_{\mathrm{Gal}\,(E/F)} \mathrm{Gal}\,(D/F)) \\ \tau & \longmapsto & (\mathrm{tr}_{E/F}(\tau)^{-1}, (\tau, 1)). \end{array}$$

It follows from the snake lemma that there is an exact sequence

$$(0) \longrightarrow \ker \mathrm{Art}_D \longrightarrow W_{E^{\mathrm{ab}}/F,D} \longrightarrow \mathrm{Gal}\,(E^{\mathrm{ab}}/F)_D \longrightarrow (0).$$

Note that

$$\begin{array}{ccccc} \mathrm{Gal}\,(D^{\mathrm{ab}}/F) & & \twoheadrightarrow & & \mathrm{Gal}\,(E^{\mathrm{ab}}/F) \\ & \searrow & & \swarrow & \\ & & \mathrm{Gal}\,(E^{\mathrm{ab}}/F)_D & & \end{array}$$

commutes.

Also note that $\mathrm{Gal}\,(D^{\mathrm{ab}}/F)$ acts 'by conjugation' on $\mathrm{Gal}\,(E^{\mathrm{ab}}/F)_D$, i.e.

$$\mathrm{conj}_\sigma[(\tau_1,\tau_2)] = [(\sigma\tau_1\sigma^{-1},\sigma\tau_2\sigma^{-1})]$$

and

$$\mathrm{conj}_\sigma[(\tau_1,(\tau_2,\tau_3))] = [(\sigma\tau_1\sigma^{-1},(\sigma\tau_2\sigma^{-1},\sigma\tau_3\sigma^{-1}))].$$

If $\rho : D \hookrightarrow \overline{F}_v$ is F-linear we will write $w(\rho)$ (resp. $u(\rho)$) for the place of E (resp. D) induced by ρ. We define

$$W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v}|_{\rho,\mathrm{Gal}\,(D/F)_{w(\rho)}} = W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v} \times_{\mathrm{Gal}\,(\rho(E)F_v/F_v),\rho_*} \mathrm{Gal}\,(D/F)_{w(\rho)},$$

where

$$\rho_* : \mathrm{Gal}\,(D/F)_{w(\rho)} \twoheadrightarrow \mathrm{Gal}\,(\rho(E)F_v/F_v)$$

is defined by $\rho_*(\sigma)\circ\rho|_E = \rho|_E \circ \sigma$. If $\sigma \in \mathrm{Gal}\,(D/F)$ then

$$c_{\rho,\rho\sigma} = \mathrm{conj}_{1\times\sigma^{-1}} : W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v}|_{\rho,\mathrm{Gal}\,(D/F)_{w(\rho)}} \xrightarrow{\sim} W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v}|_{\rho\sigma,\mathrm{Gal}\,(D/F)_{w(\rho\sigma)}}.$$

This depends only on ρ and $\rho\sigma$. It fits into a commutative diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & (\rho(E)F_v)^\times & \longrightarrow & W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v}|_{\rho,\mathrm{Gal}\,(D/F)_{w(\rho)}} & \longrightarrow & \mathrm{Gal}\,(D/F)_{w(\rho)} & \longrightarrow & (0) \\ & & || & & c_{\rho,\rho\sigma}\downarrow & & \mathrm{conj}_{\sigma^{-1}}\downarrow & & \\ (0) & \longrightarrow & ((\rho\sigma)(E)F_v)^\times & \longrightarrow & W_{((\rho\sigma)(E)F_v)^{\mathrm{ab}}/F_v}|_{\rho\sigma,\mathrm{Gal}\,(D/F)_{w(\rho\sigma)}} & \longrightarrow & \mathrm{Gal}\,(D/F)_{w(\rho\sigma)} & \longrightarrow & (0). \end{array}$$

This group has a second description. Define a semidirect product

$$((\rho(E)F_v)^\times \times \mathrm{Gal}\,(D/E)) \rtimes W_{(\rho(D)F_v)^{\mathrm{ab}}/F_v}$$

where $W_{(\rho(D)F_v)^{\mathrm{ab}}/F_v}$ acts on $\mathrm{Gal}\,(D/E)$ by the composition of conjugation with the morphism $\rho_*^{-1} : W_{(\rho(D)F_v)^{\mathrm{ab}}/F_v} \twoheadrightarrow \mathrm{Gal}\,(D/F)_{u(\rho)}$. There is a homomorphism

$$\begin{array}{rcl} \phi : ((\rho(E)F_v)^\times \times \mathrm{Gal}\,(D/E)) \rtimes W_{(\rho(D)F_v)^{\mathrm{ab}}/F_v} & \longrightarrow & W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v}|_{\rho,\mathrm{Gal}\,(D/F)_{w(\rho)}} \\ ((a,\sigma),\tau) & \longmapsto & (r_{\rho(E)F_v}(a)\tau|_{(\rho(E)F_v)^{\mathrm{ab}}}, \sigma\rho_*^{-1}(\tau)). \end{array}$$

The homomorphism ϕ is surjective. Indeed if $(\sigma,\tau) \in W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v}|_{\rho,\mathrm{Gal}\,(D/F)_{w(\rho)}}$ then we may write $\tau = \sigma_1\rho_*^{-1}(\tau_1)$ with $\sigma_1 \in \mathrm{Gal}\,(D/E)$ and $\tau_1 \in W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/F_v}$. Moreover $\tau_1|_{(\rho(E)F_v)^{\mathrm{ab}}} = r_{\rho(E)F_v}(a)\sigma$, for some $a \in (\rho(E)F_v)^\times$. Then $((a,\sigma_1),\tau_1)$ maps to (σ,τ). An element $((a,\sigma),\tau)$ lies in the kernel of ϕ if and only if $\rho_*^{-1}(\tau) = \sigma^{-1}$ (so that in particular $\tau \in W_{(\rho(D)F_v)^{\mathrm{ab}}/(\rho(E)F_v)}$) and $\tau|_{(\rho(E)F_v)^{\mathrm{ab}}} = r_{E_{w(\rho)}}(a)^{-1}$. We conclude that we have an isomorphism

$$\begin{array}{rcl} (((\rho(E)F_v)^\times \times \mathrm{Gal}\,(D/E)) \rtimes W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/F_v})/W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/(\rho(E)F_v)} & \xrightarrow{\sim} & W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v}|_{\rho,\mathrm{Gal}\,(D/F)_{w(\rho)}} \\ [((a,\sigma),\tau)] & \longmapsto & (r_{\rho(E)F_v}(a)\tau|_{(\rho(E)F_v)^{\mathrm{ab}}}, \sigma\rho_*^{-1}(\tau)), \end{array}$$

where

$$\begin{array}{rcl} W_{(\widetilde{\rho}(D)F_v)^{\rm ab}/(\rho(E)F_v)} & \longrightarrow & ((\rho(E)F_v)^\times \times \mathrm{Gal}\,(D/E)) \ltimes W_{(\widetilde{\rho}(D)F_v)^{\rm ab}/F_v} \\ \tau & \longmapsto & ((r_{\rho(E)F_v}^{-1}(\tau|_{(\rho(E)F_v)^{\rm ab}}^{-1}), \widetilde{\rho}_*^{-1}(\tau|_{\widetilde{\rho}(D)F_v}^{-1})), \tau). \end{array}$$

If in addition $\rho' : E^{\rm ab} \to \overline{F_v}$ with $\rho'|_E = \rho|_E$, then we obtain a map

$$\begin{array}{rcl} \theta_{(\rho',\rho)} : W_{(\rho(E)F_v)^{\rm ab}/F_v}|_{\rho, \mathrm{Gal}\,(D/F)_{w(\rho)}} & \longrightarrow & W_{E^{\rm ab}/F}|_{\mathrm{Gal}\,(D/F)} \\ (\sigma,\tau) & \longmapsto & (\theta_{\rho'}(\sigma),\tau). \end{array}$$

It fits into a commutative diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & (\rho(E)F_v)^\times & \longrightarrow & W_{(\rho(E)F_v)^{\rm ab}/F_v}|_{\rho,\mathrm{Gal}\,(D/F)_{w(\rho)}} & \longrightarrow & \mathrm{Gal}\,(D/F)_{w(\rho)} & \longrightarrow & (0) \\ & & \rho^{-1}\downarrow & & \theta_{(\rho',\rho)}\downarrow & & \bigcap & & \\ (0) & \longrightarrow & \mathbb{A}_E^\times/E^\times & \longrightarrow & W_{E^{\rm ab}/F}|_{\mathrm{Gal}\,(D/F)} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \end{array}$$

and we have a commutative square

$$\begin{array}{ccc} W_{(\rho(E)F_v)^{\rm ab}/F_v}|_{\rho,\mathrm{Gal}\,(D/F)_{w(\rho)}} & \longrightarrow & W_{(\rho(E)F_v)^{\rm ab}/F_v} \\ \theta_{(\rho',\rho)}\downarrow & & \theta_{\rho'}\downarrow \\ W_{E^{\rm ab}/F}|_{\mathrm{Gal}\,(D/F)} & \longrightarrow & W_{E^{\rm ab}/F}. \end{array}$$

It satisfies

$$\theta_{(\rho',\rho)\sigma} = \mathrm{conj}_{\sigma^{-1}} \circ \theta_{(\rho',\rho)} \circ c_{\rho\sigma,\rho}$$

if $\sigma \in W_{E^{\rm ab}/F}|_{\mathrm{Gal}\,(D/F)}$.

Alternatively, if $\widetilde{\rho} : D^{\rm ab} \hookrightarrow \overline{F_v}$ extends both ρ' and ρ, then $\theta_{(\rho',\rho)}$ may be given as:

$$\begin{array}{rcl} (((\rho(E)F_v)^\times \times \mathrm{Gal}\,(D/E)) \ltimes W_{(\widetilde{\rho}(D)F_v)^{\rm ab}/F_v})/W_{(\widetilde{\rho}(D)F_v)^{\rm ab}/(\rho(E)F_v)} & \longrightarrow & (\mathbb{A}_E^\times/E^\times \rtimes W_{D^{\rm ab}/F})/(\mathbb{A}_D^\times/D^\times) \\ [((a,\sigma),\tau)] & \longmapsto & [((\rho')^{-1}(a)b^{-1}, \widetilde{\sigma}\theta_{\widetilde{\rho}}(\tau))], \end{array}$$

where $\widetilde{\sigma} \in W_{D^{\rm ab}/E}$ is a lift of σ and $\widetilde{\sigma}|_{E^{\rm ab}} = r_E(b)$. In this case we also have a commutative square

$$\begin{array}{ccc} W_{(\widetilde{\rho}(D)F_v)^{\rm ab}/F_v} & \longrightarrow & W_{(\rho(E)F_v)^{\rm ab}/F_v}|_{\rho,\mathrm{Gal}\,(D/F)_{w(\rho)}} \\ \theta_{\widetilde{\rho}}\downarrow & & \theta_{(\rho',\rho)}\downarrow \\ W_{D^{\rm ab}/F} & \longrightarrow & W_{E^{\rm ab}/F}|_{\mathrm{Gal}\,(D/F)}. \end{array}$$

If $\rho : D \hookrightarrow \overline{F_v}$ is F-linear, we define

$$W_{(\rho(E)F_v)^{\rm ab}/F_v,\rho,D} = (D_{w(\rho)}^\times \rtimes W_{(\rho(E)F_v)^{\rm ab}/F_v}|_{\rho,\mathrm{Gal}\,(D/F)_{w(\rho)}})/(\rho(E)F_v)^\times,$$

where $a \in (\rho(E)F_v)^\times$ maps to $(\rho^{-1}(a)^{-1}, r_{\rho(E)F_v}(a))$. Thus we have exact sequences

$$(0) \longrightarrow D_{w(\rho)}^\times \longrightarrow W_{(\rho(E)F_v)^{\rm ab}/F_v,\rho,D} \longrightarrow \mathrm{Gal}\,(D/F)_{w(\rho)} \longrightarrow (0)$$

and

$$(0) \longrightarrow \prod_{u(\rho)\neq u|w(\rho)} D_u^\times \longrightarrow W_{(\rho(E)F_v)^{\rm ab}/F_v,\rho,D}|_{\mathrm{Gal}\,(D/F)_{u(\rho)}} \longrightarrow W_{(\rho(E)F_v)^{\rm ab}/F_v,\rho(D)F_v} \longrightarrow (0).$$

If $\rho_2 = \rho_1\sigma$, then we also have canonical isomorphisms

$$c_{\rho_1,\rho_2} = \sigma^{-1} \ltimes c_{\rho_1,\rho_2} : W_{(\rho_1(E)F_v)^{\rm ab}/F_v,\rho_1,D} \xrightarrow{\sim} W_{(\rho_2(E)F_v)^{\rm ab}/F_v,\rho_2,D},$$

which fits into a commutative diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & D^\times_{w(\rho_1)} & \longrightarrow & W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v,\rho_1,D} & \longrightarrow & \mathrm{Gal}\,(D/F)_{w(\rho_1)} & \longrightarrow & (0) \\ && \sigma^{-1}\downarrow && c_{\rho_1,\rho_2}\downarrow && \mathrm{conj}_{\sigma^{-1}}\downarrow && \\ (0) & \longrightarrow & D^\times_{w(\rho_2)} & \longrightarrow & W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v,\rho_2,D} & \longrightarrow & \mathrm{Gal}\,(D/F)_{w(\rho_2)} & \longrightarrow & (0). \end{array}$$

Moreover $\theta_{(\rho',\rho)}$ extends to a map

$$\theta_{(\rho',\rho)} : W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v,\rho,D} \longrightarrow W_{E^{\mathrm{ab}}/F,D}$$

which fits into commutative diagrams

$$\begin{array}{ccccccccc} (0) & \longrightarrow & D^\times_{w(\rho)} & \longrightarrow & W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v}|_{\rho,\mathrm{Gal}\,(D/F)_{w(\rho)}} & \longrightarrow & \mathrm{Gal}\,(D/F)_{w(\rho)} & \longrightarrow & (0) \\ && \downarrow && \theta_{(\rho',\rho)}\downarrow && \bigcap && \\ (0) & \longrightarrow & \mathbb{A}_D^\times/D^\times & \longrightarrow & W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \end{array}$$

and

$$\begin{array}{ccccc} W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v,\rho,D} & \hookleftarrow & W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v}|_{\rho,\mathrm{Gal}\,(D/F)_{w(\rho)}} & \longrightarrow & W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v} \\ \theta_{(\rho',\rho)}\downarrow && \theta_{(\rho',\rho)}\downarrow && \theta_{\rho'}\downarrow \\ W_{E^{\mathrm{ab}}/F,D} & \hookleftarrow & W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)} & \longrightarrow & W_{E^{\mathrm{ab}}/F}. \end{array}$$

It also satisfies

$$\theta_{(\rho',\rho)\sigma} = \mathrm{conj}_{\sigma^{-1}} \circ \theta_{(\rho',\rho)} \circ c_{\rho\sigma,\rho}$$

if $\sigma \in W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)}$. If $\widetilde{\rho} : D^{\mathrm{ab}} \hookrightarrow \overline{F_v}$ extends both ρ' and ρ, then

$$\begin{array}{ccc} W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/F_v} & \longrightarrow & W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v,\rho,D} \\ \theta_{\widetilde{\rho}}\downarrow && \theta_{(\rho',\rho)}\downarrow \\ W_{D^{\mathrm{ab}}/F} & \longrightarrow & W_{E^{\mathrm{ab}}/F,D} \end{array}$$

commutes. In fact we may describe $\theta_{(\widetilde{\rho}|_D,\widetilde{\rho}|_{E^{\mathrm{ab}}})}$ as the map

$$\begin{array}{ccc} ((D^\times_{w(\widetilde{\rho})} \times \mathrm{Gal}\,(D/E)) \rtimes W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/F_v})/W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/(\widetilde{\rho}(E)F_v)}) & \longrightarrow & (\mathbb{A}_D^\times/D^\times \rtimes W_{D^{\mathrm{ab}}/F})/(\mathbb{A}_D^\times/D^\times) \\ [((a,\sigma),\tau)] & \longmapsto & [(ar_E^{-1}(\widetilde{\sigma}|_{E^{\mathrm{ab}}}^{-1}), \widetilde{\sigma}\theta_{\widetilde{\rho}}(\tau))], \end{array}$$

where $\widetilde{\sigma} \in W_{D^{\mathrm{ab}}/E}$ lifts σ.

3. Kottwitz cohomology: the local case

3.1. **Generalities on algebraic cohomology.** We will need a modification of the Galois cohomology of reductive group, which is described in section 12 of [K3]. Suppose that we are given the following data

(1) An extension
$$(0) \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E} \longrightarrow \Gamma \longrightarrow (0)$$
where $\mathcal{E}^0$ is an abelian group and Γ is a group. Note that $\mathcal{E}^0$ has a Γ-action by conjugation.

(2) A group G with an action of Γ. Note that the conjugation action of G and the Γ-action on G, piece together to give a $G \rtimes \Gamma$ action on G. This, together with the G action on $\mathcal{E}^0$, gives a $G \rtimes \Gamma$-action on $\mathrm{Hom}\,(\mathcal{E}^0, G)$.

(3) A set $\mathcal{N}$ with an action of $G \rtimes \Gamma$.

(4) A $G \rtimes \Gamma$-equivariant map
$$\begin{array}{ccc} \mathcal{N} & \longrightarrow & \mathrm{Hom}\,(\mathcal{E}^0, G) \\ \nu & \longmapsto & \overline{\nu}, \end{array}$$
such that if $e \in \mathcal{E}^0$ and $\nu \in \mathcal{N}$, then
$${}^{\overline{\nu}(e)}\nu = \nu.$$

We will refer to $\mathcal{N}$ as an *pre-algebraicity condition*. In most but not all cases we will have $\mathcal{N} \subset \mathrm{Hom}\,(\mathcal{E}^0, G)$.

We define pointed sets
$$Z^1_{\mathcal{N}}(\mathcal{E}, G) = \{(\nu, \phi) \in \mathcal{N} \times Z^1(\mathcal{E}, G) :\ \phi|_{\mathcal{E}^0} = \overline{\nu} \text{ and } {}^{\phi(e)^{-1}}\nu = {}^{e}\nu \ \forall e \in \mathcal{E}\}.$$
We refer to elements of this pointed set as *algebraic cocycles*. In cases where $\mathcal{N}$ is contained $\mathrm{Hom}\,(\mathcal{E}^0, G)$ we will often use ϕ to denote a cocycle (ν, ϕ) (as $\nu = \phi|_{\mathcal{E}^0}$). The group G acts on $Z^1_{\mathcal{N}}(\mathcal{E}, G)$ by
$${}^{g}(\nu, \phi) = ({}^{g}\nu, {}^{g}\phi).$$
We define $H^1_{\mathcal{N}}(\mathcal{E}, G)$ to be the quotient of $Z^1_{\mathcal{N}}(\mathcal{E}, G)$ by G. We refer to this as the *algebraic cohomology*.

There is a left exact sequence (of pointed sets)
$$\begin{array}{ccccccc} (0) & \longrightarrow & H^1(\Gamma, G) & \longrightarrow & H^1_{\mathcal{N}}(\mathcal{E}, G) & \stackrel{\nu}{\longrightarrow} & (G\backslash\mathcal{N})^{\Gamma} \\ & & & & (\nu, \phi) & \longmapsto & [\nu]. \end{array}$$
If $\Gamma = \{1\}$, then
$$Z^1_{\mathcal{N}}(\mathcal{E}, G) \xrightarrow{\sim} \mathcal{N}$$
and
$$H^1_{\mathcal{N}}(\mathcal{E}, G) \xrightarrow{\sim} G(A_E)\backslash\mathcal{N}.$$

These sets of cocycles and cohomology sets satisfy various natural functorialities, which are a bit tedious to spell out:

(A) Suppose first that we have a Γ-equivariant map $h : G_1 \to G_2$, pre-algebraicity conditions $\mathcal{N}_i$ for $(\mathcal{E}, G_i)$ and a $G \rtimes \Gamma$-equivariant map map $n : \mathcal{N}_1 \to \mathcal{N}_2$ such that

$$\overline{n(\nu)} = h \circ \overline{\nu}$$

for all $\nu \in \mathcal{N}$. Then we obtain a natural map

$$\begin{array}{rcl} h_* = (h,n)_* : Z^1_{\mathcal{N}_1}(\mathcal{E}, G_1) & \longrightarrow & Z^1_{\mathcal{N}_2}(\mathcal{E}, G_2) \\ (\nu, \phi) & \longmapsto & (n(\nu), h \circ \phi), \end{array}$$

which induces a map

$$h_* = (h,n)_* : H^1_{\mathcal{N}_1}(\mathcal{E}, G_1) \longrightarrow H^1_{\mathcal{N}_2}(\mathcal{E}, G_2).$$

(B) Second suppose that we have maps of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathcal{E}_1^0 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \Gamma_1 & \longrightarrow & (0) \\ & & \downarrow & & f \downarrow & & \overline{f} \downarrow & & \\ (0) & \longrightarrow & \mathcal{E}_2^0 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \Gamma_2 & \longrightarrow & (0), \end{array}$$

a group homomorphism $h : G_2 \to G_1$, pre-algebraicity conditions $\mathcal{N}_i$ for $(\mathcal{E}_i, G_i)$ and a map $n : \mathcal{N}_2 \to \mathcal{N}_1$ such that

- $h({}^{\overline{f}(\sigma)}a) = {}^{\sigma}h(a)$,
- $n({}^{g}\nu) = {}^{h(g)}n(\nu)$,
- $n({}^{\overline{f}(\sigma)}\nu) = {}^{\sigma}n(\nu)$
- and $\overline{n(\nu)} = h \circ \overline{\nu} \circ f$.

Then we obtain a natural map

$$\begin{array}{rcl} f^* = (f,h,n)^* : Z^1_{\mathcal{N}_2}(\mathcal{E}_2, G_2) & \longrightarrow & Z^1_{\mathcal{N}_1}(\mathcal{E}_1, G_1) \\ (\nu, \phi) & \longmapsto & (n(\nu), h \circ \phi \circ f), \end{array}$$

which induces a map

$$f^* = (f,h,n)^* : H^1_{\mathcal{N}_2}(\mathcal{E}_2, G_2) \longrightarrow H^1_{\mathcal{N}_1}(\mathcal{E}_1, G_1).$$

(This is a minor generalization of Kottwitz's map $\Psi(n, f)$, which he defines in the special case $G_1 = G_2$ and $h = \mathrm{Id}_G$. In the special case $f = \mathrm{Id}$ we recover the map in part A.)

(If we take $\mathcal{E}_1 = \mathcal{E}_2$ and $G_1 = G_2$ and $\mathcal{N}_1 = \mathcal{N}_2$, and also take $f = \mathrm{conj}_e$ and $h = e^{-1}$ and $n = e^{-1}$ with $e \in \mathcal{E}$; then

$$\mathrm{conj}_e^*(\nu, \phi) = {}^{e^{-1}\phi(e)}(\nu, \phi).$$

In particular conj_e^* is the identity on $H^1_{\mathcal{N}_1}(\mathcal{E}_1, G_1)$.)

We have $(f_1, h_1, n_1)^* \circ (f_2, h_2, n_2)^* = (f_2 \circ f_1, h_1 \circ h_2, n_1 \circ n_2)^*$.

(C) Thirdly suppose that we have maps of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathcal{E}_1^0 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \Gamma & \longrightarrow & (0) \\ & & \downarrow & & f \downarrow & & || & & \\ (0) & \longrightarrow & \mathcal{E}_2^0 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \Gamma & \longrightarrow & (0), \end{array}$$

and a Γ-equivariant group homomorphism $h : G_1 \to G_2$ and pre-algebraicity conditions $\mathcal{N}_i$ for $(\mathcal{E}_i, G_i)$ and a map $n : \mathcal{N}_1 \to \mathcal{N}_2$ such that

- $n({}^g\nu) = {}^g n(\nu)$,
- $n({}^\sigma\nu) = {}^\sigma n(\nu)$,
- and $h \circ \overline{\nu} = \overline{n(\nu)} \circ f$.

Then we obtain a natural map

$$\begin{array}{rcl} f_* = (f,h,n)_* : Z^1_{\mathcal{N}_1}(\mathcal{E}_1, G_1) & \longrightarrow & Z^1_{\mathcal{N}_2}(\mathcal{E}_2, G_2) \\ (\nu, \phi) & \longmapsto & (n(\nu), \widetilde{\phi}), \end{array}$$

where $\widetilde{\phi}$ is defined by $\widetilde{\phi}(tf(e)) = \overline{n(\nu)}(t)(h \circ \phi(e))$ for any $t \in \mathcal{E}_2^0$ and $e \in \mathcal{E}_1$. This induces

$$f_* = (f,h,n)_* : H^1_{\mathcal{N}_1}(\mathcal{E}_1, G) \longrightarrow H^1_{\mathcal{N}_2}(\mathcal{E}_2, G).$$

(Kottwitz denotes this map $\Phi(h,n,f)$. The map in part A is a special case of this map in which f is the identity.)

We see that $(f_1,h_1,n_1)_* \circ (f_2,h_2,n_2)_* = (f_1 \circ f_2, n_1 \circ n_2)_*$.

Suppose that we are given commutative diagrams

$$\begin{array}{rcl} \mathcal{E}_2 & \stackrel{f}{\longleftarrow} & \mathcal{E}_1 \\ f_2 \downarrow & & \downarrow f_1 \\ \mathcal{E}_2' & \stackrel{f'}{\longleftarrow} & \mathcal{E}_1' \end{array}$$

and

$$\begin{array}{rcl} G_2 & \stackrel{h}{\longrightarrow} & G_1 \\ h_2 \downarrow & & \downarrow h_1 \\ G_2' & \stackrel{h'}{\longrightarrow} & G_1' \end{array}$$

and

$$\begin{array}{rcl} \mathcal{N}_2 & \stackrel{n}{\longrightarrow} & \mathcal{N}_1 \\ n_2 \downarrow & & \downarrow n_1 \\ \mathcal{N}_2' & \stackrel{n'}{\longrightarrow} & \mathcal{N}_1' \end{array}$$

such that (f,h,n) and (f',h',n') are as in part B, while (f_1,h_1,n_1) and (f_2,h_2,n_2) are as in this part. Then

$$\begin{array}{rcl} Z^1_{\mathcal{N}_2}(\mathcal{E}_2, G_2) & \stackrel{(f,h,n)^*}{\longrightarrow} & Z^1_{\mathcal{N}_1}(\mathcal{E}_1, G_1) \\ (f_2,h_2,n_2)_* \downarrow & & \downarrow (f_1,h_1,n_1)_* \\ Z^1_{\mathcal{N}_2'}(\mathcal{E}_2', G_2') & \stackrel{(f',h'n')^*}{\longrightarrow} & Z^1_{\mathcal{N}_1'}(\mathcal{E}_1', G_1') \end{array}$$

commutes.

(D) Suppose that $\Delta \subset \Gamma$ is a subgroup. If X is a set with an action of Δ we will write $\mathrm{Ind}_\Delta^\Gamma X$ for the set of functions $\varphi : \Gamma \to X$ satisfying

$$\varphi(\tau\sigma) = \tau\varphi(\sigma)$$

for all $\tau \in \Delta$ and $\sigma \in \Gamma$. It has an action of Γ via

$$(\sigma\varphi)(\sigma') = \varphi(\sigma'\sigma).$$

If X is a group and Δ acts via group automorphisms, then $\mathrm{Ind}\,_\Delta^\Gamma X$ is a group via

$$(\varphi\varphi')(\sigma) = \varphi(\sigma)\varphi'(\sigma),$$

and its Γ action is via group automorphisms. The map

$$\begin{array}{rcl} \epsilon : \mathrm{Ind}\,_\Delta^\Gamma X & \twoheadrightarrow & X \\ \varphi & \longmapsto & \varphi(1) \end{array}$$

is Δ-linear.

Suppose we have an extension

$$(0) \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E} \longrightarrow \Delta \longrightarrow (0)$$

and a group G with a Δ-action and a set $\mathcal{N}$ with a $G \rtimes \Delta$-action, together with a $G \rtimes \Delta$-invariant map $\bar{\ } : \mathcal{N} \to \mathrm{Hom}\,(\mathcal{E}^0, G)$ such that ${}^{\overline{\nu}(e)}\nu = \nu$ for all $e \in \mathcal{E}^0$ and $\nu \in \mathcal{N}$. Suppose moreover that we are given a second extension

$$(0) \longrightarrow \mathrm{Ind}\,_\Delta^\Gamma \mathcal{E}^0 \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \Gamma \longrightarrow (0)$$

such that if $\widetilde{\mathcal{E}}|_\Delta$ denotes the preimage of Δ in $\widetilde{\mathcal{E}}$, then there is a map of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathrm{Ind}\,_\Delta^\Gamma \mathcal{E}^0 & \longrightarrow & \widetilde{\mathcal{E}}|_\Delta & \longrightarrow & \Delta & \longrightarrow & (0) \\ & & \epsilon\downarrow & & \widetilde{\epsilon}\downarrow & & || & & \\ (0) & \longrightarrow & \mathcal{E}^0 & \longrightarrow & \mathcal{E} & \longrightarrow & \Delta & \longrightarrow & (0). \end{array}$$

We will write i for the natural inclusion $\widetilde{\mathcal{E}}|_\Delta \hookrightarrow \widetilde{\mathcal{E}}$. Note that if we think of $\mathcal{N}$ as a pre-algebraicity condition for $(\widetilde{\mathcal{E}}|_\Delta, G)$ with the new $\overline{\nu}$ equal to $\overline{\nu} \circ \epsilon$ for $\nu \in \mathcal{N}$, then

$$\widetilde{\epsilon}_* : Z^1_{\mathcal{N}}(\widetilde{\mathcal{E}}|_\Delta, G) \xrightarrow{\sim} Z^1_{\mathcal{N}}(\mathcal{E}, G)$$

and

$$\widetilde{\epsilon}_* : H^1_{\mathcal{N}}(\widetilde{\mathcal{E}}|_\Delta, G) \xrightarrow{\sim} H^1_{\mathcal{N}}(\mathcal{E}, G),$$

with inverse $\widetilde{\epsilon}^*$.

Note that $\mathrm{Ind}\,_\Delta^\Gamma \mathcal{N}$ has an action of $(\mathrm{Ind}\,_\Delta^\Gamma G) \rtimes \Gamma$, where

$$({}^\varphi\nu)(\sigma) = {}^{\varphi(\sigma)}(\nu(\sigma))$$

for $\varphi \in \mathrm{Ind}\,_\Delta^\Gamma G$ and $\nu \in \mathrm{Ind}\,_\Delta^\Gamma \mathcal{N}$. If $\nu \in \mathrm{Ind}\,_\Delta^\Gamma \mathcal{N}$ we define $\overline{\nu} \in \mathrm{Hom}\,(\mathrm{Ind}\,_\Delta^\Gamma \mathcal{E}^0, \mathrm{Ind}\,_\Delta^\Gamma G)$ by

$$\overline{\nu}(\varphi)(\sigma) = \overline{\nu(\sigma)}(\varphi(\sigma)).$$

It is easy to check that this makes $\mathrm{Ind}\,_\Delta^\Gamma \mathcal{N}$ a pre-algebraicity condition for $(\widetilde{\mathcal{E}}, \mathrm{Ind}\,_\Delta^\Gamma G)$. Combining lemma 12.10 of [K3] with the observation of the last paragraph we get the following result.

Lemma 3.1. *In the above situation the composite*

$$H^1_{\mathrm{Ind}\,{}_\Delta^\Gamma\mathcal{N}}(\widetilde{\mathcal{E}}, \mathrm{Ind}\,{}_\Delta^\Gamma G) \stackrel{(i,\epsilon,\epsilon)^*}{\longrightarrow} H^1_{\mathcal{N}}(\widetilde{\mathcal{E}}|_\Delta, G) \stackrel{\widetilde{\epsilon}_*}{\longrightarrow} H^1_{\mathcal{N}}(\mathcal{E}, G)$$

is an isomorphism.

It will be convenient for us to work with pre-algebraicity conditions $\mathcal{N}$ with slightly more structure. Namely we will assume that $\mathcal{N}$ is endowed with a subset $\mathcal{N}_{\mathrm{basic}} \subset \mathcal{N}^G$, an abelian group structure on $\mathcal{N}_{\mathrm{basic}}$ and an action of $\mathcal{N}_{\mathrm{basic}}$ on $\mathcal{N}$ extending the action of $\mathcal{N}_{\mathrm{basic}}$ on itself by translation; such that the following properties hold:

- Γ preserves $\mathcal{N}_{\mathrm{basic}}$ and acts on it via group automorphisms.
- The action of $G \rtimes \Gamma$ on $\mathcal{N}$ commutes with the action of $\mathcal{N}_{\mathrm{basic}}$, i.e. ${}^g(\nu\mu) = \nu\, {}^g\mu$ and ${}^\sigma(\nu\mu) = {}^\sigma\nu\, {}^\sigma\mu$ for all $\nu \in \mathcal{N}_{\mathrm{basic}}$, $\mu \in \mathcal{N}$, $g \in G$ and $\sigma \in \Gamma$.
- If $\nu \in \mathcal{N}_{\mathrm{basic}}$, then $\overline{\nu}$ factors through $Z(G)$.
- $\overline{\nu\mu} = \overline{\nu}\,\overline{\mu}$.

We will refer to this additional data as an *algebraicity condition*. Note that in this case both $\mathcal{N}$ and $\mathcal{N}_{\mathrm{basic}}$ are pre-algebraicity conditions. We will often write $Z^1_{\mathcal{N}}(\mathcal{E}, G)_{\mathrm{basic}}$ and $H^1_{\mathcal{N}}(\mathcal{E}, G)_{\mathrm{basic}}$, instead of $Z^1_{\mathcal{N}_{\mathrm{basic}}}(\mathcal{E}, G)$ and $H^1_{\mathcal{N}_{\mathrm{basic}}}(\mathcal{E}, G)$. We refer to these as the set of *basic algebraic cocycles* and the *basic algebraic cohomology*.

If G is abelian then $Z^1_{\mathcal{N}}(\mathcal{E}, G)_{\mathrm{basic}}$ and $H^1_{\mathcal{N}}(\mathcal{E}, G)_{\mathrm{basic}}$ are naturally abelian groups.

There is a natural map

$$\mathrm{ad} : Z^1_{\mathcal{N}}(\mathcal{E}, G)_{\mathrm{basic}} \longrightarrow Z^1(\Gamma, G^{\mathrm{ad}})$$

which induces a map in cohomology. If $(\nu, \zeta) \in Z^1_{\mathcal{N}_{\mathrm{basic}}}(\mathcal{E}, Z(G))$ and $(\mu, \phi) \in Z^1_{\mathcal{N}}(\mathcal{E}, G)$ then $(\nu\mu, \eta\phi) \in Z^1_{\mathcal{N}}(\mathcal{E}, G)$. This induces maps

$$(\nu, \zeta) : Z^1_{\mathcal{N}}(\mathcal{E}, G)_{\mathrm{basic}} \longrightarrow Z^1_{\mathcal{N}}(\mathcal{E}, G)_{\mathrm{basic}}$$

and

$$(\nu, \zeta) : H^1_{\mathcal{N}}(\mathcal{E}, G) \longrightarrow H^1_{\mathcal{N}}(\mathcal{E}, G)$$

and

$$(\nu, \zeta) : H^1_{\mathcal{N}}(\mathcal{E}, G)_{\mathrm{basic}} \longrightarrow H^1_{\mathcal{N}}(\mathcal{E}, G)_{\mathrm{basic}}.$$

This gives actions of $Z^1_{\mathcal{N}_{\mathrm{basic}}}(\mathcal{E}, Z(G))$ on $Z^1_{\mathcal{N}}(\mathcal{E}, G)$ and $Z^1_{\mathcal{N}}(\mathcal{E}, G)_{\mathrm{basic}}$; and of $H^1_{\mathcal{N}_{\mathrm{basic}}}(\mathcal{E}, Z(G))$ on $H^1_{\mathcal{N}}(\mathcal{E}, G)$ and $H^1_{\mathcal{N}}(\mathcal{E}, G)_{\mathrm{basic}}$. The map $H^1_{\mathcal{N}}(\mathcal{E}, G) \to H^1_{\mathcal{N}}(\mathcal{E}, G^{\mathrm{ad}})$ is constant on $H^1_{\mathcal{N}_{\mathrm{basic}}}(\mathcal{E}, Z(G))$-orbits.

We have the following additions to our various functorialities:

(A) In the situation of A if $n(\mathcal{N}_{1,\mathrm{basic}}) \subset \mathcal{N}_{2,\mathrm{basic}}$ then $(h, n)_*$ takes basic cocycles or cohomology classes to basic ones.
(B) In the situation of B if $n(\mathcal{N}_{2,\mathrm{basic}}) \subset \mathcal{N}_{1,\mathrm{basic}}$ then $(f, h, n)^*$ takes basic cocycles or cohomology classes to basic ones.
(C) In the situation of C if $n(\mathcal{N}_{1,\mathrm{basic}}) \subset \mathcal{N}_{2,\mathrm{basic}}$ then $(f, h, n)_*$ takes basic cocycles or cohomology classes to basic ones.

(D) In the situation of D if $(\mathcal{N},\mathcal{N}_{\rm basic})$ is an algebraicity condition for $(\mathcal{E},G)$, then $(\mathrm{Ind}\,^{\Gamma}_{\Delta}\mathcal{N},\mathrm{Ind}\,^{\Gamma}_{\Delta}\mathcal{N}_{\rm basic})$ is one for $(\widetilde{\mathcal{E}},\mathrm{Ind}\,^{\Gamma}_{\Delta}G)$, and lemma 3.1 is also true for the basic algebraic cohomology.

(E) Suppose that G is abelian and that $\Delta\subset\Gamma$ is a subgroup of finite index. Let $\mathcal{R}\subset\mathcal{E}$ be a set of representatives for $\mathcal{E}|_\Delta\backslash\mathcal{E}$. Then we obtain a natural map

$$\begin{array}{rcl} \mathrm{cor}\,_{\mathcal{R}}:Z^1_{\mathcal{N}}(\mathcal{E}|_\Delta,G)_{\rm basic} & \longrightarrow & Z^1_{\mathcal{N}}(\mathcal{E},G)_{\rm basic}\\ (\nu,\phi) & \longmapsto & (\prod_{r\in\mathcal{R}}{}^{r^{-1}}\nu,\widetilde{\phi}),\end{array}$$

where

$$\widetilde{\phi}(e)=\prod_{r\in\mathcal{R}}{}^{r^{-1}}(\phi(res^{-1}))$$

with each $s\in\mathcal{R}$ chosen such that $res^{-1}\in\mathcal{E}'$. It induces a map

$$\mathrm{cor}\,:H^1_{\mathcal{N}}(\mathcal{E}|_\Delta,G)_{\rm basic}\longrightarrow H^1_{\mathcal{N}}(\mathcal{E},G)_{\rm basic}$$

which is independent of the choice of $\mathcal{R}$.

If $\mathcal{E}=\mathcal{E}_\alpha$ with $\alpha\in Z^2(\Gamma,\mathcal{E}^0)$ and $\Delta=\{1\}$, then we may take $\mathcal{R}=\mathcal{R}_\alpha=\{e_\alpha(\sigma):\ \sigma\in\Gamma\}$ and we will write

$$\mathrm{cor}\,_\alpha=\mathrm{cor}\,_{\mathcal{R}_\alpha}:\mathcal{N}_{\rm basic}\longrightarrow Z^1_{\mathcal{N}}(\mathcal{E}_\alpha,G)_{\rm basic}.$$

Note that

$$(\mathrm{cor}\,_\alpha\nu)(e_\alpha(\sigma))=\prod_{\eta\in\Gamma}\eta^{-1}\overline{\nu}(\alpha(\eta,\sigma)).$$

If $\beta:\Gamma\to\mathcal{E}^0$ and if $i_\beta:\mathcal{E}_\alpha\xrightarrow{\sim}\mathcal{E}_{{}^\beta\alpha}$ is the canonical isomorphism sending $e_\alpha(\sigma)$ to $\beta(\sigma)e_{{}^\beta\alpha}(\sigma)$ then it is easily verified that

$$i_\beta^*(\mathrm{cor}\,_{{}^\beta\alpha}\nu)={}^{\prod_{\eta\in\mathrm{Gal}\,(E/F)}\eta^{-1}\overline{\nu}(\beta(\eta))^{-1}}\mathrm{cor}\,_\alpha\nu.$$

(Indeed, both sides are of the form $(\prod_{\eta\in\Gamma}{}^\eta\nu,\phi)$ for some ϕ. Moreover

$$\begin{array}{rl} & (i_\beta^*\mathrm{cor}\,_{{}^\beta\alpha}(\nu))(e_\alpha(\sigma))\\ = & \mathrm{cor}\,_{{}^\beta\alpha}(\nu)(\beta(\sigma)e_{{}^\beta\alpha}(\sigma))\\ = & \prod_{\eta\in\Gamma}{}^\eta\overline{\nu}(\beta(\sigma))\prod_{\eta\in\Gamma}\eta^{-1}\overline{\nu}({}^\beta\alpha(\eta,\sigma))\\ = & \prod_{\eta\in\Gamma}{}^\eta\overline{\nu}(\beta(\sigma))\prod_{\eta\in\Gamma}\eta^{-1}\overline{\nu}(\alpha(\eta,\sigma)\beta(\eta\sigma)\beta(\eta)^{-1}{}^\eta\beta(\sigma)^{-1})\\ = & \prod_{\eta\in\Gamma}{}^\eta\overline{\nu}(\beta(\sigma))\mathrm{cor}\,_\alpha(\nu)(e_\alpha(\sigma))\prod_{\eta\in\Gamma}\eta^{-1}\overline{\nu}(\beta(\eta\sigma))\\ & \prod_{\eta\in\Gamma}\eta^{-1}\overline{\nu}(\beta(\eta))^{-1}\prod_{\eta\in\Gamma}{}^{\eta^{-1}}\overline{\nu}(\beta(\sigma))^{-1}\\ = & \mathrm{cor}\,_\alpha(\nu)(e_\alpha(\sigma))\sigma\prod_{\eta\in\Gamma}\eta^{-1}\overline{\nu}(\beta(\eta))/\prod_{\eta\in\Gamma}\eta^{-1}\overline{\nu}(\beta(\eta)).)\end{array}$$

Now consider the case that $\Gamma=\mathrm{Gal}\,(E/F)$ and $G=H(A_E)$, where

- E/F is a finite Galois extension of fields;
- H/F is an algebraic group;
- and $A_E=A\otimes_F E$ for some F-algebra A.

If $(\nu,\phi)\in Z^1_{\mathcal{N}}(\mathcal{E},G(A_E))_{\mathrm{basic}}$ then we define ${}^{\phi}G/A$ to be the etale descent of $G\times A_E$ to $G\times A$ via the action

$$\sigma\longmapsto \mathrm{conj}_{\phi(e)}\circ\sigma,$$

where $e\in\mathcal{E}$ is any lift of σ. Thus ${}^{\phi}G\times_{A_F}A_E=G\times_F A_E$. If $g\in G(A_E)$ then

$$\mathrm{conj}_g:{}^{\phi}G\stackrel{\sim}{\longrightarrow}{}^{g\phi}G.$$

Thus ${}^{\phi}G$ depends only on $[(\nu,\phi)]$ up to an isomorphism that is unique *up to composition conjugation by an element of* $({}^{\phi}G)(A)$. (Note that we have ${}^{\phi}G(A)$ here and not ${}^{\phi}G^{\mathrm{ad}}(A)$. This is an important point.) When we are only concerned with properties of ${}^{\phi}G$ for which this ambiguity does not matter, we may write ${}^{[(\nu,\phi)]}G$. There is a bijection

$$\begin{array}{ccc} Z^1_{\mathcal{N}}(\mathcal{E},{}^{\phi}G(A_E)) & \stackrel{\sim}{\longrightarrow} & Z^1_{\mathcal{N}}(\mathcal{E},G(A_E)) \\ (\mu,\psi) & \longmapsto (\mu\nu,\psi\phi) & \end{array}$$

which takes basic subset to basic subset, and induces isomorphisms in cohomology. Note that

$${}^{gh}((\mu,\psi)(\nu,\phi))=\mathrm{conj}_g({}^{h}(\mu,\psi))^{g}(\nu,\phi).$$

If $\boldsymbol{\phi}\in H^1_{\mathcal{N}}(\mathcal{E},G(A_E))_{\mathrm{basic}}$ then

$$\{\boldsymbol{\zeta}\in H^1_{\mathcal{N}_{\mathrm{basic}}}(\mathcal{E},Z(G)(A_E)):\ \boldsymbol{\zeta\phi}=\boldsymbol{\phi}\}=\ker(H^1_{\mathcal{N}_{\mathrm{basic}}}(\mathcal{E},Z(G)(A_E))\to H^1_{\mathcal{N}}(\mathcal{E},{}^{\boldsymbol{\phi}}G(A_E)))$$

is a subgroup of $H^1_{\mathcal{N}_{\mathrm{basic}}}(\mathcal{E},Z(G)(A_E))$.

3.2. **Kottwitz cohomology for local Weil groups.** Suppose that F is a local field of charcateristic 0 and E/F is a finite Galois extension. If $\alpha\in[\alpha_{E/F}]\subset Z^2(\mathrm{Gal}\,(E/F),E^\times)$, then we get a well defined extension

$$0\longrightarrow E^\times\longrightarrow W_{E/F,\alpha}\longrightarrow \mathrm{Gal}\,(E/F)\longrightarrow 0.$$

As $H^1(\mathrm{Gal}\,(E/F),E^\times)=(0)$ the only automorphisms of this extension are conjugation by an element of $E^\times$. If α' is a second element of $[\alpha_{E/F}]$, then there is an isomorphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^\times & \longrightarrow & W_{E/F,\alpha} & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & 0 \\ & & || & & \downarrow\wr & & || & & \\ 0 & \longrightarrow & E^\times & \longrightarrow & W_{E/F,\alpha'} & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & 0. \end{array}$$

However it is only unique up to composition with conjugation by an element of $E^\times$. In particular, if $\overline{F}$ is an algebraic closure of F containing E, then the extension $W_{E/F,\alpha}$ is isomorphic to $W_{E^{\mathrm{ab}}/F}$, but this isomorphism is only unique up to composition with conjugation by an element of $E^\times$.

Let G/F denote an algebraic group. We will consider the algebraicity conditions $\mathcal{N}=X_*(G)(E)$ and $\mathcal{N}_{\mathrm{basic}}=X_*(Z(G))(E)$. We will denote the corresponding algebraic cocycles, basic cocycles, cohomology and basic cohomology as $Z^1_{\mathrm{alg}}(W_{E/F,\alpha},G(E))$, $Z^1_{\mathrm{alg}}(W_{E/F,\alpha},G(E))_{\mathrm{basic}}$, $H^1_{\mathrm{alg}}(W_{E/F},G(E))$, and $H^1_{\mathrm{alg}}(W_{E/F},G(E))_{\mathrm{basic}}$ respectively.

As the notation suggest the two cohomology groups are canonically independent of the choice of $\alpha \in [\alpha_{E/F}]$.

We will call $\boldsymbol{\phi} \in H^1_{\rm alg}(W_{E/F}, G(E))_{\rm basic}$ *compact* if ${}^{\phi}G^{\rm ad}(F)$ is compact, and we will write $H^1_{\rm alg}(W_{E/F}, G(E))^{\rm compact}_{\rm basic}$ for the set of compact elements in $H^1_{\rm alg}(\mathcal{E}(E/F), G(E))_{\rm basic}$.

Suppose that D/E is another finite extension, Galois over F. Choose representatives $\alpha_{E/F} \in [\alpha_{E/F}]$ and $\alpha_{D/F} \in [\alpha_{D/F}]$ and $\gamma_{D/E} : {\rm Gal}\,(D/F) \to D^\times$ such that $\alpha_{D/F}^{[D:E]} = {}^{\gamma_{D/E}}\alpha_{E/F} \in Z^2({\rm Gal}\,(D/F), D^\times)$. If we write $W_{E/F,\alpha_{E/F},D}$ for the push out of $W_{E/F,\alpha_{E/F}}|_{{\rm Gal}\,(D/F)}$ along $E^\times \longrightarrow D^\times$; then $i_{\gamma_{E/F}}$ gives an isomorphism from $W_{E/F,\alpha_{E/F},D}$ to the pushout of $W_{D/F,\alpha_{D/F}}$ along the $[D:E]^{th}$-power map $D^\times \to D^\times$. Thus there is a commutative diagram

$$\begin{array}{ccccccccc}
(0) & \longrightarrow & D^\times & \longrightarrow & W_{D/F,\alpha_{D/F}} & \longrightarrow & {\rm Gal}\,(D/F) & \longrightarrow & (0) \\
 & & [D:E]\downarrow & & \eta_{D/E,\gamma_{D/E}}\downarrow & & || & & \\
(0) & \longrightarrow & D^\times & \longrightarrow & W_{E/F,\alpha_{E/F},D} & \longrightarrow & {\rm Gal}\,(D/F) & \longrightarrow & (0) \\
 & & \uparrow & & \uparrow & & || & & \\
(0) & \longrightarrow & E^\times & \longrightarrow & W_{E/F,\alpha_{E/F}}|_{{\rm Gal}\,(D/F)} & \longrightarrow & {\rm Gal}\,(D/F) & \longrightarrow & (0) \\
 & & || & & \downarrow & & \downarrow & & \\
(0) & \longrightarrow & E^\times & \longrightarrow & W_{E/F,\alpha_{E/F}} & \longrightarrow & {\rm Gal}\,(E/F) & \longrightarrow & (0).
\end{array}$$

Using successively functorialities (B) then (C) then (B) again from the end of section 3.1, we obtain a map

$$\begin{aligned}
{\rm inf}_{D/E,\gamma_{D/E}} : Z^1_{\rm alg}(W_{E/F,\alpha_{E/F}}, G(E)) \longrightarrow Z^1_{\rm alg}(W_{E/F,\alpha_{E/F}}|_{{\rm Gal}\,(D/F)}, G(D)) \longrightarrow Z^1_{\rm alg}(W_{E/F,\alpha_{E/F},D}, G(D)) \\
\longrightarrow Z^1_{\rm alg}(W_{D/F,\alpha_{D/F}}, G(D)).
\end{aligned}$$

(Where we use the algebraicity conditions $\mathcal{N} = X_*(G)(D)$ and $\mathcal{N}_{\rm basic} = X_*(Z(G))(D)$ for the middle two sets of cocycles.) All these maps take basic elements to basic elements. The composite sends (ν, ϕ) to $(\nu^{[D:E]}, \widetilde{\phi})$, where, if $\eta_{D/E,\gamma_{D/E}}(e) = de'$ with $d \in D^\times$ and $e' \in W_{E/F,\alpha_{E/F}}|_{{\rm Gal}\,(D/F)}$ then

$$\widetilde{\phi}(e) = \nu(d)\phi(\overline{e}')$$

where $\overline{e}'$ denotes the image of e' in $W_{E/F,\alpha_{E/F}}$. This composite is injective. The map $\gamma_{E/D}$ can only be replaced by

$${}^d\gamma_{D/E} : \sigma \mapsto \gamma_{D/E}(\sigma)d/{}^{\sigma}d,$$

for some $d \in D^\times$. (As $H^1({\rm Gal}\,(D/F), D^\times) = (0)$.) We have

$$\eta_{D/E,{}^d\gamma_{D/E}} = {\rm conj}_{d^{-1}} \circ \eta_{D/E,\gamma_{D/E}},$$

and so the induced map

$$\inf_{D/E,\gamma_{D/E}} : H^1_{\rm alg}(W_{E/F}, G(E)) \longrightarrow H^1_{\rm alg}(W_{D/F}, G(D))$$

is independent of the choice of $\gamma_{D/E}$ and so we will denote it simply $\inf_{D/E}$. The maps

$$\begin{aligned}\inf_{D/E} : H^1_{\mathrm{alg}}(W_{E/F,\alpha_{E/F}}, G(E)) \longrightarrow H^1_{\mathrm{alg}}(\mathcal{E}', G(D)) \longrightarrow H^1_{\mathrm{alg}}(W_{E/F,\alpha_{E/F},D}, G(D)) \\ \longrightarrow H^1_{\mathrm{alg}}(W_{D/F,\alpha_{D/F}}, G(D))\end{aligned}$$

are all injective. (The first because the usual inflation map is injective on H^1, and the second and third immediately from the definitions.)

Kottwitz defines

$$B(F,G) = \lim_{\to E} H^1_{\mathrm{alg}}(W_{E/F}, G(E))$$

and

$$B(F,G)_{\mathrm{basic}} = \lim_{\to E} H^1_{\mathrm{alg}}(W_{E/F}, G(E))_{\mathrm{basic}}.$$

If $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in H^1_{\mathrm{alg}}(W_{E/F}, G(E))$ have the same image in $H^1(W_{E/F}, G^{\mathrm{ad}}(E))$, then we can find a finite extension D/E Galois over F such that $\inf \boldsymbol{\phi}_i \in H^1_{\mathrm{alg}}(W_{D/F}, G(D))$ can be represented by cocycles ϕ_i with $\mathrm{ad}\,\phi_1 = \mathrm{ad}\,\phi_2$. (If ϕ_i' is a representative of $\boldsymbol{\phi}_i$ and if $g \in G^{\mathrm{ad}}(E)$ with ${}^g\mathrm{ad}\,\phi_1' = \mathrm{ad}\,\phi_2'$, then we may choose such a field D and a $\widetilde{g} \in G(D)$ lifting g. Then ${}^{\widetilde{g}}\inf \phi_1'$ and $\inf \phi_2'$ will do.) Thus $\inf \boldsymbol{\phi}_1, \inf \boldsymbol{\phi}_2 \in H^1_{\mathrm{alg}}(W_{D/F}, G(D))$ differ by an element of $H^1_{\mathrm{alg}}(W_{D/F}, Z(G)(D))$.

3.3. **Reductive groups.** If T/F is a torus split by a finite Galois extension E/F, then Kottwitz shows that

$$\mathrm{cor} : X_*(T)_{\mathrm{Gal}\,(E/F)} \xrightarrow{\sim} H^1_{\mathrm{alg}}(W_{E/F}, T).$$

Moreover if G is any reductive group which splits over E, then Kottwitz constructs a map

$$\kappa_G : H^1_{\mathrm{alg}}(W_{E/F}, G(E)) \longrightarrow \Lambda_{G,\mathrm{Gal}\,(E/F)}$$

with the following properties

- it is functorial in G;
- if $G = T$ is a torus, then $\kappa_T = \mathrm{cor}^{-1}$ is an isomorphism;
- if D/E is a finite extension Galois over F, then

$$\begin{array}{ccc} H^1_{\mathrm{alg}}(W_{E/F}, G(E)) & \xrightarrow{\kappa_G} & \Lambda_{G,\mathrm{Gal}\,(\overline{F}/F)} \\ \inf_{D/E} \downarrow & \kappa_G \nearrow & \\ H^1_{\mathrm{alg}}(W_{D/F}, G(D)) & & \end{array}$$

  commutes.

(See [K3].)In the limit Kottwitz obtains a map

$$\kappa_G : B(F,G) \longrightarrow \Lambda_{G,\mathrm{Gal}\,(\overline{F}/F)}.$$

Lemma 3.2. *If $\boldsymbol{\phi} \in H^1_{\mathrm{alg}}(W_{E/F}, G)_{\mathrm{basic}}$ and $\boldsymbol{\psi} \in H^1_{\mathrm{alg}}(W_{E/F}, {}^{\phi}G)$ then*

$$\kappa_G(\boldsymbol{\psi}\boldsymbol{\phi}) = \kappa_{{}^{\phi}G}(\boldsymbol{\psi})\kappa_G(\boldsymbol{\phi}).$$

Proof: This follows easily from the construction of κ_G in section 9.3 of [K3]. If G is a torus it simply expresses the fact that cor is an abelian group homomorphism in this case. If $G^{\rm der}$ is simply connected it follows for the corresponding fact for $C(G)$. In the general case it follows from the corresponding fact for a suitable z-extension of G. □

If $T \subset G$ is a maximal torus then we define $H^1_{\rm alg}(W_{E/F}, T(E))_{G-{\rm basic}}$ to be those elements $[(\nu, \phi)]$ where ν factors through $Z(G)$. If $F = \mathbb{R}$ suppose that T is fundamental, while if F is p-adic assume that E is elliptic. If E splits T then

$$H^1_{\rm alg}(W_{E/F}, T(E))_{G-{\rm basic}} \twoheadrightarrow H^1_{\rm alg}(W_{E/F}, G(E))_{\rm basic}.$$

(See proposition 13.1 and lemma 13.2 of [K3].) We deduce that if E splits an elliptic (in the p-adic case) or fundamental (in the real case) torus, then

$$H^1_{\rm alg}(W_{E/F}, G(E))_{\rm basic} \stackrel{\sim}{\to} B(F,G)_{\rm basic}.$$

We further deduce that for E sufficiently large the quotient of $H^1_{\rm alg}(W_{E/F}, G(E))_{\rm basic}$ by its action of $H^1_{\rm alg}(W_{E/F}, Z(G)(E))$ embeds into $H^1_{\rm alg}(W_{E/F}, G^{\rm ad}(E))_{\rm basic}$. (First choose E_0 such that $H^1_{\rm alg}(W_{E/F}, G(E))_{\rm basic} \stackrel{\sim}{\to} B(F,G)_{\rm basic}$ for any $E \supset E_0$, and then $E \supset E_0$ such that every element of $G^{\rm ad}(E_0)$ has a lift in $G(E)$. If $Z(G)$ is a torus then we may take $E = E_0$.)

If F is a p-adic field then κ_G is infact a bijection

$$\kappa_G : B(F,G)_{\rm basic} \stackrel{\sim}{\longrightarrow} \Lambda_{G, {\rm Gal}\,(\overline{F}/F)}.$$

and so $B(F,G)_{\rm basic}$ becomes an abelian group. (See proposition 13.1 of [K3].) We deduce that if E splits some maximal torus of G defined over F, then

$$\kappa_G : H^1_{\rm alg}(W_{E/F}, G(E))_{\rm basic} \stackrel{\sim}{\longrightarrow} \Lambda_{G, {\rm Gal}\,(\overline{F}/F)}.$$

In particular if G is semi-simple, simply connected, then $H^1_{\rm alg}(W_{E/F}, G(E))_{\rm basic} = \{1\}$. If G is semisimple then

$$\kappa_G : H^1(F,G) \stackrel{\sim}{\longrightarrow} B(F,G)_{\rm basic} \stackrel{\sim}{\longrightarrow} \Lambda_{G, {\rm Gal}\,(\overline{F}/F)}.$$

If $F = \mathbb{C}$ then $W_{\mathbb{C}/\mathbb{C}} = \mathbb{C}^\times$, $H^1(\mathbb{C}, G) = (0)$ and $B(\mathbb{C}, G)_{\rm basic} = X_*(Z(G))$ and $\kappa_G : B(\mathbb{C}, G)_{\rm basic} = X_*(Z(G)) \hookrightarrow \Lambda_G$.

3.4. **The real case.** Now suppose that $F = \mathbb{R}$. Choose a representative $\alpha^0_{\mathbb{C}/\mathbb{R}}$ for $[\alpha_{\mathbb{C}/\mathbb{R}}]$ defined by

$$\alpha^0_{\mathbb{C}/\mathbb{R}}(\sigma_1, \sigma_2) = \begin{cases} -1 & \text{if } \sigma_1 = \sigma_2 = c \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$W_{\mathbb{C}/\mathbb{R}, \alpha^0_{\mathbb{C}/\mathbb{R}}} \cong \langle \mathbb{C}^\times, j : \; j^2 = -1 \text{ and } jzj^{-1} = {}^c z \rangle,$$

with $e(1) = 1$ and $e(c) = j$. Thus an element of $Z^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R},\alpha^0_{\mathbb{C}/\mathbb{R}}}, G(\mathbb{C}))$ is a pair (ν, J) where $\nu \in X_*(G)$ and $J \in G(\mathbb{C})$ satisfy

$$\nu = J^c \nu J^{-1}$$

and

$$J^c J = \nu(-1).$$

Moreover $[(\nu, J)] = [(\nu', J')] \in H^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R}}, G(\mathbb{C}))$ if there exists $g \in G(\mathbb{C})$ such that

$$\nu' = g\nu g^{-1}$$

and

$$J' = gJ^c g^{-1}.$$

If $\mu \in X_*(G)$ and the image of μ commutes with that of ${}^c\mu$, then we obtain an element $\widehat{\lambda}_G(\mu) \in Z^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R},\alpha^0_{\mathbb{C}/\mathbb{R}}}, G(\mathbb{C}))$ defined by

$$\widehat{\lambda}_G(\mu) = (\mu^c\mu, \mu(-1)).$$

Note that $\kappa_G \widehat{\lambda}_G(\mu)$ equals the image of μ in $\Lambda_{G,\mathrm{Gal}\,(\mathbb{C}/\mathbb{R})}$. (To see this note that it suffices to treat the case that $G = T$ is a torus, in which case $\widehat{\lambda}_G(\mu) = \mathrm{cor}_{\,\alpha^0_{\mathbb{C}/\mathbb{R}}}\mu$.) If $\mu^c\mu$ factors through $Z(G)$, then $\widehat{\lambda}_G(\mu) \in Z^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R},\alpha^0_{\mathbb{C}/\mathbb{R}}}, G(\mathbb{C}))_{\mathrm{basic}}$. This induces maps

$$\widehat{\boldsymbol{\lambda}}_G : G(\mathbb{R})\backslash\{\mu \in X_*(G) : \text{ the images of } \mu \text{ and } {}^c\mu \text{ commute}\} \longrightarrow H^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R}}, G(\mathbb{C}))$$

and

$$\widehat{\boldsymbol{\lambda}}_G : G(\mathbb{R})\backslash\{\mu \in X_*(G) : \; {}^c\mu = \mu^{-1} \in X_*(G^{\mathrm{ad}})\} \longrightarrow H^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R}}, G(\mathbb{C}))_{\mathrm{basic}}.$$

The image $\widehat{\boldsymbol{\lambda}}_G(\mu)$ depends only on the $G(\mathbb{R})$-conjugacy class of μ, so we will sometimes write $\widehat{\boldsymbol{\lambda}}_G([\mu]_{G(\mathbb{R})})$.

If $Y \subset X_*(G)$ is a basic $G(\mathbb{R})$-conjugacy class then for any $\mu \in Y$ we have $\mu^c\mu = {}^c\mu\mu \in X_*(Z(G))$, and this homomorphism is independent of the choice of $\mu \in Y$. We will denote it $\boldsymbol{\nu}_Y$. We have $\boldsymbol{\nu}_{\widehat{\boldsymbol{\lambda}}_G(Y)} = \boldsymbol{\nu}_Y$.

The group ${}^{\widehat{\lambda}_G(\mu)}G$ comes with basic ${}^{\widehat{\lambda}_G(\mu)}G(\mathbb{R})$-conjugacy classes $[\mu]_{{}^{\widehat{\lambda}_G(\mu)}G(\mathbb{R})}$ and $[\mu^{-1}]_{{}^{\widehat{\lambda}_G(\mu)}G(\mathbb{R})}$. If $g \in G(\mathbb{R})$ then $\mathrm{conj}_g : {}^{\widehat{\lambda}_G(\mu)}G \xrightarrow{\sim} {}^{\widehat{\lambda}_G(\mathrm{conj}_g\circ\mu)}G$ and it takes $[\mu]_{{}^{\widehat{\lambda}_G(\mu)}G(\mathbb{R})}$ to $[\mathrm{conj}_g \circ \mu]_{{}^{\widehat{\lambda}_G(\mathrm{conj}_g\circ\mu)}G(\mathbb{R})}$ and $[\mu^{-1}]_{{}^{\widehat{\lambda}_G(\mu)}G(\mathbb{R})}$ to $[\mathrm{conj}_g \circ \mu^{-1}]_{{}^{\widehat{\lambda}_G(\mathrm{conj}_g\circ\mu)}G(\mathbb{R})}$. We will write $Y([\mu]_{G(\mathbb{R})})_{{}^{\widehat{\lambda}_G(\mu)}G}$ and $Y([\mu^{-1}]_{G(\mathbb{R})})_{{}^{\widehat{\lambda}_G(\mu)}G}$ for the corresponding ${}^{\widehat{\boldsymbol{\lambda}}_G(\mu)}G(\mathbb{R})$-conjugacy classes in $X_*({}^{\widehat{\boldsymbol{\lambda}}_G(\mu)}G)$. (This makes sense as much as ${}^{\widehat{\boldsymbol{\lambda}}_G(\mu)}G$ does. More precisely, if $\phi \in \widehat{\boldsymbol{\lambda}}_G(\mu)$ then $Y([\mu^{\pm 1}]_{G(\mathbb{R})})_{{}^{\phi}G}$ are well defined ${}^{\phi}G(\mathbb{R})$-conjugacy classes; and if $g \in G(\mathbb{C})$, then $\mathrm{conj}_g Y([\mu^{\pm 1}]_{G(\mathbb{R})})_{{}^{\phi}G} = Y([\mu^{\pm 1}]_{G(\mathbb{R})})_{{}^{g\phi}G}$.) Note that $\widehat{\boldsymbol{\lambda}}_{{}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G}(Y(Y^{\pm 1})_{{}^{\widehat{\lambda}_G(\mu)}G}) = \widehat{\boldsymbol{\lambda}}_G(Y)^{\pm 1}$.

If $G/\mathbb{R}$ is reductive and $G^{\mathrm{ad}}(\mathbb{R})$ is compact, then any G-conjugacy class C of cocharacters contains a unique basic $G(\mathbb{R})$-conjugacy class C^0. In this case we define $\widetilde{\boldsymbol{\lambda}}_G(C) = \widehat{\boldsymbol{\lambda}}_G((C^0)^{-1})$ and $Y(C)_{\tilde{\boldsymbol{\lambda}}_G(C)G}$ for $Y(C^0)_{\hat{\boldsymbol{\lambda}}_G(C^{0,-1})G}$, a ${}^{\tilde{\boldsymbol{\lambda}}_G(C)}G(\mathbb{R})$-conjugacy class in $X_*({}^{\tilde{\boldsymbol{\lambda}}_G(C)}G)$. We have

$$\widehat{\boldsymbol{\lambda}}_{{}^{\tilde{\boldsymbol{\lambda}}_G(C)}G}(Y(C)_{\tilde{\boldsymbol{\lambda}}_G(C)G}) = \widetilde{\boldsymbol{\lambda}}_G(C)^{-1}$$

and

$$\kappa_G(\widetilde{\boldsymbol{\lambda}}_G(C)) = \lambda_G(C)^{-1}.$$

Now suppose that $G/\mathbb{R}$ is reductive, that $Y \subset X_*(G)$ is a compactifying $G(\mathbb{R})$-conjugacy class and that C is a G-conjugacy class in $X_*(G)$. Then C is canonically a ${}^{\hat{\boldsymbol{\lambda}}_G(Y)}G$-conjugacy class in $X_*({}^{\hat{\boldsymbol{\lambda}}_G(Y)}G)$, and so we have $\widetilde{\boldsymbol{\lambda}}_{{}^{\hat{\lambda}_G(Y)}G}(C) \in H^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R}}, {}^{\hat{\lambda}_G(Y)}G)_{\mathrm{basic}}$. We set

$$\widehat{\boldsymbol{\lambda}}_G(Y - C) = \widetilde{\boldsymbol{\lambda}}_{{}^{\hat{\lambda}_G(Y)}G}(C)\widehat{\boldsymbol{\lambda}}_G(Y) \in H^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R}}, G)_{\mathrm{basic}}.$$

The group ${}^{\hat{\boldsymbol{\lambda}}_G(Y-C)}G$ comes with a ${}^{\hat{\boldsymbol{\lambda}}_G(Y-C)}G(\mathbb{R})$-conjugacy class of cocharacters

$$Y(C)_{\hat{\boldsymbol{\lambda}}_G(Y-C)G} = Y(C)_{\tilde{\boldsymbol{\lambda}}_{{}^{\hat{\boldsymbol{\lambda}}_G(Y)}G}(C)}({}^{\hat{\boldsymbol{\lambda}}_G(Y)}G).$$

Note that

$$\widehat{\boldsymbol{\lambda}}_{{}^{\hat{\boldsymbol{\lambda}}_G(Y-C)}G}(Y(C)_{\hat{\boldsymbol{\lambda}}_G(Y-C)G}) = \widetilde{\boldsymbol{\lambda}}_{{}^{\hat{\boldsymbol{\lambda}}_G(Y)}G}(C)^{-1}$$

and

$$\kappa_G(\widehat{\boldsymbol{\lambda}}_G(Y - C)) = \lambda_G(Y)/\lambda_G(C)$$

and

$$\boldsymbol{\nu}_{\widehat{\boldsymbol{\lambda}}_G(Y-C)} = \boldsymbol{\nu}_Y/\boldsymbol{\nu}_{C^0}.$$

Moreover if $G^{\mathrm{ad}}(\mathbb{R})$ is compact, then

$$\widehat{\boldsymbol{\lambda}}_{{}^{\tilde{\boldsymbol{\lambda}}_G(C_1)}G}(C_2 - Y(C_1)_{\tilde{\boldsymbol{\lambda}}_G(C_1)G})\widetilde{\boldsymbol{\lambda}}_G(C_1) = \widetilde{\boldsymbol{\lambda}}_G(C_2)$$

and

$$Y(C_2)_{\hat{\boldsymbol{\lambda}}_{{}^{\tilde{\boldsymbol{\lambda}}_G(C_1)}G}(C_2 - Y(C_1)_{\tilde{\boldsymbol{\lambda}}_G(C_1)G})}({}^{\tilde{\boldsymbol{\lambda}}_G(C_1)}) = Y(C_2)_{\tilde{\boldsymbol{\lambda}}_G(C_2)G}.$$

More concretely choose $\mu \in Y$ and $\mu' \in C$ such that $(\mathrm{conj}_{\mu(-1)} \circ {}^c\mu')\mu'$ is central, and set

$$\widehat{\lambda}_G(\mu - \mu') = \widehat{\lambda}_{{}^{\hat{\lambda}_G(\mu)}G}((\mu')^{-1})\widehat{\lambda}_G(\mu) \in Z^1_{\mathrm{alg}}(W_{E/F}, G(E))_{\mathrm{basic}}$$

and

$$Y(\mu')_{\hat{\lambda}_G(\mu-\mu')G} = [\mu']_{\hat{\lambda}_G(\mu-\mu')G(\mathbb{R})}.$$

We can replace μ' by $\mathrm{conj}_h \circ \mu'$ for any $h \in G(\mathbb{C})$ with $\mathrm{conj}_{\mu(-1)^{-1}}({}^c h) = h$. We can also replace μ by $\mathrm{conj}_g \circ \mu$ with $g \in G(\mathbb{R})$ as long as we replace $\mathrm{conj}_h \circ \mu'$ by $\mathrm{conj}_{gh} \circ \mu'$.

This is the only freedom we have in the choice of μ and μ'. Then

$$\begin{array}{rcl} \widehat{\lambda}_G(\mathrm{conj}_g \circ \mu - \mathrm{conj}_{gh} \circ \mu') & = & \widehat{\lambda}_{{}^{g\widehat{\lambda}_G(\mu)}G}((\mathrm{conj}_{ghg^{-1}} \circ \mathrm{conj}_g \circ \mu')^{-1})^g\widehat{\lambda}_G(\mu) \\ & = & \mathrm{conj}_g(\widehat{\lambda}_{{}^{\widehat{\lambda}_G(\mu)}G}((\mathrm{conj}_h \circ \mu')^{-1}))^g\widehat{\lambda}_G(\mu) \\ & = & \mathrm{conj}_g({}^h\widehat{\lambda}_{{}^{\widehat{\lambda}_G(\mu)}G}((\mu')^{-1}))^g\widehat{\lambda}_G(\mu) \\ & = & {}^{gh}\widehat{\lambda}_G(\mu - \mu'), \end{array}$$

so that

$$\mathrm{conj}_{gh} : ({}^{\widehat{\lambda}_G(\mu-\mu')}G, Y(\mu')_{{}^{\widehat{\lambda}_G(\mu-\mu')}G}) \xrightarrow{\sim} ({}^{\widehat{\lambda}_G(\mathrm{conj}_g \circ\mu - \mathrm{conj}_{gh}\circ\mu')}G, Y(\mathrm{conj}_{gh}\circ\mu')_{{}^{\widehat{\lambda}_G(\mathrm{conj}_g\circ\mu-\mathrm{conj}_{gh}\circ\mu')}G}).$$

4. Kottwitz cohomology: the adelic case

4.1. **The extension $\mathcal{E}^{\mathrm{loc}}(E/F)$.** Suppose that E/F is a finite Galois extension of number fields and that S a set of places of F. We set

$$\mathcal{E}^{\mathrm{loc}}(E/F)^0_S = \prod_{v\in S} E_v^\times = \prod_{w\in S_E} E_w^\times.$$

There is a unique class

$$[\alpha^{\mathrm{loc}}_{E/F,S}] \in H^2(\mathrm{Gal}\,(E/F), \prod_{v\in S} E_v^\times) \cong \prod_{v\in S} H^2(\mathrm{Gal}\,(E/F), E_v^\times) \cong \prod_{v\in S} H^2(\mathrm{Gal}\,(E_w/F_v), E_w^\times)$$

corresponding to $\prod_{v\in S}[\alpha_{E_w/F_v}]$, where for each $v\in S$ we choose a place w of E above v, and where the latter isomorphism arises from Shapiro's lemma. By the functoriality of $[\alpha_{E_w/F_v}]$ under morphisms of fields, this is independent of the choices of $w|v$. If D/E is a finite extension Galois over F then we have

$$\eta^0_{D/E,*}[\alpha^{\mathrm{loc}}_{D/F,S}] = \inf[\alpha^{\mathrm{loc}}_{E/F,S}] \in H^2(\mathrm{Gal}\,(D/F), \prod_{v\in S} D_v^\times),$$

where

$$\begin{array}{rcl} \eta^0_{D/E} : \prod_{v\in S} D_v^\times & \longrightarrow & \prod_{v\in S} D_v^\times \\ (x_v) & \longmapsto & (x_v^{[D_u:E_w]}), \end{array}$$

with $u|w|v$.

If $\alpha\in[\alpha^{\mathrm{loc}}_{E/F,S}]$, we get an extension

$$0 \longrightarrow \prod_{v\in S} E_v^\times \longrightarrow \mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha} \longrightarrow \mathrm{Gal}\,(E/F) \longrightarrow 0.$$

If $\alpha' \in [\alpha^{\mathrm{loc}}_{E/F,S}]$ then the extensions $\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha}$ and $\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha'}$ are isomorphic. However the isomorphism is not unique. Because $H^1(\mathrm{Gal}\,(E/F), \prod_{v\in S} E_v^\times) \cong \prod_{v\in S} H^1(\mathrm{Gal}\,(E_w/F_v), E_w^\times) \cong (0)$, we see that the isomorphism is unique up to conjugation by an element of $\prod_{v\in S} E_v^\times$. If $S=V_F$, the set of all places of F, we will drop it from the notation. If $S=V_F-S'$ we will sometimes replace a lower S by an upper S'. An element $\alpha\in[\alpha^{\mathrm{loc}}_{E/F}]$ gives rise (under the map induced by $\prod_{v\in V_F} E_v\times \twoheadrightarrow \prod_{v\in S} E_v^\times$) to a classe $\alpha_S \in Z^2(\mathrm{Gal}\,(E/F), \prod_{v\in S} E_v^\times)$ representing $[\alpha^{\mathrm{loc}}_{E/F,S}]$; and gives rise (under the map induced by $\mathrm{Gal}\,(E_w/F_v) \hookrightarrow \mathrm{Gal}\,(E/F)$ and $\prod_{v\in V_F} E_v^\times \to E_w^\times$) to a class $\alpha_w \in Z^2(\mathrm{Gal}\,(E_w/F_v), E_w^\times)$ representing $[\alpha_{E_w/F_v}]$. If $S'\subset S$ we get a natural map of extensions

$$\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha_S} \longrightarrow \mathcal{E}^{\mathrm{loc}}(E/F)_{S',\alpha_{S'}}$$

with composite the identity. Moreover if $w|v\in S$ then there is a natural map of extensions

$$\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha_S}|_{\mathrm{Gal}\,(E_w/F_v)} \longrightarrow W_{E_w/F_v,\alpha_w}.$$

It will be useful to us to have a more explicit form for a 2-cocyle defining $\mathcal{E}^{\mathrm{loc}}(E/F)_S$, although this of course depends on a number of choices. So, for each place $v\in S$

fix a place w of E above each v, a 2-cocycle α_w representing $[\alpha_{E_w/F_v}]$, and a section $s_w : \mathrm{Gal}\,(E/F)/\mathrm{Gal}\,(E_w/F_v) \to \mathrm{Gal}\,(E/F)$ with $s_w(1) = 1$. Then

$$\alpha(\sigma_1, \sigma_2) = \prod_{v \in S} \prod_{\eta \in \mathrm{Gal}\,(E/F)/\mathrm{Gal}\,(E_w/F_v)} s_w(\eta)\alpha_w(s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))$$

is a representative of $[\alpha^{\mathrm{loc}}_{E/F,S}]$. (If one restricts the class to $\mathrm{Gal}\,(E_w/F_v)$ and projects to $E_w^\times$ one certainly recovers α_w, so the only thing to check is that $\alpha(\sigma_1, \sigma_2)$ is a 2-cocycle. Writing out the cocycle relation and changing the variable from η to $\sigma_1\eta$ in one of the terms, what we need to check is that

$$\begin{array}{rl} & (s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta))\alpha_w(s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta), s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)^{-1}\sigma_3 s_w(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\eta)) \\ & \alpha_w(s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2\sigma_3 s_w(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\eta)) \\ = & \alpha_w(s_w(\eta)^{-1}\sigma_1\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta), s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)^{-1}\sigma_3 s_w(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\eta)) \\ & \alpha_w(s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)), \end{array}$$

which is just the cocycle relation for α_w.)

We will be interested in algebraicity conditions

$$\mathcal{N}_S = \{(\nu_w)_{w \in S_E} :\ \nu_w \in X_*(G)(E_w) \text{ and } \nu_w = 1 \text{ for all but finitely many } w\}$$

and

$$\mathcal{N}_{S,\mathrm{basic}} = \{(\nu_w) \in \mathcal{N}_S :\ \nu_w \text{ factors through } Z(G)\ \forall w \in S_E\}.$$

If $\alpha \in [\alpha^{\mathrm{loc}}_{E/F,S}]$, we will write $Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha}, G(\mathbb{A}_{E,S}))$ and $Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha}, G(\mathbb{A}_{E,S}))_{\mathrm{basic}}$ and $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_S, G(\mathbb{A}_{E,S}))$ and $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_S, G(\mathbb{A}_{E,S}))_{\mathrm{basic}}$ for the corresponding pointed sets of cocycles and cohomology classes. Note that the cohomology sets are canonically independent of the choice of α.

If $S' \supset S$ and $\alpha \in [\alpha^{\mathrm{loc}}_{E/F,S'}]$, then there are natural maps

$$Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha_S}, G(\mathbb{A}_{E,S})) \longrightarrow Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S',\alpha}, G(\mathbb{A}_{E,S'})) \stackrel{\mathrm{res}_S}{\longrightarrow} Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha_S}, G(\mathbb{A}_{E,S}))$$

and

$$H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha_S}, G(\mathbb{A}_{E,S})) \longrightarrow H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S',\alpha}, G(\mathbb{A}_{E,S'})) \stackrel{\mathrm{res}_S}{\longrightarrow} H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha_S}, G(\mathbb{A}_{E,S}))$$

which preserves the basic subsets, and with composite the identity. These give an isomorphisms

$$Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha}, G(\mathbb{A}_{E,S})) \stackrel{\sim}{\longrightarrow} {\prod_{v \in S}}' Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{\{v\},\alpha_{\{v\}}}, G(E_v))$$

and

$$H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_S, G(\mathbb{A}_{E,S})) \stackrel{\sim}{\longrightarrow} {\prod_{v \in S}}' H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{\{v\}}, G(E_v))$$

where the product is restricted with respect to the subsets (defined for almost all v) $Z^1(\mathrm{Gal}\,(E/F), G(\mathcal{O}_{E,v}))$ and $H^1(\mathrm{Gal}\,(E/F), G(\mathcal{O}_{E,v}))$ respectively. If G is connected, the right hand side of the second of these isomorphisms becomes simply $\oplus_{v \in S} H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{\{v\},\alpha_{\{v\}}}, G(E_v))$. (See the corollary to theorem 6.8 of [PR].)

If $w|v \in S$ is a place of E, then we get a map

$$\begin{array}{rcl} \mathrm{res}_w : Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha_S}, G(E_v)) & \longrightarrow & Z^1_{X_*(G)(E_w)}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha_S}|_{\mathrm{Gal}\,(E_w/F_v)}, G(E_w)) \\ & \longrightarrow & Z^1_{\mathrm{alg}}(W_{E_w/F_v,\alpha_w}, G(E_w)) \end{array}$$

as in D. This induces

$$\mathrm{res}_w : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_S, G(\mathbb{A}_{E,S}) \longrightarrow H^1_{\mathrm{alg}}(W_{E_w/F_v}, G(E_w)).$$

Both these maps preserve basic subsets. It follows from lemma 3.1 that if $w|v$ then

$$\mathrm{res}_w : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{\{v\}}, G(E_v) \xrightarrow{\sim} H^1_{\mathrm{alg}}(W_{E_w/F_v}, G(E_w)),$$

and similarly for basic subsets. Note that $\mathrm{res}_{\sigma w} = \sigma_* \circ \mathrm{res}_w$. We deduce that if E^0_v/F_v is a finite extension abstractly isomorphic to E_w/F_v for any, and hence all, $w|v \in S$, then we obtain a natural map

$$\mathrm{res}_{E^0_v} : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_S, G(E_v)) \longrightarrow H^1_{\mathrm{alg}}(W_{E^0_v/F_v}, G(E^0_v)),$$

defined as $\tau_* \circ \mathrm{res}_w$ for any $w|v$ and any $\tau : E_w \xrightarrow{\sim} E^0_v$. (The point being that $\mathrm{res}_{E^0_w}$ does not depend on the choice of w or τ.) This preserves basic subsets and is an isomorphism if $S = \{v\}$. If G is connected, then

$$H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_S, G(\mathbb{A}_{E,S})) \xrightarrow{\sim} \bigoplus_{v \in S} H^1_{\mathrm{alg}}(W_{E^0_v/F_v}, G(E^0_v)).$$

Suppose that $D \supset E$ is also a finite Galois extension of F. Choose representatives $\alpha_E \in [\alpha^{\mathrm{loc}}_{E/F,S}]$ and $\alpha_D \in [\alpha^{\mathrm{loc}}_{D/F,S}]$. Write $\mathcal{E}^{\mathrm{loc}}(E/F)|_{\mathrm{Gal}\,(D/E)}$ for the pull back of $\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha_{E/F}}$ along $\mathrm{Gal}\,(D/F) \twoheadrightarrow \mathrm{Gal}\,(E/F)$, and $\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha_E,D}$ for the push out of $\mathcal{E}^{\mathrm{loc}}(E/F)|_{\mathrm{Gal}\,(D/E)}$ along $\prod_{v\in S} E_v^\times \longrightarrow \prod_{v \in S} D_v^\times$. Define

$$\begin{array}{rcl} \eta^0_{D/E,S} : \prod_{v\in S} D_v^\times & \longrightarrow & \prod_{v\in S} D_v^\times \\ (x_v) & \longmapsto & (x_v^{[D_u:E_w]}) \end{array}$$

where $u|w|v$. We can choose $\gamma : \mathrm{Gal}\,(D/F) \to \prod_{v\in S} D_v^\times$ such that $\eta^0_{D/E,S}\alpha_{D/F} = {}^\gamma\alpha_{E/F} \in Z^2(\mathrm{Gal}\,(D/F), \prod_{v\in S} D_v^\times)$ which gives us a map of extensions

$$\eta_{D/E,S,\gamma} : \mathcal{E}^{\mathrm{loc}}(D/F)_{S,\alpha_D} \longrightarrow \mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha_E,D}$$

extending $\eta^0_{D/E,S}$. As

$$H^1(\mathrm{Gal}\,(D/F), \prod_{v\in S} D_v^\times) \cong \prod_{v \in S} H^1(\mathrm{Gal}\,(D_u/F_v), D_u^\times) = (0)$$

we may only replace γ by ${}^d\gamma$ for some $d \in \prod_{v\in S} D_v^\times$. Thus, if we vary γ, then $\eta_{D/E,S,\gamma}$ varies by post-composition with conjugation by an element of $\prod_{v\in S} D_v^\times$. We have a commutative diagram

$$\begin{array}{ccccccccc}
(0) & \longrightarrow & \prod_{v\in S} D_v^\times & \longrightarrow & \mathcal{E}^{\rm loc}(D/F)_{S,\alpha_D} & \longrightarrow & {\rm Gal}\,(D/F) & \longrightarrow & (0) \\
 & & \eta^0_{D/E}\downarrow & & \eta_{D/E,S,\gamma}\downarrow & & || & & \\
(0) & \longrightarrow & \prod_{v\in S} D_v^\times & \longrightarrow & \mathcal{E}^{\rm loc}(E/F)_{S,\alpha_E,D} & \longrightarrow & {\rm Gal}\,(D/F) & \longrightarrow & (0) \\
 & & \uparrow & & \uparrow & & || & & \\
(0) & \longrightarrow & \prod_{v\in S} E_v^\times & \longrightarrow & \mathcal{E}^{\rm loc}(E/F)|_{{\rm Gal}\,(D/E)} & \longrightarrow & {\rm Gal}\,(D/F) & \longrightarrow & (0) \\
 & & || & & \downarrow & & \downarrow & & \\
(0) & \longrightarrow & \prod_{v\in S} E_v^\times & \longrightarrow & \mathcal{E}^{\rm loc}(E/F)_{S,\alpha_E} & \longrightarrow & {\rm Gal}\,(E/F) & \longrightarrow & (0).
\end{array}$$

Using successively functorialities (B) then (C) then (B) from the end of section 3.1 we obtain a map Then there is an injective map

$$\inf_{D/E,S,\gamma} : Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/F)_{S,\alpha_E}, G(\mathbb{A}_{E,S})) \longrightarrow Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(D/F)_{S,\alpha_D}, G(\mathbb{A}_{D,S}))$$

which sends $(\{\nu_w\}_w,\phi)$ to $(\{\nu_{u|_E}^{[D_u:E_{u|_E}]}\}_u,\phi')$, where, if $\eta_{D/E,\gamma}(e) = de'$ with $d \in \prod_{w|v\in S} D_w^\times$ and $e' \in \mathcal{E}'$ then

$$\phi'(e) = \prod_w \nu_w(d_w)\phi(\overline{e}')$$

where $\overline{e}'$ denotes the image of e' in $\mathcal{E}^{\rm loc}(E/F)_{S,\alpha_E}$. This map takes basic elements to basic elements and induces maps in cohomology. The map

$$\inf_{D/E,S,\gamma} : H^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/F)_S, G(\mathbb{A}_{E,S})) \longrightarrow H^1_{\rm alg}(\mathcal{E}^{\rm loc}(D/F)_S, G(\mathbb{A}_{D,S}))$$

is independent of the choice of α_E, α_D and γ, so we will denote it simply $\inf_{D/E,S}$. As in the local case, this map is injective. We have a commutative diagram

$$\begin{array}{ccccc}
Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/F)_{S',\alpha_{E,S'}}, G(\mathbb{A}_{E,S'})) & \longrightarrow & Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/F)_{S,\alpha_E}, G(\mathbb{A}_{E,S})) & \longrightarrow & Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/F)_{S',\alpha_{E,S'}}, G(\mathbb{A}_{E,S'})) \\
\downarrow & & \downarrow & & \downarrow \\
Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(D/F)_{S',\alpha_{D,S'}}, G(\mathbb{A}_{D,S'})) & \longrightarrow & Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(D/F)_{S,\alpha_D}, G(\mathbb{A}_{D,S})) & \longrightarrow & Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(D/F)_{S',\alpha_{D,S'}}, G(\mathbb{A}_{D,S'}))
\end{array}$$

where we use $\gamma_{S'}$ induced from γ via projection. We also have a commutative diagram

$$\begin{array}{ccc}
Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/F)_{\{v\},\alpha_E}, G(E_v)) & \longrightarrow & Z^1_{\rm alg}(W_{E_w/F_v,\alpha_{E_w}}, G(E_v^0)) \\
\downarrow & & \downarrow \\
Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(D/F)_{\{v\},\alpha_D}, G(D_v)) & \longrightarrow & Z^1_{\rm alg}(W_{D_u/F_v,\alpha_{D_u}}, G(D_v^0))
\end{array}$$

where $u|w|v$ and where the vertical maps are defined by γ and its restriction γ_w.

We define

$$B^{\rm loc}(F,G)_S = \lim_{\to E} H^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/F)_S, G(\mathbb{A}_{E,S}))$$

and

$$B^{\rm loc}(F,G)_{S,{\rm basic}} = \lim_{\to E} H^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/F)_S, G(\mathbb{A}_{E,S}))_{\rm basic}$$

If $S' \supset S$ we obtain maps

$$B^{\rm loc}(F,G)_S \longrightarrow B^{\rm loc}(F,G)_{S'} \longrightarrow B^{\rm loc}(F,G)_S$$

with composite the identity, and which preserve basic elements. Moreover restriction gives a well defined isomorphism

$$B^{\mathrm{loc}}(F,G)_{\{v\}} \xrightarrow{\sim} B(F_v,G).$$

Note that

$$B^{\mathrm{loc}}(F,G)_S \xrightarrow{\sim} \prod_{v\in S}{}' B^{\mathrm{loc}}(F_v,G)_v,$$

where the product is restricted with respect to $\{H^1(\mathrm{Gal}\,(F_v^{\mathrm{nr}}/F_v), G(\mathcal{O}_{F_v^{\mathrm{nr}}}))\}$, where F_v^{nr} denotes the maximal unramified extension of F_v. If G is connected, then

$$B^{\mathrm{loc}}(F,G)_S \xrightarrow{\sim} \bigoplus_{v\in S} B^{\mathrm{loc}}(F_v,G)_v.$$

Suppose that G is reductive. Suppose also that

- G contains a maximal torus (defined over F) which splits over E,
- and E is totally complex.

In this case we see that

$$H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_S, G(\mathbb{A}_{E,S})) \xrightarrow{\sim} B^{\mathrm{loc}}(F,G)_S.$$

Moreover, if E_v^0/F_v is an extension abstractly isomorphic to E_w/F_v for any $w|v$, we get a map

$$\kappa_{G,v} : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_S, G(\mathbb{A}_E)_S) \longrightarrow H^1_{\mathrm{alg}}(W_{E_v^0/F_v}, G(E_v^0)) \xrightarrow{\kappa_G} \Lambda_{G,\mathrm{Gal}\,(E_v^0/F_v)}.$$

Thus we get a well defined map

$$\kappa_G : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_S, G(\mathbb{A}_{E,S})) \longrightarrow \bigoplus_{v\in S} \Lambda_{G,\mathrm{Gal}\,(E_v^0/F_v)} = (\bigoplus_{w\in S_E} \Lambda_G)_{\mathrm{Gal}\,(E/F)}.$$

We also define

$$\begin{array}{rcl} \overline{\kappa}_G : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F), G(\mathbb{A}_E)) & \longrightarrow & \Lambda_{G,\mathrm{Gal}\,(E/F)} \\ x & \longmapsto & \sum_w \kappa_G(x)_w. \end{array}$$

We have the following technical lemma.

Lemma 4.1. *Suppose that T/F is a torus split by a finite Galois extension E/F, and that for all $w\in V_E$ we have a character $\chi_w:\mathbb{G}_m\to T$ with $\chi_w=1$ for all but finitely many w. Then $\chi=\prod_w \chi_w : \prod_w E_w^\times \to T(\mathbb{A}_E)$. Suppose that $\alpha\in[\alpha^{\mathrm{loc}}_{E/F}]$ gives rise to $\alpha_w\in[\alpha_{E_w/F_v}]$. Now fix $w|v$ and let H denote a set of representatives for* $\mathrm{Gal}\,(E/F)/\mathrm{Gal}\,(E_w/F_v)$. *Then*

$$\mathrm{res}_w \mathrm{cor}_{\,\alpha}\chi = {}^{y^{-1}}\mathrm{cor}_{\,\alpha_w} \prod_{\eta\in H} ({}^{\eta^{-1}}\chi_{\eta w})$$

where

$$y = \prod_{\eta\in H} \prod_{\xi\in\mathrm{Gal}\,(E_w/F_v)} \xi^{-1}\eta^{-1}\chi_{\eta w}(\alpha(\eta,\xi)_{\eta w}).$$

In particular

$$\mathrm{res}_w[\mathrm{cor}\,\chi] = [\mathrm{cor}\prod_{\eta\in H}({}^{\eta^{-1}}\chi_{\eta w})].$$

Proof: If $x \in E_w^\times$, then

$$\begin{array}{rcl} (\mathrm{cor}\,_{\alpha}\chi)(x)_w &=& \prod_{\eta\in \mathrm{Gal}\,(E/F)}({}^{\eta}\chi)(x)_w \\ &=& \prod_{\eta\in \mathrm{Gal}\,(E/F)}({}^{\eta}\chi_{\eta^{-1}w})(x) \\ &=& (\mathrm{cor}\,_{\alpha_{w(v)/v}}\prod_{\eta\in H}{}^{\eta}\chi_{\eta^{-1}w})(x). \end{array}$$

Moreover, if $\tau \in \mathrm{Gal}\,(E_w/F_v)$ then

$$\begin{array}{rl} & (\mathrm{cor}\,_{\alpha}\chi)(e(\tau))_w \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)}\eta^{-1}\chi(\alpha(\eta,\tau))_{\eta w} \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)}\eta^{-1}\chi_{\eta w}(\alpha(\eta,\tau)_{\eta w}) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)/\mathrm{Gal}\,(E_w/F_v)}\prod_{\xi\in \mathrm{Gal}\,(E_w/F_v)}\xi^{-1}\eta^{-1}\chi_{\eta w}(\alpha(\eta\xi,\tau)_{\eta w}) \\ = & \prod_{\eta\in H}\prod_{\xi\in \mathrm{Gal}\,(E_w/F_v)}\xi^{-1}\eta^{-1}\chi_{\eta w}((\eta\alpha(\xi,\tau)_w)\alpha(\eta,\xi\tau)_{\eta w}/\alpha(\eta,\xi)_{\eta w}) \\ = & \prod_{\eta\in H}\prod_{\xi\in \mathrm{Gal}\,(E_w/F_v)}\xi^{-1}\eta^{-1}\chi_{\eta w}(\eta\alpha(\xi,\tau)_w) \\ & \prod_{\xi\in \mathrm{Gal}\,(E_w/F_v)}\xi^{-1}\eta^{-1}\chi_{\eta w}(\alpha(\eta,\xi\tau)_{\eta w})/\prod_{\xi\in \mathrm{Gal}\,(E_w/F_v)}\xi^{-1}\eta^{-1}\chi_{\eta w}(\alpha(\eta,\xi)_{\eta w}) \\ = & \prod_{\eta\in H}\prod_{\xi\in \mathrm{Gal}\,(E_w/F_v)}\xi^{-1}({}^{\eta^{-1}}\chi_{\eta w})(\alpha(\xi,\tau)_w) \\ & \prod_{\xi\in \mathrm{Gal}\,(E_w/F_v)}\tau\xi^{-1}\eta^{-1}\chi_{\eta w}(\alpha(\eta,\xi)_{\eta w})/\prod_{\xi\in \mathrm{Gal}\,(E_w/F_v)}\xi^{-1}\eta^{-1}\chi_{\eta w}(\alpha(\eta,\xi)_{\eta w}) \\ = & (\mathrm{cor}\,_{\alpha_w}\prod_{\eta\in H}({}^{\eta^{-1}}\chi_{\eta w}))(e(\tau)) \\ & {}^{(\tau-1)}\prod_{\eta\in H)}\prod_{\xi\in \mathrm{Gal}\,(E_w/F_v)}\xi^{-1}\eta^{-1}\chi_{\eta w}(\alpha(\eta,\xi)_{\eta w}). \end{array}$$

The lemma follows. □

4.2. **Some pro-tori.** Suppose that E/F is a Galois extension of algebraic extensions of $\mathbb{Q}$ and that S is a set of places of F. We define pro-tori $T_{2,E,S}$ and $T_{3,E,S}$ over F by specifying their character groups to be

$$X^*(T_{2,E,S}) = \mathbb{Z}[V_{E,S}] \quad \text{and} \quad X^*(T_{2,E,S}) = \mathbb{Z}[V_{E,S}]_0.$$

If $S = V_F$ we will write simply $T_{2,E}$ and $T_{3,E}$. The short exact sequence

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathbb{Z}[V_{E,S}]_0 & \longrightarrow & \mathbb{Z}[V_{E,S}] & \longrightarrow & \mathbb{Z} & \longrightarrow & (0) \\ &&&& \sum_w m_w w & \longmapsto & \sum_w m_w &&\end{array}$$

gives an exact sequence

$$(0) \longrightarrow \mathbb{G}_m \longrightarrow T_{2,E,S} \longrightarrow T_{3,E,S} \longrightarrow (0).$$

For each place $w \in V_{E,S}$ there are natural maps (defined over E) $\iota_w : \mathbb{G}_m \to T_{2,E,S}$ and $\pi_w : T_{2,E,S} \to \mathbb{G}_m$ such that $\pi_w\iota_w = \mathrm{Id}$. ($X^*(\iota_w)(\sum m_v v) = m_w$ and $X^*(\pi_w)(m) = mw$.) For any F-algebra A, this gives an identification

$$T_{2,E,S}(A_E) \stackrel{\sim}{\longrightarrow} \prod_{w\in S_E} A_E^\times$$

where $A_E = A \otimes_F E$, but with the twisted Galois action

$$\sigma(x_w)_{w\in S_E} = ((1\otimes\sigma)x_{\sigma^{-1}w})_{w\in S_E}.$$

If $S' \supset S$ there are natural maps

$$T_{2,E,S} \hookrightarrow T_{2,E,S'} \twoheadrightarrow T_{2,E,S}$$

with composite the identity.

If $D \supset E$ is an extension, Galois over F, then the map

$$\begin{array}{ccc} \mathbb{Z}[V_{D,S}] & \longrightarrow & \mathbb{Z}[V_{E,S}] \\ \sum_u m_u u & \longmapsto & \sum_u m_u u|_E \end{array}$$

gives rise to a commutative diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathbb{G}_m & \longrightarrow & T_{2,E,S} & \longrightarrow & T_{3,E,S} & \longrightarrow & (0) \\ & & || & & \iota^0_{D/E}\downarrow & & \downarrow \iota^0_{D/E} & & \\ (0) & \longrightarrow & \mathbb{G}_m & \longrightarrow & T_{2,D,S} & \longrightarrow & T_{3,D,S} & \longrightarrow & (0). \end{array}$$

If further D/E is finite, then the map

$$\begin{array}{ccc} \mathbb{Z}[V_{E,S}] & \longrightarrow & \mathbb{Z}[V_{D,S}] \\ \sum_w m_w w & \longmapsto & \sum_u [D_u : E_{u|_E}] m_{u|_E} u \end{array}$$

gives rise to a commutative diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathbb{G}_m & \longrightarrow & T_{2,D,S} & \longrightarrow & T_{3,D,S} & \longrightarrow & (0) \\ & & [D:E]\downarrow & & \eta^0_{D/E}\downarrow & & \downarrow \eta^0_{D/E} & & \\ (0) & \longrightarrow & \mathbb{G}_m & \longrightarrow & T_{2,E,S} & \longrightarrow & T_{3,E,S} & \longrightarrow & (0), \end{array}$$

and

$$\eta^0_{D/E} \circ \iota^0_{D/E} = [D:E].$$

There is a natural $\mathrm{Gal}\,(E/F)$-equivariant map

$$\iota : \prod_{w\in V_E} E_w^\times \hookrightarrow T_{2,E}(\mathbb{A}_E)$$

which maps (a_w) to (a_w) where we think of $E_w^\times \subset \mathbb{A}_E^\times$ in the w-component.

Lemma 4.2. *Suppose that $D \supset E \supset F$ are number fields with D and E Galois over F. For each place v of F choose places $u(v)|w(v)|v$ of D and E respectively.*

(1) $H^i(\mathrm{Gal}\,(D/F), T_{2,E}(D)) \cong \prod_{v\in V_F} H^i(\mathrm{Gal}\,(D/F)_{w(v)}, D^\times)$ *and* $H^1(\mathrm{Gal}\,(D/F), T_{2,E}(D)) = (0)$.

(2) $H^i(\mathrm{Gal}\,(D/F), T_{2,E}(\mathbb{A}_D)) \cong \prod_{v\in V_F} H^i(\mathrm{Gal}\,(D/F)_{w(v)}, \mathbb{A}_D^\times)$ *and* $H^1(\mathrm{Gal}\,(D/F), T_{2,E}(\mathbb{A}_D)) = (0)$.

(3) $H^i(\mathrm{Gal}\,(D/F), T_{2,E}(\mathbb{A}_D)/T_{2,E}(D)) \cong \prod_{v\in V_F} H^i(\mathrm{Gal}\,(D/F)_{w(v)}, \mathbb{A}_D^\times/D^\times)$ *and* $H^1(\mathrm{Gal}\,(D/F), T_{2,E}(\mathbb{A}_D)/T_{2,E}(D)) = (0)$.

(4) $H^1(\mathrm{Gal}\,(D/F), T_{3,E}(D)) = (0)$.

Proof: The first part is an application of Shapiro's lemma

$$H^i(\mathrm{Gal}\,(D/F), T_{2,E}(A_D)) \cong \prod_{v\in V_F} H^i(\mathrm{Gal}\,(D/F), \prod_{w\in\{v\}_E} A_D^\times) \cong \prod_{v\in V_F} H^i(\mathrm{Gal}\,(D/F)_{w(v)}, A_D^\times)$$

combined with Hilbert's theorem 90.

The second and third parts are proved similarly using the vanishing of $H^1(\mathrm{Gal}\,(D/E)_{w(v)}, \mathbb{A}_D^\times)$ and $H^1(\mathrm{Gal}\,(D/E)_{w(v)}, \mathbb{A}_D^\times/D^\times)$.

Consider the fourth part. As D splits $T_{2,E}$ there is an exact sequence

$$(0) \longrightarrow D^\times \longrightarrow T_{2,E}(D) \longrightarrow T_{3,E}(D) \longrightarrow (0),$$

and so it suffices to show that

$$H^2(\mathrm{Gal}\,(D/F), D^\times) \longrightarrow H^2(\mathrm{Gal}\,(D/F), T_{2,E}(D)) \cong \prod_{v\in V_F} H^2(\mathrm{Gal}\,(D/F)_{w(v)}, D^\times)$$

is injective. In fact it suffices to show that the composite with the map

$$\prod_{v\in V_F} H^2(\mathrm{Gal}\,(D/F)_{w(v)}, D^\times) \longrightarrow \prod_{v\in V_F} H^2(\mathrm{Gal}\,(D/F)_{u(v)}, D_{u(w)}^\times)$$

is injective. However this injectivity follows from the fact that the Brauer group of F (of which $H^2(\mathrm{Gal}\,(D/F), D^\times)$ is a subgroup) injects into the product of the Brauer groups of all completions of F. □

4.3. **The extension $\mathcal{E}_2(E/F)$.** If $\alpha \in [\alpha_{E/F}^{\mathrm{loc}}]$, then Kottwitz defines an extension

$$0 \longrightarrow T_{2,E}(\mathbb{A}_E) \longrightarrow \mathcal{E}_2(E/F)_\alpha \longrightarrow \mathrm{Gal}\,(E/F) \longrightarrow 0$$

by pushing out $\mathcal{E}^{\mathrm{loc}}(E/F)$ along

$$\iota : \prod_{w\in V_E} E_w^\times \hookrightarrow T_{2,E}(\mathbb{A}_E).$$

If $\alpha' \in [\alpha_{E/F}^{\mathrm{loc}}]$, then $\mathcal{E}_2(E/F)_\alpha$ and $\mathcal{E}_2(E/F)_{\alpha'}$ are isomorphic extensions, but the isomorphism is only unique up to conjugation by an element of $\prod_{w\in V_E} E_w^\times \subset T_{2,E}(\mathbb{A}_E)$.

We will consider the following algebraicity conditions for the cohomology of $\mathcal{E}_2(E/F)$:

$$\mathcal{N} = \{\nu \in \mathrm{Hom}\,(T_{2,E}, G)(\mathbb{A}_E) : \ \nu \text{ is } G(\mathbb{A}_E)\text{−conjugate to an element of } \mathrm{Hom}\,(T_{2,E}, G)(E)\}$$

and

$$\mathcal{N}_{\mathrm{basic}} = \mathrm{Hom}\,(T_{2,E}, Z(G))(E).$$

We denoted the corresponding pointed sets of cocycles and cohomology classes $Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_\alpha, G(\mathbb{A}_E))$ and $Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_\alpha, G(\mathbb{A}_E))_{\mathrm{basic}}$ and $H^1_{\mathrm{alg}}(\mathcal{E}_2(E/F), G(\mathbb{A}_E))$ and $H^1_{\mathrm{alg}}(\mathcal{E}_2(E/F), G(\mathbb{A}_E))_{\mathrm{basic}}$. The latter two are canonically independent of the choice of $\alpha \in [\alpha_{E/F}^{\mathrm{loc}}]$. We get maps

$$\mathrm{res}_S : Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_\alpha, G(\mathbb{A}_E)) \longrightarrow Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha_S}, G(\mathbb{A}_{E,S}))$$

and

$$\mathrm{res}_w : Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_\alpha, G(\mathbb{A}_E)) \to Z^1_{\mathrm{alg}}(W_{E_w/F_v,\alpha_w}, G(E_w)).$$

These maps respect basic elements and pass to cohomology. If E_v^0/F_v is abstractly isomorphic to E_w/F_v for any $w|v$, then we also get a well defined map

$$\mathrm{res}_{E_v^0} : H^1_{\mathrm{alg}}(\mathcal{E}_2(E/F), G(\mathbb{A}_E)) \to H^1_{\mathrm{alg}}(W_{E_v^0/F_v}, G(E_v^0)).$$

Suppose that $D \supset E$ is also a finite Galois extension of F. We write $\mathcal{E}_2(E/F)_\alpha|_{\mathrm{Gal}\,(D/F)}$ for the pull back of $\mathcal{E}_2(E/F)_\alpha$ along $\mathrm{Gal}\,(D/F) \twoheadrightarrow \mathrm{Gal}\,(E/F)$ and $\mathcal{E}_2(E/F)_{\alpha,D}$ for the push out of $\mathcal{E}_2(E/F)_\alpha|_{\mathrm{Gal}\,(D/F)}$ along $T_{2,E}(\mathbb{A}_E) \longrightarrow T_{2,E}(\mathbb{A}_D)$. If $\alpha_D \in [\alpha^{\mathrm{loc}}_{D/F}]$, then we can find $\gamma : \mathrm{Gal}\,(D/F) \to \prod_v D_v^\times$ such that $\eta^0_{D/E}\alpha_D = {}^\gamma\alpha \in Z^2(\mathrm{Gal}\,(D/F), \prod_v D_v^\times)$. The choice of γ gives rise to a map of extensions $\eta_{D/E,\gamma} : \mathcal{E}_2(D/E)_{\alpha_D} \to \mathcal{E}_2(E/F)_{\alpha,D}$ extending $\eta^0_{D/E}$. If we make a different choice of γ then $\eta_{D/E,\gamma}$ changes by post composition with conjugation by an element of $\prod_v D_v^\times$. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
(0) & \longrightarrow & T_{2,D}(\mathbb{A}_D) & \longrightarrow & \mathcal{E}_2(D/F)_{\alpha_D} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \\
 & & \eta^0_{D/E}\downarrow & & \eta_{D/E,\gamma}\downarrow & & || & & \\
(0) & \longrightarrow & T_{2,E}(\mathbb{A}_D) & \longrightarrow & \mathcal{E}_2(E/F)_{\alpha,D} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \\
 & & \uparrow & & \uparrow & & || & & \\
(0) & \longrightarrow & T_{2,E}(\mathbb{A}_E) & \longrightarrow & \mathcal{E}_2(E/F)_\alpha|_{\mathrm{Gal}\,(D/F)} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \\
 & & || & & \downarrow & & \downarrow & & \\
(0) & \longrightarrow & T_{2,E}(\mathbb{A}_E) & \longrightarrow & \mathcal{E}_2(E/F)_\alpha & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & (0).
\end{array}$$

The map $\eta_{E/F}$ is determined up to composition with conjugation by an element of $\prod_{w\in S_E} D_w^\times$. Making use successively of functorialities (B) then (C) then (B) again from the end of section 3.1, we obtain a map

$$\inf_{D/E,\gamma} : Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_\alpha, G(\mathbb{A}_E)) \longrightarrow Z^1_{\mathrm{alg}}(\mathcal{E}_2(D/F)_{\alpha_D}, G(\mathbb{A}_D))$$

which sends (ν, ϕ) to $(\nu \circ \eta^0_{D/E}, \phi')$, where, if $\eta_{D/E,\gamma}(e) = de'$ with $d \in T_{2,E}(\mathbb{A}_D)$ and $e' \in \mathcal{E}'$ then

$$\phi'(e) = \nu(d)x(\overline{e}')$$

where $\overline{e}'$ denotes the image of e' in $\mathcal{E}_2(E/F)_{\alpha_E}$. This map takes basic elements to basic elements and induces maps in cohomology. The latter is independent of the choice of α, α_D and γ, so we write simply

$$\inf_{D/E} : H^1_{\mathrm{alg}}(\mathcal{E}_2(E/F), G(\mathbb{A}_E)) \longrightarrow H^1_{\mathrm{alg}}(\mathcal{E}_2(D/F), G(\mathbb{A}_D))$$

We have commutative diagrams

$$\begin{array}{ccc}
Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_\alpha, G(\mathbb{A}_E)) & \stackrel{\inf_{D/E,\gamma}}{\longrightarrow} & Z^1_{\mathrm{alg}}(\mathcal{E}_2(D/F)_{\alpha_D}, G(\mathbb{A}_D)) \\
\mathrm{res}_S \downarrow & & \downarrow \mathrm{res}_S \\
Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\alpha}, G(\mathbb{A}_{E,S})) & \stackrel{\inf_{D/E,\gamma}}{\longrightarrow} & Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(D/F)_{S,\alpha_D}, G(\mathbb{A}_{D,S}))
\end{array}$$

and, if $u|w|v$,

$$\begin{array}{ccc} Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_\alpha, G(\mathbb{A}_E)) & \stackrel{\mathrm{inf}_{D/E,\gamma}}{\longrightarrow} & Z^1_{\mathrm{alg}}(\mathcal{E}_2(D/F)_{\alpha_D}, G(\mathbb{A}_D)) \\ \mathrm{res}_w \downarrow & & \downarrow \mathrm{res}_u \\ Z^1_{\mathrm{alg}}(W_{E_w/F_v,\alpha_w}, G(E_w)) & \stackrel{\mathrm{inf}_{D/E,\gamma}}{\longrightarrow} & Z^1_{\mathrm{alg}}(W_{D_u/F_v,\alpha_u}, G(D_u)). \end{array}$$

If $D_v^0/E_v^0/F_v$ is a tower of extensions abstractly isomorphic to $D_u/E_w/F_v$ for any $u|w|v$, then

$$\begin{array}{ccc} H^1_{\mathrm{alg}}(\mathcal{E}_2(E/F), G(\mathbb{A}_E)) & \stackrel{\mathrm{inf}_{D/E}}{\longrightarrow} & H^1_{\mathrm{alg}}(\mathcal{E}_2(D/F), G(\mathbb{A}_D)) \\ \mathrm{res}_{E_v^0} \downarrow & & \downarrow \mathrm{res}_{D_u^0} \\ H^1_{\mathrm{alg}}(W_{E_v^0/F_v}, G(E_v^0)) & \stackrel{\mathrm{inf}_{D/E}}{\longrightarrow} & H^1_{\mathrm{alg}}(W_{D_v^0/F_v}, G(D_v^0)) \end{array}$$

also commutes.

We define

$$B(\mathbb{A}_F, G) = \lim_{\to E} H^1_{\mathrm{alg}}(\mathcal{E}_2(E/F), G(\mathbb{A}_E))$$

and

$$B(\mathbb{A}_F, G)_{\mathrm{basic}} = \lim_{\to E} H^1_{\mathrm{alg}}(\mathcal{E}_2(E/F), G(\mathbb{A}_E))_{\mathrm{basic}}.$$

We obtain maps (independent of choices)

$$\mathrm{res}_S : B(\mathbb{A}_F, G) \longrightarrow B^{\mathrm{loc}}(F, G)_S,$$

and

$$\mathrm{res}_v : B(\mathbb{A}_F, G) \longrightarrow B^{\mathrm{loc}}(F_v, G)$$

which preserve basic elements.

5. Kottwitz cohomology: the global case

5.1. **The extensions $\mathcal{E}^{\rm glob}(E/F)$ and $\mathcal{E}_3(E/F)$.** If E/F is a finite Galois extension of number fields and if $\alpha \in [\alpha^W_{E/F}] \subset Z^2(\mathrm{Gal}\,(E/F), \mathbb{A}_E^\times/E^\times)$, then we obtain a well-defined extension:

$$(0) \longrightarrow \mathbb{A}_E^\times/E^\times \longrightarrow W_{E/F,\alpha} \longrightarrow \mathrm{Gal}\,(E/F) \longrightarrow (0).$$

As $H^1(\mathrm{Gal}\,(E/F), \mathbb{A}_E^\times/E^\times) = (0)$ the only automorphisms of this extension are conjugation by elements of $\mathbb{A}_E^\times/E^\times$. If α' is a second element of $[\alpha_{E/F}]$, then there is an isomorphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{A}_E^\times/E^\times & \longrightarrow & W_{E/F,\alpha} & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & 0 \\ && || && \downarrow \wr && || && \\ 0 & \longrightarrow & \mathbb{A}_E^\times/E^\times & \longrightarrow & W_{E/F,\alpha'} & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & 0. \end{array}$$

However it is only unique up to composition with conjugation by an element of $\mathbb{A}_E^\times/E^\times$. In particular, if $\overline{F}$ is an algebraic closure of F containing E, then the extension $W_{E/F,\alpha}$ is isomorphic to $W_{E^{\rm ab}/F}$, but this isomorphism is only unique up to composition with conjugation by an element of $\mathbb{A}_E^\times/E^\times$.

Define

$$\begin{array}{rcl} \mathcal{E}^{\rm glob}(E/F)^0 & = & T_{2,E}(\mathbb{A}_E) \times_{T_{2,E}(\mathbb{A}_E)/T_{2,E}(E)} \mathbb{A}_E^\times/E^\times \\ & = & \{a \in T_{2,E}(\mathbb{A}_E) :\ \pi_w(a) \bmod E^\times \text{ is independent of } w\}. \end{array}$$

Thus there are exact sequences

$$(0) \longrightarrow \mathbb{A}_E^\times \longrightarrow \mathcal{E}^{\rm glob}(E/F)^0 \longrightarrow T_{3,E}(E) \longrightarrow (0)$$

and

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathcal{E}^{\rm glob}(E/F)^0 & \longrightarrow & T_{2,E}(\mathbb{A}_E) \times \mathbb{A}_E^\times/E^\times & \longrightarrow & T_{2,E}(\mathbb{A}_E)/T_{2,E}(E) & \longrightarrow & (0) \\ &&&& ((t_w)_{w\in V_E}, tE^\times) & \longmapsto & (t_w t^{-1})_{w \in V_E}. && \end{array}$$

From the first and lemma 4.2, we deduce that $H^1(\mathrm{Gal}\,(E/F), \mathcal{E}^{\rm glob}(E/F)^0) = (0)$. From the second and lemma 4.2, we deduce that there is a left exact sequence

$$\begin{array}{r} (0) \longrightarrow H^2(\mathrm{Gal}\,(E/F), \mathcal{E}^{\rm glob}(E/F)^0) \longrightarrow H^2(\mathrm{Gal}\,(E/F), T_{2,E}(\mathbb{A}_E)) \oplus H^2(\mathrm{Gal}\,(E/F), \mathbb{A}_E^\times/E^\times) \\ \longrightarrow \prod_{v\in V_F} H^2(\mathrm{Gal}\,(E_{w(v)}/F_v), \mathbb{A}_E^\times/E^\times), \end{array}$$

where we choose a place $w(v)$ of E above each place v of F. Tate [T2] remarks that by the above mentioned funtoriality $([\alpha^{\rm loc}_{E/F}], [\alpha^W_{E/F}])$ maps to 0 in $\prod_{v\in V_F} H^2(\mathrm{Gal}\,(E_w/F_v)$, and so lifts to a unique class $[\alpha^{\rm glob}_{E/F}] \in H^2(\mathrm{Gal}\,(E/F), \mathcal{E}^{\rm glob}(E/F)^0)$.

If $\alpha^{\rm glob} \in [\alpha^{\rm glob}_{E/F}]$, we get an extension

$$0 \longrightarrow \mathcal{E}^{\rm glob}(E/F)^0 \longrightarrow \mathcal{E}^{\rm glob}(E/F)_{\alpha^{\rm glob}} \longrightarrow \mathrm{Gal}\,(E/F) \longrightarrow 0$$

and a natural map

$$\mathcal{E}^{\rm glob}(E/F)_{\alpha^{\rm glob}} \twoheadrightarrow W_{E/F,\alpha^{\rm glob}}.$$

which is well defined up to conjugation by $\mathcal{E}^{\mathrm{glob}}(E/F)^0$. If further $\alpha^{\mathrm{loc}} \in [\alpha^{\mathrm{loc}}_{E/F}]$ and $\beta : \mathrm{Gal}\,(E/F) \to T_{2,E}(\mathbb{A}_E)$ with

$$^{\beta}\alpha^{\mathrm{loc}} = \alpha^{\mathrm{glob}} \in Z^2(\mathrm{Gal}\,(E/F), T_{2,E}(\mathbb{A}_E)),$$

then we get a map

$$\mathrm{loc}_{\beta} : \mathcal{E}^{\mathrm{glob}}(E/F)_{\alpha^{\mathrm{glob}}} \hookrightarrow \mathcal{E}_2(E/F)_{\alpha^{\mathrm{loc}}}$$

characterized by

$$\mathrm{loc}_{\beta} e_{\alpha^{\mathrm{glob}}}(\sigma) = \beta(\sigma)^{-1} e_{\alpha^{\mathrm{loc}}}(\sigma).$$

Note that if $t \in T_{2,E}(\mathbb{A}_E)$, then $\mathrm{loc}_{^t\beta} = \mathrm{conj}_{t^{-1}} \circ \mathrm{loc}_{\beta}$. Kottwitz further defines $\mathcal{E}_3(E/F)_{\alpha^{\mathrm{glob}}}$ to be the pushout of $\mathcal{E}^{\mathrm{glob}}(E/F)_{\alpha^{\mathrm{glob}}}$ along the map $\mathcal{E}^{\mathrm{glob}}(E/F)^0 \to T_{3,E}(E)$, so that we have the extension

$$0 \longrightarrow T_{3,E}(E) \longrightarrow \mathcal{E}_3(E/F)_{\alpha^{\mathrm{glob}}} \longrightarrow \mathrm{Gal}\,(E/F) \longrightarrow 0.$$

Thus to the triple $\boldsymbol{\alpha} = (\alpha^{\mathrm{glob}}, \alpha^{\mathrm{loc}}, \beta)$ with

- $\alpha^{\mathrm{glob}} \in [\alpha^{\mathrm{glob}}_{E/F}]$,
- $\alpha^{\mathrm{loc}} \in [\alpha^{\mathrm{loc}}_{E/F}]$,
- and $\beta : \mathrm{Gal}\,(E/F) \to T_{2,E}(\mathbb{A}_E)$ with

$$^{\beta}\alpha^{\mathrm{loc}} = \alpha^{\mathrm{glob}} \in Z^2(\mathrm{Gal}\,(E/F), T_{2,E}(\mathbb{A}_E)),$$

we can associate a diagram of extensions

$$\begin{array}{ccccccc} & & \mathcal{E}_3(E/F)_{\boldsymbol{\alpha}} & \twoheadleftarrow & \mathcal{E}^{\mathrm{glob}}(E/F)_{\boldsymbol{\alpha}} & \twoheadrightarrow & W_{E/F,\boldsymbol{\alpha}} \\ & & & & \mathrm{loc}_{\boldsymbol{\alpha}} \downarrow & & \\ & & \mathcal{E}^{\mathrm{loc}}(E/F)_{\boldsymbol{\alpha}} & \hookrightarrow & \mathcal{E}_2(E/F)_{\boldsymbol{\alpha}} & & \\ & & \bigcup & & & & \\ W_{E_w/F_v,\boldsymbol{\alpha}} & \twoheadleftarrow & \mathcal{E}^{\mathrm{loc}}(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(E/F)_w} & & & & \end{array}$$

together with distinguished elements $e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma) \in \mathcal{E}^{\mathrm{loc}}(E/F)$ and $e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma) \in \mathcal{E}^{\mathrm{glob}}(E/F)$ for all $\sigma \in \mathrm{Gal}\,(E/F)$. We will write $\mathcal{Z}(E/F)$ for the set of such triples $\boldsymbol{\alpha}$.

The set $\mathcal{Z}(E/F)$ has various actions we will consider:

(1) We will write $\mathcal{B}(E/F)$ for the abelian group consisting of pairs $\boldsymbol{\gamma} = (\gamma^{\mathrm{glob}}, \gamma^{\mathrm{loc}})$ where $\gamma^{\mathrm{glob}} : \mathrm{Gal}\,(E/F) \to \mathcal{E}^{\mathrm{glob}}(E/F)^0$ and $\gamma^{\mathrm{loc}} : \mathrm{Gal}\,(E/F) \to \prod_{w \in V_E} E_w^{\times}$; with pointwise multiplication. Then $\mathcal{B}(E/F)$ acts on $\mathcal{Z}(E/F)$ via

$$^{(\gamma^{\mathrm{glob}},\gamma^{\mathrm{loc}})}(\alpha^{\mathrm{glob}}, \alpha^{\mathrm{loc}}, \beta) = ({}^{\gamma^{\mathrm{glob}}}\alpha^{\mathrm{glob}}, {}^{\gamma^{\mathrm{loc}}}\alpha^{\mathrm{loc}}, \gamma^{\mathrm{glob}}\beta(\gamma^{\mathrm{loc}})^{-1}).$$

We write $\mathcal{H}(E/F)$ for the set of orbits of $\mathcal{B}(E/F)$ on $\mathcal{Z}(E/F)$. We will call two elements of $\mathcal{Z}(E/F)$ in the same orbit *equivalent*.

(2) $T_{2,E}(\mathbb{A}_E)$ acts on $\mathcal{Z}(E/F)$ via

$$^{t}(\alpha^{\mathrm{glob}}, \alpha^{\mathrm{loc}}, \beta) = (\alpha^{\mathrm{glob}}, \alpha^{\mathrm{loc}}, {}^{t}\beta).$$

This action commutes with the action of $\mathcal{B}(E/F)$, and the induced action of $\mathcal{B}(E/F) \times T_{2,E}(\mathbb{A}_E)$ is transitive. The stabilizer in $\mathcal{B}(E/F) \times T_{2,E}(\mathbb{A}_E)$ of any $\boldsymbol{\alpha}$ is the group of

$$(({}^a1, {}^b1), bc/a)$$

with $a \in \mathcal{E}^{\mathrm{glob}}(E/F)^0$ and $b \in \prod_{w\in V_E} E_w^\times$ and $c \in T_{2,E}(\mathbb{A}_F)$. (This is because $H^1(\mathrm{Gal}\,(E/F), \mathcal{E}^{\mathrm{glob}}(E/F)^0) = (0)$ and $H^1(\mathrm{Gal}\,(E/F), \prod_{w\in V_E} E_w^\times) = (0)$.)

Lemma 5.1. *(1) If $w_1 \neq w_2$ then the intersection of the images of $E_{w_1}^\times$ and $E_{w_2}^\times$ in $\mathbb{A}_E^\times/E^\times$ is trivial. Thus $\mathcal{E}^{\mathrm{glob}}(E/F)^0 \cap \prod_w E_w^\times = \{1\}$.*

(2) If $t \in \mathcal{E}^{\mathrm{glob}}(E/F)^0$ and $s \in \prod_w E_w^\times$ and $st \in T_{2,E}(\mathbb{A}_F)$, then s and $t \in T_{2,E}(\mathbb{A}_F)$.

Proof: For the first part if $t_i \in E_{w_i}^\times$ have the same image in $\mathbb{A}_E^\times/E^\times$ for $i = 1, 2$, then $t_1/t_2 \in E^\times \cap E_{w_1}^\times E_{w_2}^\times = \{1\}$ and so $t_1 = t_2 = 1$. Then $\mathcal{E}^{\mathrm{glob}}(E/F)^0 \cap \prod_w E_w^\times = T_{2,E}(E) \cap \prod_w E_w^\times = \{1\}$.

For the second part, write $t = (t_w)$ and $s = (s_w)$ as w runs over places of E. Also write $\bar{t}$ for the common image of the t_w in $\mathbb{A}_E^\times/E^\times$. We see that for all $\sigma \in \mathrm{Gal}\,(E/F)$ we have

$$\sigma t_{\sigma^{-1}w} \equiv t_w \bmod E_w^\times.$$

From the first part of the lemma we see that ${}^\sigma\bar{t}/\bar{t} = 1$, and so

$$\bar{t} \in (\mathbb{A}_E^\times/E^\times)^{\mathrm{Gal}\,(E/F)} = \mathbb{A}_F^\times/F^\times.$$

Thus we may write $t_w = t_0 t'_w$ with $t_0 \in \mathbb{A}_F^\times$ independent of w and $t'_w \in E^\times$. If $\sigma \in \mathrm{Gal}\,(E/F)$, then

$${}^\sigma(t'_w)/(t'_w) = (s_w)/{}^\sigma(s_w) \in \prod_w E^\times \cap \prod_w E_w^\times \subset \prod_w \mathbb{A}_E^\times,$$

from which we can conclude that ${}^\sigma(t'_w) = (t'_w)$ and ${}^\sigma(s_w) = (s_w)$. The lemma follows. □

Corollary 5.2. *If $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_1 \in \mathcal{Z}(E/F)$ are equivalent, then there are unique functions $\gamma^{\mathrm{glob}} : \mathrm{Gal}\,(E/F) \to \mathcal{E}^{\mathrm{glob}}(E/F)^0$ and $\gamma^{\mathrm{loc}} : \mathrm{Gal}\,(E/F) \to \prod_{w\in V_E} E_w^\times$ with*

$$(\alpha_1^{\mathrm{glob}}, \alpha_1^{\mathrm{loc}}, \beta_1) = ({}^{\gamma^{\mathrm{glob}}}\alpha^{\mathrm{glob}}, {}^{\gamma^{\mathrm{loc}}}\alpha^{\mathrm{loc}}, \gamma^{\mathrm{glob}}\beta(\gamma^{\mathrm{loc}})^{-1}).$$

Proof: Any other such pair must be of the form ${}^a\gamma^{\mathrm{glob}}$ and ${}^b\gamma^{\mathrm{loc}}$ with $a \in \mathcal{E}^{\mathrm{glob}}(E/F)^0$ and $b \in \prod_{w\in V_E} E_w^\times$ and $a/b \in T_{2,E}(\mathbb{A}_F)$. Thus, by the lemma, $a, b \in T_{2,E}(\mathbb{A}_F)$ so that ${}^a\gamma^{\mathrm{glob}} = \gamma^{\mathrm{glob}}$ and ${}^b\gamma^{\mathrm{loc}} = \gamma^{\mathrm{loc}}$. □

Corollary 5.3. *The stabilizer in $T_{2,E}(\mathbb{A}_E)$ of a class $\mathfrak{a} \in \mathcal{H}(E/F)$ is $\mathcal{E}^{\mathrm{loc}}(E/F)^0\mathcal{E}^{\mathrm{glob}}(E/F)^0 T_{2,E}(\mathbb{A}_F)$.*

Proof: Suppose $\boldsymbol{\alpha} = (\alpha^{\rm glob}, \alpha^{\rm loc}, \beta) \in \mathfrak{a}$. If ${}^t\mathfrak{a} = \mathfrak{a}$, then

$$(\alpha^{\rm glob}, \alpha^{\rm loc}, {}^t\beta) = ({}^{\gamma^{\rm glob}}\alpha^{\rm glob}, {}^{\gamma^{\rm loc}}\alpha^{\rm loc}, \gamma^{\rm glob}\beta(\gamma^{\rm loc})^{-1})$$

for some $\boldsymbol{\gamma} = (\gamma^{\rm glob}, \gamma^{\rm loc}) \in \mathcal{B}(E/F)$. Then $\gamma^{\rm glob} = {}^a 1$ for some $a \in \mathcal{E}^{\rm glob}(E/F)^0$ (because $H^1(\mathrm{Gal}\,(E/F), \mathcal{E}^{\rm glob}(E/F)^0) = (0)$) and $\gamma^{\rm loc} = {}^b 1$ for some $b \in \mathcal{E}^{\rm loc}(E/F)^0$ (because $H^1(\mathrm{Gal}\,(E/F), \mathcal{E}^{\rm loc}(E/F)^0) = (0)$) and ${}^t\beta = {}^{a/b}\beta$, so that $tb/a \in T_{2,E}(\mathbb{A}_F)$ and $t \in \mathcal{E}^{\rm loc}(E/F)^0\mathcal{E}^{\rm glob}(E/F)^0 T_{2,E}(\mathbb{A}_F)$. The converse is easier. □

It follows from the first corollary above that up to canonical isomorphism the extensions $W_{E_w/F_v,\mathfrak{a}}$, $\mathcal{E}^{\rm loc}(E/F)_{\mathfrak{a}}$, $\mathcal{E}_2(E/F)_{\mathfrak{a}}$, $W_{E/F,\mathfrak{a}}$, $\mathcal{E}^{\rm glob}(E/F)_{\mathfrak{a}}$, $\mathcal{E}_3(E/F)_{\mathfrak{a}}$ and the diagram

$$\begin{array}{ccccccc} & & \mathcal{E}_3(E/F)_{\mathfrak{a}} & \twoheadleftarrow & \mathcal{E}^{\rm glob}(E/F)_{\mathfrak{a}} & \twoheadrightarrow & W_{E/F,\mathfrak{a}} \\ & & & & \mathrm{loc}_{\mathfrak{a}} \downarrow & & \\ & & \mathcal{E}^{\rm loc}(E/F)_{\mathfrak{a}} & \hookrightarrow & \mathcal{E}_2(E/F)_{\mathfrak{a}} & & \\ & & \bigcup & & & & \\ W_{E_w/F_v,\mathfrak{a}} & \twoheadleftarrow & \mathcal{E}^{\rm loc}(E/F)_{\mathfrak{a}}|_{\mathrm{Gal}\,(E/F)_w}. & & & & \end{array}$$

only depends on a class $\mathfrak{a} \in \mathcal{H}(E/F)$.

Suppose that $\boldsymbol{\alpha} = (\alpha^{\rm glob}, \alpha^{\rm loc}, \beta) \in \mathcal{Z}(E/F)$ and that w is a place of E above a place v of F. Then

$$\pi_w \alpha^{\rm glob}|_{\mathrm{Gal}\,(E/F)_w^2} = {}^{\pi_w\beta|_{\mathrm{Gal}\,(E/F)_w}}\alpha_w^{\rm loc} \in H^2(\mathrm{Gal}\,(E/F)_w, \mathbb{A}_E^\times).$$

Pushing out $\mathcal{E}^{\rm glob}_{\boldsymbol{\alpha}}(E/F)|_{\mathrm{Gal}\,(E/F)_w}$ along π_w gives an extension, which fits into a diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & E_w^\times & \longrightarrow & W_{E_w/F_v,\boldsymbol{\alpha}} & \longrightarrow & \mathrm{Gal}\,(E_w/F_v) & \longrightarrow & (0) \\ & & \downarrow & & \iota_w^{\boldsymbol{\alpha}} \downarrow & & || & & \\ (0) & \longrightarrow & \mathbb{A}_E^\times & \longrightarrow & \pi_{w,*}\mathcal{E}^{\rm glob}_{\boldsymbol{\alpha}}(E/F)|_{\mathrm{Gal}\,(E/F)_w} & \longrightarrow & \mathrm{Gal}\,(E/F)_w & \longrightarrow & (0) \\ & & \downarrow & & \downarrow & & || & & \\ (0) & \longrightarrow & \mathbb{A}_E^\times/E^\times & \longrightarrow & W_{E/F,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(E/F)_w} & \longrightarrow & \mathrm{Gal}\,(E/F)_w & \longrightarrow & (0), \end{array}$$

where

$$\iota_w^{\boldsymbol{\alpha}}(e^{\rm loc}_{\boldsymbol{\alpha}}(\sigma)) = \pi_w(\beta(\sigma))e^{\rm glob}_{\boldsymbol{\alpha}}(\sigma).$$

One sees immediately that $\iota_w^{\boldsymbol{\alpha}}$, up to the canonical identifications, only depends on the image $\mathfrak{a}$ of $\boldsymbol{\alpha} \in \mathcal{H}(E/F)$, so we will denote it

$$\iota_w^{\mathfrak{a}} : W_{E_w/F_v,\mathfrak{a}} \longrightarrow \pi_{w,*}\mathcal{E}^{\rm glob}_{\mathfrak{a}}(E/F)|_{\mathrm{Gal}\,(E/F)_w} \longrightarrow W_{E/F,\mathfrak{a}}.$$

Any other map of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & E_w^\times & \longrightarrow & W_{E_w/F_v,\boldsymbol{\alpha}} & \longrightarrow & \mathrm{Gal}\,(E_w/F_v) & \longrightarrow & (0) \\ & & \downarrow & & i \downarrow & & || & & \\ (0) & \longrightarrow & \mathbb{A}_E^\times/E^\times & \longrightarrow & W_{E/F,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(E/F)_w} & \longrightarrow & \mathrm{Gal}\,(E/F)_w & \longrightarrow & (0) \end{array}$$

must be of the form $\mathrm{conj}_a \circ \iota_w^{\mathfrak{a}}$ for some $a \in \mathbb{A}_E^\times/E^\times$. (If $i(e_{\alpha_w^{\rm loc}}(\sigma)) = \pi_w(\beta(\sigma))\gamma(\sigma)e^{\rm glob}(\sigma)$, then $\gamma \in Z^1(\mathrm{Gal}\,(E/F)_w, \mathbb{A}_E^\times/E^\times)$ and hence is of the form $\gamma(\sigma) = a/{}^\sigma a$.) If

$\varphi : W_{E/F,\mathfrak{a}} \xrightarrow{\sim} W_{E^{\mathrm{ab}}/F}$ and $\varphi_w : W_{E_w/F_v,\mathfrak{a}} \xrightarrow{\sim} W_{E_w^{\mathrm{ab}}/F_v}$ are isomorphisms of extensions, and if u is a place of E^{ab} above w, then we conclude that $\varphi \circ \iota_w^{\mathfrak{a}} = \mathrm{conj}_a \circ \theta_u \circ \varphi_w$ for some $a \in \mathbb{A}_E^\times/E^\times$. Thus $\mathrm{conj}_a^{-1} \circ \varphi \circ \iota_w^{\mathfrak{a}}$ and θ_u have the same image. In particular the image of $\varphi \circ \iota_w^{\mathfrak{a}}$ is the decomposition group for some place of E^{ab} above w. This suggests that the choice of $\mathfrak{a}$ is not dissimilar from the choice of a place of E^{ab} above each place of E.

Lemma 5.4. *If $\tau \in \mathrm{Gal}\,(E/F)$, then $\mathrm{conj}_{e_{\alpha^{\mathrm{loc}}}(\tau)}$ induces an isomorphism $W_{E_w/F_v,\boldsymbol{\alpha}} \xrightarrow{\sim} W_{E_{\tau w}/F_v,\boldsymbol{\alpha}}$, and $\mathrm{conj}_{e_{\alpha^{\mathrm{glob}}}(\tau)}$ induces a map $W_{E/F,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(E/F)_w} \xrightarrow{\sim} W_{E/F,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(E/F)_{\tau w}}$. Moreover*

$$\iota_{\tau w}^{\mathfrak{a}} \circ \mathrm{conj}_{e_{\alpha^{\mathrm{loc}}}(\tau)} = \mathrm{conj}_{\pi_{\tau w}\beta(\tau)} \circ \mathrm{conj}_{e_{\alpha^{\mathrm{glob}}}(\tau)} \circ \iota_w^{\mathfrak{a}} : W_{E_w/F_v,\boldsymbol{\alpha}} \longrightarrow W_{E/F,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(E_{\tau w}/F_v)}.$$

Proof: Both maps send $x \in E_w^\times$ to ${}^\tau x \in E_{\tau w}^\times$.

If $\sigma \in \mathrm{Gal}\,(E_w/F_v)$, then $\mathrm{conj}_{e_{\alpha^{\mathrm{loc}}}(\tau)}(e_{\alpha^{\mathrm{loc}}}(\sigma))$ is the image of

$$\begin{aligned} & e_{\alpha^{\mathrm{loc}}}(\tau)e_{\alpha^{\mathrm{loc}}}(\sigma)e_{\alpha^{\mathrm{loc}}}(\tau)^{-1} \\ = \; & e_{\alpha^{\mathrm{loc}}}(\tau)e_{\alpha^{\mathrm{loc}}}(\sigma)e_{\alpha^{\mathrm{loc}}}(\tau^{-1})(e_{\alpha^{\mathrm{loc}}}(\tau)e_{\alpha^{\mathrm{loc}}}(\tau^{-1}))^{-1} \\ = \; & \alpha^{\mathrm{loc}}(\tau,\sigma)\alpha^{\mathrm{loc}}(\tau\sigma,\tau^{-1})e_{\alpha^{\mathrm{loc}}}(\tau\sigma\tau^{-1})(\alpha^{\mathrm{loc}}(\tau,\tau^{-1})\alpha^{\mathrm{loc}}(1,1))^{-1} \end{aligned}$$

and so

$$\mathrm{conj}_{e_{\alpha^{\mathrm{loc}}}(\tau)}(e_{\alpha^{\mathrm{loc}}}(\sigma)) = \pi_{\tau w}(\alpha^{\mathrm{loc}}(\tau,\sigma)\alpha^{\mathrm{loc}}(\tau\sigma,\tau^{-1})/{}^{\tau\sigma\tau^{-1}}(\alpha^{\mathrm{loc}}(\tau,\tau^{-1})\alpha^{\mathrm{loc}}(1,1)))e_{\alpha^{\mathrm{loc}}}(\tau\sigma\tau^{-1}).$$

Thus

$$\begin{aligned} & (\iota_{\tau w}^{\mathfrak{a}} \circ \mathrm{conj}_{e_{\alpha^{\mathrm{loc}}}(\tau)})(e_{\alpha^{\mathrm{loc}}}(\sigma)) \\ = \; & \pi_{\tau w}(\alpha^{\mathrm{loc}}(\tau,\sigma)\alpha^{\mathrm{loc}}(\tau\sigma,\tau^{-1})\beta(\tau\sigma\tau^{-1})/{}^{\tau\sigma\tau^{-1}}(\alpha^{\mathrm{loc}}(\tau,\tau^{-1})\alpha^{\mathrm{loc}}(1,1)))e_{\alpha^{\mathrm{glob}}}(\tau\sigma\tau^{-1}). \end{aligned}$$

On the other hand

$$\begin{aligned} & (\mathrm{conj}_{\pi_{\tau w}\beta(\tau)} \circ \mathrm{conj}_{e_{\alpha^{\mathrm{glob}}}(\tau)} \circ \iota_w^{\mathfrak{a}})(e_{\alpha^{\mathrm{loc}}}(\sigma)) \\ = \; & \pi_{\tau w}(\beta(\tau))^\tau\pi_w(\beta(\sigma))e_{\alpha^{\mathrm{glob}}}(\tau)e_{\alpha^{\mathrm{glob}}}(\sigma)e_{\alpha^{\mathrm{glob}}}(\tau)^{-1}\pi_{\tau w}(\beta(\tau))^{-1} \\ = \; & \pi_{\tau w}(\beta(\tau)^\tau\beta(\sigma)/{}^{\tau\sigma\tau^{-1}}\beta(\tau))e_{\alpha^{\mathrm{glob}}}(\tau)e_{\alpha^{\mathrm{glob}}}(\sigma)e_{\alpha^{\mathrm{glob}}}(\tau)^{-1} \\ = \; & \pi_{\tau w}(\beta(\tau)^\tau\beta(\sigma)/{}^{\tau\sigma\tau^{-1}}\beta(\tau))e_{\alpha^{\mathrm{glob}}}(\tau)e_{\alpha^{\mathrm{glob}}}(\sigma)e_{\alpha^{\mathrm{glob}}}(\tau^{-1})(e_{\alpha^{\mathrm{glob}}}(\tau)e_{\alpha^{\mathrm{glob}}}(\tau^{-1}))^{-1} \\ = \; & \pi_{\tau w}(\beta(\tau)^\tau\beta(\sigma)\alpha^{\mathrm{glob}}(\tau,\sigma)\alpha^{\mathrm{glob}}(\tau\sigma,\tau^{-1})/{}^{\tau\sigma\tau^{-1}}\beta(\tau))e_{\alpha^{\mathrm{glob}}}(\tau\sigma\tau^{-1})\pi_{\tau w}(\alpha^{\mathrm{glob}}(\tau,\tau^{-1})\alpha^{\mathrm{glob}}(1,1))^{-1}. \end{aligned}$$

Thus to prove the lemma it suffices to check that

$$\begin{aligned} & \alpha^{\mathrm{loc}}(\tau,\sigma)\alpha^{\mathrm{loc}}(\tau\sigma,\tau^{-1})\beta(\tau\sigma\tau^{-1})/{}^{\tau\sigma\tau^{-1}}(\alpha^{\mathrm{loc}}(\tau,\tau^{-1})\alpha^{\mathrm{loc}}(1,1)) \\ = \; & \beta(\tau)^\tau\beta(\sigma)\alpha^{\mathrm{glob}}(\tau,\sigma)\alpha^{\mathrm{glob}}(\tau\sigma,\tau^{-1})/{}^{\tau\sigma\tau^{-1}}(\beta(\tau)\alpha^{\mathrm{glob}}(\tau,\tau^{-1})\alpha^{\mathrm{glob}}(1,1)), \end{aligned}$$

or equivalently that

$$\begin{aligned} & \beta(\tau\sigma\tau^{-1}) \\ = \; & \beta(\tau)^\tau\beta(\sigma)\beta(\tau\sigma)\beta(\tau\sigma\tau^{-1})^{\tau\sigma\tau^{-1}}(\beta(\tau)^\tau\beta(\tau^{-1})\beta(1)^2)/\beta(\tau)^\tau\beta(\sigma)\beta(\tau\sigma)^{\tau\sigma}\beta(\tau^{-1})^{\tau\sigma\tau^{-1}}(\beta(\tau)\beta(1)^2), \end{aligned}$$

which is clear. □

By a *decomposition group* in $W_{E/F,\mathfrak{a}}$ we mean a subgroup conjugate to the image of some $\iota_w^{\mathfrak{a}}$. By the lemma we need only consider one place w above each place v of F.

If $\mathfrak{a},\mathfrak{a}'\in\mathcal{H}(E/F)$, then $\mathfrak{a}'={}^t\mathfrak{a}$ for some $t\in T_{2,E}(\mathbb{A}_E)$ which is well defined modulo $T_{2,E}(\mathbb{A}_F)\mathcal{E}^{\rm glob}(E/F)^0\prod_w E_w^\times$. Then we may choose representatives $(\alpha^{\rm glob},\alpha^{\rm loc},\beta)\in\mathfrak{a}$ and $(\alpha^{\rm glob},\alpha^{\rm loc},{}^t\beta)\in\mathfrak{a}'$. This gives identifications

$$\mathfrak{z}_t:\mathcal{E}^{\rm loc}(E/F)_{\mathfrak{a}}\cong\mathcal{E}^{\rm loc}(E/F)_{\alpha^{\rm loc}}\cong\mathcal{E}^{\rm loc}(E/F)_{\mathfrak{a}}$$

and similarly for $W_{E_w/F_v,\mathfrak{a}}$ and $\mathcal{E}_2(E/F)_{\mathfrak{a}}$; i.e. we get a commutative diagram of isomorphisms

$$\begin{array}{ccccccc} W_{E_w/F_v,\mathfrak{a}} & \longleftarrow & \mathcal{E}^{\rm loc}(E/F)_{\mathfrak{a}}|_{{\rm Gal}\,(E/F)_w} & \subset & \mathcal{E}^{\rm loc}(E/F)_{\mathfrak{a}} & \hookrightarrow & \mathcal{E}_2(E/F)_{\mathfrak{a}} \\ \mathfrak{z}_t\downarrow\wr & & \mathfrak{z}_t\downarrow\wr & & \mathfrak{z}_t\downarrow\wr & & \mathfrak{z}_t\downarrow\wr \\ W_{E_w/F_v,\mathfrak{a}'} & \longleftarrow & \mathcal{E}^{\rm loc}(E/F)_{\mathfrak{a}'}|_{{\rm Gal}\,(E/F)_w} & \subset & \mathcal{E}^{\rm loc}(E/F)_{\mathfrak{a}'} & \hookrightarrow & \mathcal{E}_2(E/F)_{\mathfrak{a}'}. \end{array}$$

Similarly it gives identifications

$$\mathfrak{z}_t:\mathcal{E}^{\rm glob}(E/F)_{\mathfrak{a}}\cong\mathcal{E}^{\rm glob}(E/F)_{\alpha^{\rm glob}}\cong\mathcal{E}^{\rm glob}(E/F)_{\mathfrak{a}},$$

and similarly for $W_{E/F,\mathfrak{a}}$ and $\mathcal{E}_3(E/F)_{\mathfrak{a}}$; i.e. we get a commutative diagram of isomorphisms

$$\begin{array}{ccccccc} \mathcal{E}_3(E/F)_{\mathfrak{a}} & \longleftarrow & \mathcal{E}^{\rm glob}(E/F)_{\mathfrak{a}} & \twoheadrightarrow & W_{E/F,\mathfrak{a}} & \stackrel{\varphi_{\mathfrak{a}}}{\longrightarrow} & {\rm Gal}\,(E^{\rm ab}/F) \\ \mathfrak{z}_t\downarrow\wr & & \mathfrak{z}_t\downarrow\wr & & \mathfrak{z}_t\downarrow\wr & & || \\ \mathcal{E}_3(E/F)_{\mathfrak{a}'} & \longleftarrow & \mathcal{E}^{\rm glob}(E/F)_{\mathfrak{a}'} & \twoheadrightarrow & W_{E/F,\mathfrak{a}'} & \stackrel{\varphi_{\mathfrak{a}}}{\longrightarrow} & {\rm Gal}\,(E^{\rm ab}/F). \end{array}$$

The maps $\mathfrak{z}_t$ do not depend on the choice of representative $(\alpha^{\rm glob},\alpha^{\rm loc},\beta)\in\mathfrak{a}$, but only on $\mathfrak{a}$ and t. (Indeed if $(\alpha_1^{\rm glob},\alpha_1^{\rm loc},\beta_1)\in\mathfrak{a}$ and $(\alpha_1^{\rm glob},\alpha_1^{\rm loc},{}^t\beta_1)\in\mathfrak{a}'$ are another pair of such representatives, then we can find unique functions $\gamma^{\rm glob}:{\rm Gal}\,(E/F)\to\mathcal{E}^{\rm glob}(E/F)^0$ and $\gamma^{\rm loc}:{\rm Gal}\,(E/F)\to\prod_w E_w^\times$ such that

$$(\alpha_1^{\rm glob},\alpha_1^{\rm loc},\beta_1)=({}^{\gamma^{\rm glob}}\alpha^{\rm glob},{}^{\gamma^{\rm loc}}\alpha^{\rm loc},\gamma^{\rm glob}\beta(\gamma^{\rm loc})^{-1}).$$

In this case

$$(\alpha_1^{\rm glob},\alpha_1^{\rm loc},{}^t\beta_1,\varphi_1)=({}^{\gamma^{\rm glob}}\alpha^{\rm glob},{}^{\gamma^{\rm loc}}\alpha^{\rm loc},\gamma^{\rm glob}({}^t\beta)(\gamma^{\rm loc})^{-1},\varphi\circ i_{\gamma^{\rm glob}}^{-1}).$$

Then

$$\begin{array}{ccc} \mathcal{E}^{\rm loc}(E/F)_{\alpha^{\rm loc}} & \stackrel{i_{\gamma^{\rm loc}}}{\longrightarrow} & \mathcal{E}^{\rm loc}(E/F)_{\alpha_1^{\rm loc}} \\ || & & || \\ \mathcal{E}^{\rm loc}(E/F)_{\alpha^{\rm loc}} & \stackrel{i_{\gamma^{\rm loc}}}{\longrightarrow} & \mathcal{E}^{\rm loc}(E/F)_{\alpha_1^{\rm loc}} \end{array}$$

obviously commutes, and $(\alpha^{\rm glob},\alpha^{\rm loc},\beta)$ and $(\alpha_1^{\rm glob},\alpha_1^{\rm loc},\beta_1)$ give the same isomorphism $\mathcal{E}^{\rm loc}(E/F)_{\mathfrak{a}}\stackrel{\sim}{\to}\mathcal{E}^{\rm loc}(E/F)_{\mathfrak{a}'}$. Similarly for the other extensions we are considering.) We have

$$\mathfrak{z}_t\circ{\rm loc}_{\mathfrak{a}}={\rm conj}_t\circ{\rm loc}_{{}^t\mathfrak{a}}\circ\mathfrak{z}_t$$

and

$${\rm conj}_{t_w}\circ\mathfrak{z}_t\circ\iota_w^{\mathfrak{a}}=\iota_w^{{}^t\mathfrak{a}}\circ\mathfrak{z}_t.$$

Moreover $\mathfrak{z}_t$ takes decomposition groups to decomposition groups. If $a \in T_{2,E}(\mathbb{A}_F)$ and $b \in \mathcal{E}^{\mathrm{glob}}(E/F)^0$ and $c \in \prod_w E_w^\times$, then $\mathfrak{z}_{abct} = \mathrm{conj}_{b^{-1}} \circ \mathfrak{z}_t$ on $\mathcal{E}_3(E/F)_{\mathfrak{a}}$ and $\mathcal{E}^{\mathrm{glob}}(E/F)_{\mathfrak{a}}$ and $W_{E/F,\mathfrak{a}}$, while $\mathfrak{z}_{abct} = \mathrm{conj}_c \circ \mathfrak{z}_t$ on $W_{E_w/F_v,\mathfrak{a}}$ and $\mathcal{E}^{\mathrm{loc}}(E/F)_{\mathfrak{a}}$ and $\mathcal{E}_2(E/F)_{\mathfrak{a}}$.

We want to explain how given some auxiliary data we can write down a triple $(\alpha^{\mathrm{glob}}, \alpha^{\mathrm{loc}}, \beta) \in \mathcal{Z}(E/F)$ more explicitly. Fix the following data

- α representing $[\alpha^{\mathrm{glob}}_{E/F}]$,
- a place $w = w(v)$ above each place v of F,
- α_w representing $[\alpha_{E_w/F_v}]$ for each such w,
- a section $s_w : \mathrm{Gal}\,(E/F)/\mathrm{Gal}\,(E_w/F_v) \to \mathrm{Gal}\,(E/F)$ with image H_w for each such w,
- and, for each such w, a function $\gamma_w : \mathrm{Gal}\,(E_w/F_v) \to \mathbb{A}_E^\times$ such that ${}^{\gamma_w}(\pi_w \circ \alpha)|_{\mathrm{Gal}\,(E_w/F_v)} = i_w\alpha_w \in Z^2(\mathrm{Gal}\,(E_w/F_v), \mathbb{A}_E^\times)$. (This is possible as α is equivalent to some α^{loc} in $Z^2(\mathrm{Gal}\,(E/F), T_{2,E}(\mathbb{A}_E))$.)

Set $\delta_v(\sigma) = \sigma^{-1}s_v(\sigma)$. Then $[\alpha^{\mathrm{loc}}_{E/F}]$ can be represented by

$$\alpha^{\mathrm{loc}}(\sigma_1, \sigma_2) = \prod_{v \in V_F} \prod_{\eta \in \mathrm{Gal}\,(E/F)/\mathrm{Gal}\,(E_w/F_v)} s_w(\eta)\alpha_w(s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)).$$

Define

$$\beta : \mathrm{Gal}\,(E/F) \longrightarrow T_{2,E}(\mathbb{A}_E)$$

by

$$\pi_{\eta w}\beta(\sigma) = \pi_{\eta w}(\alpha(s_w(\eta), s_w(\eta)^{-1}\sigma s_w(\sigma^{-1}\eta))/\alpha(\sigma, s_w(\sigma^{-1}\eta)))^{s_w(\eta)}\gamma_w(s_w(\eta)^{-1}\sigma s_w(\sigma^{-1}\eta))^{-1}.$$

We claim that $(\alpha, \alpha^{\mathrm{loc}}, \beta) \in \mathcal{Z}(E/F)$, i.e. that

$${}^{\beta}\alpha^{\mathrm{loc}} = \alpha.$$

Our verification is a rather ugly cocyle computation. We need to check that

$$\beta(\sigma_1\sigma_2)\beta(\sigma_1)^{-1\sigma_1}\beta(\sigma_2)^{-1}\alpha^{\mathrm{loc}}(\sigma_1, \sigma_2) = \alpha(\sigma_1, \sigma_2)$$

or, after projecting under $\pi_{\eta w}$, that

$$\begin{aligned} & \pi_{\eta w}\alpha(\sigma_1, \sigma_2) \\ = \; & \pi_{\eta w}(\beta(\sigma_1\sigma_2)\beta(\sigma_1)^{-1})^{\sigma_1}\pi_{\sigma_1^{-1}\eta w}(\beta(\sigma_2))^{-1} \\ & {}^{s_w(\eta)}\alpha_w(s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)). \end{aligned}$$

The right hand side equals

$$\begin{aligned} & \pi_{\eta w}(\alpha(s_w(\eta), s_w(\eta)^{-1}\sigma_1\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))\alpha(\sigma_1, s_w(\sigma_1^{-1}\eta))) \\ & \pi_{\eta w}(\alpha(s_w(\eta), s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta))\alpha(\sigma_1\sigma_2, s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)))^{-1} \\ & {}^{\sigma_1}\pi_{\sigma_1^{-1}\eta w}(\alpha(\sigma_2, s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))/\alpha(s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))) \\ & {}^{s_w(\eta)}\alpha_w(s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)) \\ & {}^{s_w(\eta)}(\gamma_w(s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta))/\gamma_w(s_w(\eta)^{-1}\sigma_1\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)))^{\sigma_1 s_w(\sigma_1^{-1})}\gamma_w(s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)) \end{aligned}$$

or

$$\begin{array}{l}\pi_{\eta w}(\alpha(s_w(\eta), s_w(\eta)^{-1}\sigma_1\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))\alpha(\sigma_1, s_w(\sigma_1^{-1}\eta))) \\ \pi_{\eta w}(\alpha(s_w(\eta), s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta))\alpha(\sigma_1\sigma_2, s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)))^{-1} \\ \pi_{\eta w}({}^{\sigma_1}\alpha(\sigma_2, s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))/{}^{\sigma_1}\alpha(s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))) \\ {}^{s_w(\eta)}\pi_w\alpha(s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)),\end{array}$$

which in turn equals $\pi_{\eta w}$ of

$$\begin{array}{l}\alpha(s_w(\eta), s_w(\eta)^{-1}\sigma_1\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))\alpha(\sigma_1, s_w(\sigma_1^{-1}\eta))/\alpha(s_w(\eta), s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta))\alpha(\sigma_1\sigma_2, s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)) \\ {}^{\sigma_1}\alpha(\sigma_2, s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))/{}^{\sigma_1}\alpha(s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)) \\ {}^{s_w(\eta)}\alpha(s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)).\end{array}$$

We can rewrite this

$$\begin{array}{l}\alpha(s_w(\eta), s_w(\eta)^{-1}\sigma_1\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))\alpha(\sigma_1, s_w(\sigma_1^{-1}\eta)) \\ (\alpha(s_w(\eta), s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta))\alpha(\sigma_1\sigma_2, s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)))^{-1} \\ (\alpha(\sigma_1, \sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))\alpha(\sigma_1\sigma_2, s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))\alpha(\sigma_1, \sigma_2)) \\ (\alpha(\sigma_1 s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))\alpha(\sigma_1, s_w(\sigma_1^{-1}\eta))\alpha(\sigma_1, \sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta)))^{-1} \\ \alpha(\sigma_1 s_w(\sigma_1^{-1}\eta), s_w(\sigma_1^{-1}\eta)^{-1}\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))\alpha(s_w(\eta), s_w(\eta)^{-1}\sigma_1 s_w(\sigma_1^{-1}\eta)) \\ \alpha(s_w(\eta), s_w(\eta)^{-1}\sigma_1\sigma_2 s_w(\sigma_2^{-1}\sigma_1^{-1}\eta))^{-1}\end{array}$$

in which almost everything cancels leaving just $\alpha(\sigma_1, \sigma_2)$. The claim follows.

Let us be still more explicit in a special case. Assume that $F = \mathbb{Q}$ and that E is totally imaginary. We may and will assume that

- $\alpha(1, 1) = 1$,
- $\alpha_w(1, 1) = 1$ for all w,
- $1 \in H_w$ for all w,
- $\alpha_{w(\infty)}(\sigma_1, \sigma_2) = \begin{cases} -1 & \text{if } \sigma_1 = \sigma_2 = c_{w(\infty)} \\ 1 & \text{otherwise,} \end{cases}$
- $\gamma_{w(\infty)} \equiv 1$,
- and $\alpha(\sigma, c_{w(\infty)}) = 1$ if $\sigma \in H_\infty$.

(To achieve the last of these we replace α by ${}^{\gamma}\alpha$ where $\gamma(1) = \gamma(c_{w(\infty)}) = 1$ and $\gamma(\sigma c_{w(\infty)}) = \alpha(\sigma, c_{w(\infty)})\gamma(\sigma)$ if $\sigma \in H_\infty$.) Then

- $\alpha(\sigma, 1) = \alpha(1, \sigma) = 1$;
- $\alpha(c_{w(\infty)}, c_{w(\infty)})_{w(\infty)} = -1_{w(\infty)}$;
- $\alpha(\sigma, c_{w(\infty)})_{\sigma w(\infty)} = -1_{\sigma w(\infty)}$ if $\sigma \notin H_\infty$;
- $\alpha(\sigma_1, \sigma_2 c_{w(\infty)})_{\sigma_1\sigma_2 w(\infty)} = \alpha(\sigma_1, \sigma_2)_{\sigma_1\sigma_2 w(\infty)} \begin{cases} -1_{\sigma_1\sigma_2 w(\infty)} & \text{if } \delta_\infty(\sigma_2) \neq \delta_\infty(\sigma_1\sigma_2) \\ 1 & \text{if } \delta_\infty(\sigma_2) = \delta_\infty(\sigma_1\sigma_2). \end{cases}$

(For the penultimate of these note that $\alpha(\sigma, c_{w(\infty)}) = {}^{\sigma}\alpha(c_{w(\infty)}, c_{w(\infty)})\alpha(\sigma, 1)/\alpha(\sigma c_{w(\infty)}, c_{w(\infty)})$, and for the ultimate one use the cocycle relation.) Thus

$$\begin{array}{rcl}\pi_{\eta w(\infty)}\beta(\sigma) & = & \pi_{\eta w(\infty)}(\alpha(\eta\delta_\infty(\eta), \delta_\infty(\eta)^{-1}\delta_\infty(\sigma^{-1}\eta))/\alpha(\sigma, \sigma^{-1}\eta\delta_\infty(\sigma^{-1}\eta))) \\ & \equiv & \pi_{\eta w(\infty)}(\alpha(\sigma, \sigma^{-1}\eta))^{-1} \begin{cases} -1_{\eta w(\infty)} & \text{if } \delta_\infty(\sigma^{-1}\eta) \neq 1 \\ 1 & \text{if } \delta_\infty(\sigma^{-1}\eta) = 1 \end{cases}\end{array}$$

for all $\eta \in H_\infty$ and $\sigma \in \mathrm{Gal}\,(E/\mathbb{Q})$.

Although this doesn't fix $\boldsymbol{\alpha}$ uniquely, we will use $\boldsymbol{\alpha}_0$ to denote the element of $\mathcal{Z}(E/\mathbb{Q})$ arising from such a choice. Also choose $\rho_0 : E_{w(\infty)} \xrightarrow{\sim} \mathbb{C}$. Then we get a canonical identification

$$\widetilde{\Theta}_0 : W_{\mathbb{C}/\mathbb{R}} \xrightarrow{\sim} W_{E_{w(\infty)}/\mathbb{Q}_\infty, \boldsymbol{\alpha}_0}.$$

Moreover

$$\iota^{\boldsymbol{\alpha}_0}_{w(\infty)}(e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(c_{w(\infty)})) = e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(c_{w(\infty)}).$$

5.2. **Change of field.** Suppose that $D \supset E \supset F$ are extensions of number fields with E and D both Galois over F. The comparison of the various extensions $\mathcal{E}(E/F)$ and $\mathcal{E}(D/F)$ seems to be less straight forward than in the case of Weil groups. We will need to introduce another extension $\mathcal{E}(E/F)_D$, which we will compare with both $\mathcal{E}(E/F)$ and $\mathcal{E}(D/F)$.

We will write

$$[\alpha^{\mathrm{loc}}_{E/F,D}] = \inf^{\mathrm{Gal}\,(D/F)}_{\mathrm{Gal}\,(E/F)}[\alpha^{\mathrm{loc}}_{E/F}] = \eta^0_{D/E,*}[\alpha^{\mathrm{loc}}_{D/F}] \in H^2(\mathrm{Gal}\,(D/F), \prod_{v\in V_F} D_v^\times).$$

If $\alpha \in [\alpha^{\mathrm{loc}}_{E/F,D}]$ we will write $\mathcal{E}_2(E/F)_{D,\alpha}$ for the push out of $\mathcal{E}^{\mathrm{loc}}(E/F)_{D,\alpha}$ along $\prod_{w\in V_E} D_w^\times \to T_{2,E}(\mathbb{A}_D)$. If w is a place of E below a place u of D we will write $W_{E_w/F_v,D,\alpha}$ (resp. $W_{E_w/F_v,D_u,\alpha}$) for the pushout of $\mathcal{E}^{\mathrm{loc}}(E/F)_{D,\alpha}|_{\mathrm{Gal}\,(D/F)_w}$ along $\prod_{w\in V_E} D_w^\times \to D_w^\times$ (resp. the pushout of $\mathcal{E}^{\mathrm{loc}}(E/F)_{D,\alpha}|_{\mathrm{Gal}\,(D/F)_u}$ along $\prod_{w\in V_E} D_w^\times \to D_u^\times$).

We define

$$\mathcal{E}^{\mathrm{glob}}(E/F)^0_D = T_{2,E}(\mathbb{A}_D) \times_{T_{2,E}(\mathbb{A}_D)/T_{2,E}(D)} \mathbb{A}_D^\times/D^\times.$$

Thus $\mathrm{Gal}\,(D/F)$ acts on $\mathcal{E}^{\mathrm{glob}}(E/F)^0_D$ and

$$(\mathcal{E}^{\mathrm{glob}}(E/F)^0_D)^{\mathrm{Gal}\,(D/E)} = \mathcal{E}^{\mathrm{glob}}(E/F)^0.$$

Moreover we have an exact sequence

$$(0) \longrightarrow \mathbb{A}_D^\times \longrightarrow \mathcal{E}^{\mathrm{glob}}(E/F)^0_D \longrightarrow T_{3,E}(D) \longrightarrow (0),$$

from which we deduce that

$$H^1(\mathrm{Gal}\,(D/F), \mathcal{E}^{\mathrm{glob}}(E/F)^0_D) = (0).$$

We also have an exact sequence

$$(0) \longrightarrow \mathcal{E}^{\mathrm{glob}}(E/F)^0_D \longrightarrow T_{2,E}(\mathbb{A}_D) \times \mathbb{A}_D^\times/D^\times \longrightarrow T_{2,E}(\mathbb{A}_D)/T_{2,E}(D) \longrightarrow (0)$$

from which we deduce that there is a left exact sequence

$$\begin{aligned}(0) \longrightarrow H^2(\mathrm{Gal}\,(D/F), \mathcal{E}^{\mathrm{glob}}(E/F)^0_D) \longrightarrow H^2(\mathrm{Gal}\,(D/F), T_{2,E}(\mathbb{A}_D)) \oplus H^2(\mathrm{Gal}\,(D/F), \mathbb{A}_D^\times/D^\times) \\ \longrightarrow \textstyle\prod_{v\in V_F} H^2(\mathrm{Gal}\,(D/F)_{w(v)}, \mathbb{A}_D^\times/D^\times),\end{aligned}$$

where $w(v)$ is a choice of place of E above v. Moreover

$$\eta^0_{D/E} : \mathcal{E}^{\mathrm{glob}}(D/F)^0 \longrightarrow \mathcal{E}^{\mathrm{glob}}(E/F)^0_D.$$

This is compatible with the $[D:E]$-power map $\mathbb{A}_D^\times/D^\times$ to itself. Then we see that

$$\inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)}[\alpha_{E/F}^{\mathrm{glob}}] = \eta_{D/E,*}^0[\alpha_{D/F}^{\mathrm{glob}}] \in H^2(\mathrm{Gal}\,(D/F), \mathcal{E}^{\mathrm{glob}}(E/F)_D^0).$$

(Indeed from the above injectivity, this reduces to showing that

$$\inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)}[\alpha_{E/F}^{\mathrm{loc}}] = \eta_{D/E,*}^0[\alpha_{D/F}^{\mathrm{loc}}] \in H^2(\mathrm{Gal}\,(D/F), T_{2,E}(\mathbb{A}_D))$$

and

$$\inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)}[\alpha_{E/F}^{W}] = [D:E][\alpha_{D/F}^{W}] \in H^2(\mathrm{Gal}\,(D/F), \mathbb{A}_D^\times/D^\times);$$

both of which we have already observed.) We will denote this class

$$[\alpha_{E/F,D}^{\mathrm{glob}}] \in H^2(\mathrm{Gal}\,(D/F), \mathcal{E}^{\mathrm{glob}}(E/F)_D^0).$$

If $\alpha \in [\alpha_{E/F,D}^{\mathrm{glob}}]$, then we get extensions

$$\mathcal{E}_3(E/F)_{D,\alpha} \longleftarrow \mathcal{E}^{\mathrm{glob}}(E/F)_{D,\alpha} \longrightarrow W_{E/F,D,\alpha}$$

where

$$(0) \longrightarrow \mathcal{E}^{\mathrm{glob}}(E/F)_D^0 \longrightarrow \mathcal{E}^{\mathrm{glob}}(E/F)_{D,\alpha} \longrightarrow \mathrm{Gal}\,(D/F) \longrightarrow (0)$$

and

$$(0) \longrightarrow T_{3,E}(D) \longrightarrow \mathcal{E}_3(E/F)_{D,\alpha} \longrightarrow \mathrm{Gal}\,(D/F) \longrightarrow (0)$$

and

$$(0) \longrightarrow \mathbb{A}_D^\times/D^\times \longrightarrow W_{E/F,D,\alpha} \longrightarrow \mathrm{Gal}\,(D/F) \longrightarrow (0).$$

Moreover there is an isomorphism of extensions $W_{E/F,D,\alpha} \xrightarrow{\sim} W_{E^{\mathrm{ab}}/F,D}$, well defined up to composition with conjugation by an element of $\mathbb{A}_D^\times/D^\times$.

Also note that

$$\mathrm{res}_{\mathrm{Gal}\,(D/E)}^{\mathrm{Gal}\,(D/F)}[\alpha_{D/F}^{\mathrm{glob}}] = [\alpha_{D/E}^{\mathrm{glob}}] \in H^2(\mathrm{Gal}\,(D/E), \mathcal{E}^{\mathrm{glob}}(D/F)^0).$$

(This is proved in the same manner as the corresponding assertions in section 5.1.)

We will consider the collection $\mathcal{Z}(E/F)_D$ of triples

$$\boldsymbol{\alpha} = (\alpha^{\mathrm{glob}}, \alpha^{\mathrm{loc}}, \beta)$$

where

- $\alpha^{\mathrm{glob}} \in [\alpha_{E/F,D}^{\mathrm{glob}}]$,
- $\alpha^{\mathrm{loc}} \in [\alpha_{E/F,D}^{\mathrm{loc}}]$,
- and $\beta : \mathrm{Gal}\,(D/F) \to T_{2,E}(\mathbb{A}_D)$ with

$$^{\beta}\alpha^{\mathrm{loc}} = \alpha^{\mathrm{glob}} \in Z^2(\mathrm{Gal}\,(D/F), T_{2,E}(\mathbb{A}_D)).$$

To $\boldsymbol{\alpha}$ we can associate a diagram of extensions:

$$\begin{array}{ccccc} & & \mathcal{E}_3(E/F)_{D,\boldsymbol{\alpha}} & \twoheadleftarrow & \mathcal{E}^{\rm glob}(E/F)_{D,\boldsymbol{\alpha}} \twoheadrightarrow W_{E/F,D,\boldsymbol{\alpha}} \\ & & & & \mathrm{loc}_{\boldsymbol{\alpha}} = i_\beta^{-1} \downarrow \\ & & \mathcal{E}^{\rm loc}(E/F)_{D,\boldsymbol{\alpha}} & \hookrightarrow & \mathcal{E}_2(E/F)_{D,\boldsymbol{\alpha}} \\ & & \bigcup & & \\ W_{E_w/F_v,D,\boldsymbol{\alpha}} & \twoheadleftarrow & \mathcal{E}^{\rm loc}(E/F)_{D,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_w} & & \\ & & \bigcup & & \\ W_{E_w/F_v,D_u,\boldsymbol{\alpha}} & \twoheadleftarrow & \mathcal{E}^{\rm loc}(E/F)_{D,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_u}. & & \end{array}$$

We will call two triples $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_1 \in \mathcal{Z}(E/F)_D$ *equivalent* if there exist $\gamma^{\rm glob}$: $\mathrm{Gal}\,(D/F) \to \mathcal{E}^{\rm glob}(E/F)_D^0$ and $\gamma^{\rm loc} : \mathrm{Gal}\,(D/F) \to \prod_{u\in V_D} D_u^\times$ with

$$(\alpha_1^{\rm glob}, \alpha_1^{\rm loc}, \beta_1) = ({}^{\gamma^{\rm glob}}\alpha^{\rm glob}, {}^{\gamma^{\rm loc}}\alpha^{\rm loc}, \gamma^{\rm glob}\beta(\gamma^{\rm loc})^{-1}).$$

In this case the choice of $\gamma^{\rm glob}$ and $\gamma^{\rm loc}$ is unique. (Indeed as $H^1(\mathrm{Gal}\,(D/F), \prod_u D_u^\times) = (0)$ and $H^1(\mathrm{Gal}\,(D/F), \mathcal{E}^{\rm glob}(E/F)_D^0) = (0)$ the only possibility would be to replace $\gamma^{\rm glob}$ by ${}^a\gamma^{\rm glob}$ and $\gamma^{\rm loc}$ by ${}^b\gamma^{\rm loc}$ for some $a \in \mathcal{E}^{\rm glob}(E/F)_D^0$ and $b \in \prod_u D_u^\times$ with $a/b \in T_{2,E}(\mathbb{A}_F)$. Using the embedding $\iota_{D/E}^0 : \mathcal{E}^{\rm glob}(E/F)_D^0 \hookrightarrow \mathcal{E}^{\rm glob}(D/F)^0$ and lemma 5.1 we see that $a, b \in T_{2,E}(\mathbb{A}_F)$ and so ${}^a\gamma^{\rm glob} = \gamma^{\rm glob}$ and ${}^b\gamma^{\rm loc} = \gamma^{\rm loc}$.) Thus the various extensions attached to $\boldsymbol{\alpha}$ only depend, up to canonical isomorphism, on the equivalence class of $\boldsymbol{\alpha}$. We will write $\mathcal{H}(E/F)_D$ for the set of equivalence classes. Thus to $\mathfrak{a} \in \mathcal{H}(E/F)_D$ we have a well-defined diagram of extensions

$$\begin{array}{ccccc} & & \mathcal{E}_3(E/F)_{D,\mathfrak{a}} & \twoheadleftarrow & \mathcal{E}^{\rm glob}(E/F)_{D,\mathfrak{a}} \twoheadrightarrow W_{E/F,D,\mathfrak{a}} \\ & & & & \mathrm{loc}_{\mathfrak{a}} \downarrow \\ & & \mathcal{E}^{\rm loc}(E/F)_{D,\mathfrak{a}} & \hookrightarrow & \mathcal{E}_2(E/F)_{D,\mathfrak{a}} \\ & & \bigcup & & \\ W_{E_w/F_v,D,\mathfrak{a}} & \twoheadleftarrow & \mathcal{E}^{\rm loc}(E/F)_{D,\mathfrak{a}}|_{\mathrm{Gal}\,(D/F)_w} & & \\ & & \bigcup & & \\ W_{E_w/F_v,D_u,\mathfrak{a}} & \twoheadleftarrow & \mathcal{E}^{\rm loc}(E/F)_{D,\mathfrak{a}}|_{\mathrm{Gal}\,(D/F)_u}. & & \end{array}$$

The set of such triples $\mathcal{Z}(E/F)_D$ has an action of $T_{2,E}(\mathbb{A}_D)$ via

$${}^t(\alpha^{\rm glob}, \alpha^{\rm loc}, \beta) = (\alpha^{\rm glob}, \alpha^{\rm loc}, {}^t\beta).$$

This action descends to an action on $\mathcal{H}(E/F)_D$, and is transitive on the set of equivalence classes. (The latter assertion because $H^1(\mathrm{Gal}\,(D/F), T_{2,E}(\mathbb{A}_D)) = (0)$.) The 4-tuples $\boldsymbol{\alpha}$ and ${}^t\boldsymbol{\alpha}$ are equivalent if and only if $t \in T_{2,E}(\mathbb{A}_F)\mathcal{E}^{\rm glob}(E/F)_D^0 \prod_u D_u^\times$. (This follows because $H^1(\mathrm{Gal}\,(D/F), \prod_u D_u^\times) = (0)$ and $H^1(\mathrm{Gal}\,(D/F), \mathcal{E}^{\rm glob}(E/F)_D^0) = (0)$, so that in any equivalence $\gamma^{\rm loc}$ and $\gamma^{\rm glob}$ must be coboundaries.) There are canonical identifications

$$\mathfrak{z}_t : \mathcal{E}_{\boldsymbol{\alpha}}(E/F)_D \xrightarrow{\sim} \mathcal{E}_{{}^t\boldsymbol{\alpha}}(E/F)_D$$

for each of the extensions we have considered, simply because they are defined by the same cocycles. It is easily checked that these identifications are compatible with the identifications of the extensions for two equivalent 4-tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_1$. Thus for $\mathfrak{a} \in \mathcal{H}(E/F)_D$ we get canonical identifications

$$\mathfrak{z}_t : \mathcal{E}_{\mathfrak{a}}(E/F)_D \xrightarrow{\sim} \mathcal{E}_{{}^t\mathfrak{a}}(E/F)_D$$

for each of the extensions we have considered. We have commutative diagrams

$$\begin{array}{ccccccc} W_{E_w/F_v,D,\mathfrak{a}} & \longleftarrow & \mathcal{E}^{\mathrm{loc}}(E/F)_{D,\mathfrak{a}}|_{\mathrm{Gal}\,(D/F)_w} & \subset & \mathcal{E}^{\mathrm{loc}}(E/F)_{D,\mathfrak{a}} & \hookrightarrow & \mathcal{E}_2(E/F)_{D,\mathfrak{a}} \\ \mathfrak{z}_t \downarrow \wr & & \mathfrak{z}_t \downarrow \wr & & \mathfrak{z}_t \downarrow \wr & & \mathfrak{z}_t \downarrow \wr \\ W_{E_w/F_v,D,{}^t\mathfrak{a}} & \longleftarrow & \mathcal{E}^{\mathrm{loc}}(E/F)_{D,{}^t\mathfrak{a}}|_{\mathrm{Gal}\,(D/F)_w} & \subset & \mathcal{E}^{\mathrm{loc}}(E/F)_{D,{}^t\mathfrak{a}} & \hookrightarrow & \mathcal{E}_2(E/F)_{D,{}^t\mathfrak{a}}, \end{array}$$

and

$$\begin{array}{ccccc} \mathcal{E}_3(E/F)_{D,\mathfrak{a}} & \longleftarrow & \mathcal{E}^{\mathrm{glob}}(E/F)_{D,\mathfrak{a}} & \twoheadrightarrow & W_{E/F,D,\mathfrak{a}} \\ \mathfrak{z}_t \downarrow \wr & & \mathfrak{z}_t \downarrow \wr & & \mathfrak{z}_t \downarrow \wr \\ \mathcal{E}_3(E/F)_{D,{}^t\mathfrak{a}} & \longleftarrow & \mathcal{E}^{\mathrm{glob}}(E/F)_{D,{}^t\mathfrak{a}} & \twoheadrightarrow & W_{E/F,D,{}^t\mathfrak{a}}. \end{array}$$

However

$$\mathfrak{z}_t \circ \mathrm{loc}_{\mathfrak{a}} = \mathrm{conj}_t \circ \mathrm{loc}_{{}^t\mathfrak{a}} \circ \mathfrak{z}_t.$$

Suppose that w is a place of E above a place v of F. Then

$$(\alpha^{\mathrm{glob}} \bmod T_{2,E}(D))|_{\mathrm{Gal}\,(D/F)_w^2} = {}^{\pi_w\beta|_{\mathrm{Gal}\,(D/F)_w}}\alpha_w^{\mathrm{loc}},$$

so that there is a map of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & D_w^\times & \longrightarrow & W_{E_w/F_v,D,\boldsymbol{\alpha}} & \longrightarrow & \mathrm{Gal}\,(D/F)_w & \longrightarrow & (0) \\ & & \downarrow & & \iota_w^{\boldsymbol{\alpha}} \downarrow & & || & & \\ (0) & \longrightarrow & \mathbb{A}_D^\times/D^\times & \longrightarrow & W_{E/F,D,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_w} & \longrightarrow & \mathrm{Gal}\,(D/F)_w & \longrightarrow & (0), \end{array}$$

where

$$\iota_w^{\boldsymbol{\alpha}}(e_{\alpha_w^{\mathrm{loc}}}(\sigma)) = \pi_w(\beta(\sigma))e^{\mathrm{glob}}(\sigma).$$

We have

$$i_{\gamma^{\mathrm{glob}}} \circ \iota_w^{\boldsymbol{\alpha}} = \iota_w^{(\gamma^{\mathrm{glob}},\gamma^{\mathrm{glob}})\boldsymbol{\alpha}} \circ i_{\pi_w \circ \gamma^{\mathrm{loc}}}$$

and so we can write

$$\iota_w^{\mathfrak{a}} : W_{E_w/F_v,D,\mathfrak{a}} \longrightarrow W_{E/F,D,\mathfrak{a}}|_{\mathrm{Gal}\,(D/F)_w}.$$

Moreover if $t \in T_{2,E}(\mathbb{A}_D)$ then

$$\mathrm{conj}_{t_w} \circ \mathfrak{z}_t \circ \iota_w^{\mathfrak{a}} = \iota_w^{{}^t\mathfrak{a}} \circ \mathfrak{z}_t.$$

There are isomorphisms of extensions $W_{E/F,D,\mathfrak{a}} \xrightarrow{\sim} W_{E^{\mathrm{ab}}/F,D}$ well defined up to composition with conjugation by an elements of $\mathbb{A}_D^\times/D^\times$; and $W_{E_w/F_v,D,\mathfrak{a}}|_{\mathrm{Gal}\,(D/F)_u} \xrightarrow{\sim} W_{E_w^{\mathrm{ab}}/F_v,D}$ well defined up to composition with conjugation by an elements of $D_w^\times$; and $W_{E_w/F_v,D_u,\mathfrak{a}} \xrightarrow{\sim} W_{E_w^{\mathrm{ab}}/F_v,D_u}$ well defined up to composition with conjugation by an elements of $D_u^\times$.

If $\boldsymbol{\alpha} = (\alpha^{\mathrm{glob}}, \alpha^{\mathrm{loc}}, \beta) \in \mathcal{Z}(E/F)$, then we define

$$\inf_{D/E} \boldsymbol{\alpha} = (\inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)} \alpha^{\mathrm{glob}}, \inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)} \alpha^{\mathrm{loc}}, \inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)} \beta) \in \mathcal{Z}(E/F)_D.$$

Thus
$$\mathcal{E}^{\mathrm{loc}}(E/F)_{D,\inf_{D/E}\boldsymbol{\alpha}} \cong (\prod_{u\in V_D} D_u^\times \rtimes \mathcal{E}^{\mathrm{loc}}(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)})/\prod_{w\in V_E} E_w^\times,$$
where
$$\begin{array}{rcl} \prod_{w\in V_E} E_w^\times & \longrightarrow & \prod_{u\in V_D} D_u^\times \rtimes \mathcal{E}^{\mathrm{loc}}(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)} \\ (e_w) & \longmapsto & ((e_{u|_E})^{-1},((e_w),1)); \end{array}$$
and
$$\mathcal{E}_2(E/F)_{D,\inf_{D/E}\boldsymbol{\alpha}} \cong (T_{2,E}(\mathbb{A}_D) \rtimes \mathcal{E}_2(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)})/T_{2,E}(\mathbb{A}_E),$$
where
$$\begin{array}{rcl} T_{2,E}(\mathbb{A}_E) & \longrightarrow & T_{2,E}(\mathbb{A}_D) \rtimes \mathcal{E}_2(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)} \\ a & \longmapsto & (a^{-1},(a,1)); \end{array}$$
and
$$W_{E_w/F_v,D,\inf_{D/E}\boldsymbol{\alpha}} \cong (D_w^\times \rtimes W_{E_w/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_w})/E_w^\times,$$
where
$$\begin{array}{rcl} E_w^\times & \longrightarrow & D_w^\times \rtimes W_{E_w/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_w} \\ e & \longmapsto & (e^{-1},(e,1)); \end{array}$$
and
$$W_{E_w/F_v,D_u,\inf_{D/E}\boldsymbol{\alpha}} \cong (D_u^\times \rtimes W_{E_w/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_u})/E_w^\times,$$
where
$$\begin{array}{rcl} E_w^\times & \longrightarrow & D_u^\times \rtimes W_{E_w/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_u} \\ e & \longmapsto & (e^{-1},(e,1)); \end{array}$$
and
$$\mathcal{E}^{\mathrm{glob}}(E/F)_{D,\inf_{D/E}\boldsymbol{\alpha}} \cong (\mathcal{E}^{\mathrm{glob}}(E/F)^0_D \rtimes \mathcal{E}^{\mathrm{glob}}(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/E)})/\mathcal{E}^{\mathrm{glob}}(E/F)^0,$$
where
$$\begin{array}{rcl} \mathcal{E}^{\mathrm{glob}}(E/F)^0 & \longrightarrow & \mathcal{E}^{\mathrm{glob}}(E/F)^0_D \rtimes \mathcal{E}^{\mathrm{glob}}(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/E)} \\ a & \longmapsto & (a^{-1},(a,1)); \end{array}$$
and
$$\mathcal{E}_3(E/F)_{D,\inf_{D/E}\boldsymbol{\alpha}} \cong (T_{E,3}(D) \rtimes \mathcal{E}_3(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)})/T_{E,3}(E),$$
where
$$\begin{array}{rcl} T_{E,3}(E) & \longrightarrow & T_{E,3}(D) \rtimes \mathcal{E}_3(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)} \\ a & \longmapsto & (a^{-1},(a,1)); \end{array}$$
and
$$W_{E/F,D,\inf_{D/E}\boldsymbol{\alpha}} \cong (\mathbb{A}_D^\times/D^\times \rtimes W_{E/F,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)})/(\mathbb{A}_E^\times/E^\times),$$
where
$$\begin{array}{rcl} \mathbb{A}_E^\times/E^\times & \longrightarrow & \mathbb{A}_D^\times/D^\times \rtimes W_{E/F,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)} \\ a & \longmapsto & (a^{-1},(a,1)). \end{array}$$

In each case we have natural maps of extensions
$$\xi_{D/E} : \mathcal{E}^?(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)} \longrightarrow \mathcal{E}^?(E/F)_{D,\inf_{D/E}\boldsymbol{\alpha}}.$$

These give commutative diagrams

$$\begin{array}{ccccccc} W_{E_w/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_w} & \longleftarrow & \mathcal{E}^{\mathrm{loc}}(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_w} & \subset & \mathcal{E}^{\mathrm{loc}}(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)} & \hookrightarrow & \mathcal{E}_2(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)} \\ \xi_{D/E}\downarrow && \xi_{D/E}\downarrow && \xi_{D/E}\downarrow && \xi_{D/E}\downarrow \\ W_{E_w/F_v,D,\inf_{D/E}\boldsymbol{\alpha}} & \longleftarrow & \mathcal{E}^{\mathrm{loc}}(E/F)_{D,\inf_{D/E}\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_w} & \subset & \mathcal{E}^{\mathrm{loc}}(E/F)_{D,\inf_{D/E}\boldsymbol{\alpha}} & \hookrightarrow & \mathcal{E}_2(E/F)_{D,\inf_{D/E}\boldsymbol{\alpha}}, \end{array}$$

and

$$\begin{array}{ccccc} \mathcal{E}_3(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)} & \longleftarrow & \mathcal{E}^{\mathrm{glob}}(E/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)} & \twoheadrightarrow & W_{E/F,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)} \\ \xi_{D/E}\downarrow && \xi_{D/E}\downarrow && \xi_{D/E}\downarrow \\ \mathcal{E}_3(E/F)_{D,\inf_{D/E}\boldsymbol{\alpha}} & \longleftarrow & \mathcal{E}^{\mathrm{glob}}(E/F)_{D,\inf_{D/E}\boldsymbol{\alpha}} & \twoheadrightarrow & W_{E/F,D,\inf_{D/E}\boldsymbol{\alpha}}. \end{array}$$

Moreover they commute with the maps $\mathrm{loc}_{\boldsymbol{\alpha}}$ and $\mathrm{loc}_{\inf_{D/E}\boldsymbol{\alpha}}$.

We have $\inf_{D/E} {}^{\gamma}\boldsymbol{\alpha} = {}^{\inf_{D/E}\gamma} \inf_{D/E}\boldsymbol{\alpha}$, and so we get a map

$$\inf_{D/E} : \mathcal{H}(E/F) \longrightarrow \mathcal{H}(E/F)_D.$$

Moreover $i_{\inf_{D/E}\boldsymbol{\gamma}} = (1,(i_{\boldsymbol{\gamma}},1))$. In particular the maps $\xi_{D/E}$ for different $\boldsymbol{\alpha} \in \mathfrak{a}$ become identified. We also have

$$\inf_{D/E} {}^t\boldsymbol{\alpha} = {}^t \inf_{D/E} \boldsymbol{\alpha}.$$

The map $\mathfrak{z}_t$ for $\inf_{D/E}\boldsymbol{\alpha}$ is identified with $(1,(\mathfrak{z}_t,1))$; and the map $\iota_w^{\inf_{D/E}\boldsymbol{\alpha}}$ is identified with $(1,(\iota_w^{\boldsymbol{\alpha}},1))$. Thus

$$\xi_{D/E}\circ\mathfrak{z}_t = \mathfrak{z}_t\circ\xi_{D/E}$$

and

$$\iota_w^{\inf_{D/E}\boldsymbol{\alpha}}\circ\xi_{D/E} = \xi_{D/E}\circ\iota_w^{\boldsymbol{\alpha}}.$$

Next suppose that $\boldsymbol{\alpha} = (\alpha^{\mathrm{glob}},\alpha^{\mathrm{loc}},\beta) \in \mathcal{Z}(D/F)$. We define

$$\eta_{D/E,*}\boldsymbol{\alpha} = (\eta_{D/E,*}\alpha^{\mathrm{glob}},\eta_{D/E,*}\alpha^{\mathrm{loc}},\eta_{D/E}(\beta)) \in \mathcal{Z}(E/F)_D.$$

Thus

$$\mathcal{E}^{\mathrm{loc}}(E/F)_{D,\eta_{D/E,*}\boldsymbol{\alpha}} \cong (\prod_{u\in V_D} D_u^\times \rtimes \mathcal{E}^{\mathrm{loc}}(D/F)_{\boldsymbol{\alpha}})/\prod_{u\in V_D} D_u^\times,$$

where

$$\begin{array}{rcl} \prod_{u\in V_D} D_u^\times & \longrightarrow & \prod_{u\in V_D} D_u^\times \rtimes \mathcal{E}^{\mathrm{loc}}(D/F)_{\boldsymbol{\alpha}} \\ (d_u) & \longmapsto & ((d_u^{-[D_u:E_{u|_E}]}),(d_u)); \end{array}$$

and

$$\mathcal{E}_2(E/F)_{D,\eta_{D/E,*}\boldsymbol{\alpha}} \cong (T_{2,E}(\mathbb{A}_D)\rtimes\mathcal{E}_2(D/F)_{\boldsymbol{\alpha}})/T_{2,D}(\mathbb{A}_D),$$

where

$$\begin{array}{rcl} T_{2,D}(\mathbb{A}_D) & \longrightarrow & T_{2,E}(\mathbb{A}_D)\rtimes\mathcal{E}_2(D/F)_{\boldsymbol{\alpha}} \\ a & \longmapsto & (\eta_{D/E}(a)^{-1},a); \end{array}$$

and

$$W_{E_w/F_v,D,\eta_{D/E,*}\boldsymbol{\alpha}} \cong (D_w^\times \rtimes \mathcal{E}^{\mathrm{loc}}(D/F)_{\boldsymbol{\alpha}})/\prod_{u\in V_D} D_u^\times,$$

where

$$\begin{array}{rcl} \prod_{u\in V_D} D_u^\times & \longrightarrow & D_w^\times \rtimes \mathcal{E}^{\mathrm{loc}}(D/F)_{\boldsymbol{\alpha}} \\ (d_u) & \longmapsto & ((d_u^{-[D_u:E_w]})_{u|w}, (d_u)); \end{array}$$

and

$$W_{E_w/F_v,D_u,\eta_{D/E,*}\boldsymbol{\alpha}} \cong (D_u^\times \rtimes W_{D_u/F_v,\boldsymbol{\alpha}})/D_u^\times,$$

where

$$\begin{array}{rcl} D_u^\times & \longrightarrow & D_u^\times \rtimes W_{D_u/F_v,\boldsymbol{\alpha}} \\ d & \longmapsto & (d^{-[D_u:E_w]}), d); \end{array}$$

and

$$\mathcal{E}^{\mathrm{glob}}(E/F)_{D,\eta_{D/E,*}\boldsymbol{\alpha}} \cong (\mathcal{E}^{\mathrm{glob}}(E/F)^0_D \rtimes \mathcal{E}^{\mathrm{glob}}(D/F)_{\boldsymbol{\alpha}})/\mathcal{E}^{\mathrm{glob}}(D/F)^0,$$

where

$$\begin{array}{rcl} \mathcal{E}^{\mathrm{glob}}(D/F)^0 & \longrightarrow & \mathcal{E}^{\mathrm{glob}}(E/F)^0_D \rtimes \mathcal{E}^{\mathrm{glob}}(D/F)_{\boldsymbol{\alpha}} \\ a & \longmapsto & (\eta_{D/E}(a)^{-1}, a); \end{array}$$

and

$$\mathcal{E}_3(E/F)_{D,\eta_{D/E,*}\boldsymbol{\alpha}} \cong (T_{E,3}(D) \rtimes \mathcal{E}_3(D/F)_{\boldsymbol{\alpha}})/T_{D,3}(D),$$

where

$$\begin{array}{rcl} T_{D,3}(D) & \longrightarrow & T_{E,3}(D) \rtimes \mathcal{E}_3(D/F)_{\boldsymbol{\alpha}} \\ a & \longmapsto & (\eta_{D/E}(a)^{-1}, a); \end{array}$$

and

$$W_{E/F,D,\eta_{D/E,*}\boldsymbol{\alpha}} \cong (\mathbb{A}_D^\times/D^\times \rtimes W_{D/F,\boldsymbol{\alpha}})/(\mathbb{A}_D^\times/D^\times),$$

where

$$\begin{array}{rcl} \mathbb{A}_D^\times/D^\times & \longrightarrow & \mathbb{A}_D^\times/D^\times \rtimes W_{D/F,\boldsymbol{\alpha}} \\ a & \longmapsto & (a^{-[D:E]}, a). \end{array}$$

In each case we have natural maps of extensions

$$\zeta_{D/E} : \mathcal{E}^?(D/F)_{\boldsymbol{\alpha}} \longrightarrow \mathcal{E}^?(E/F)_{D,\eta_{D/E,*}\boldsymbol{\alpha}}.$$

These give commutative diagrams

$$\begin{array}{ccccccc} W_{D_u/F_v,\boldsymbol{\alpha}} & \longleftarrow & \mathcal{E}^{\mathrm{loc}}(D/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_u} & \subset & \mathcal{E}^{\mathrm{loc}}(D/F)_{\boldsymbol{\alpha}} & \hookrightarrow & \mathcal{E}_2(D/F)_{\boldsymbol{\alpha}} \\ \zeta_{D/E}\downarrow & & \zeta_{D/E}\downarrow & & \zeta_{D/E}\downarrow & & \zeta_{D/E}\downarrow \\ W_{E_w/F_v,D_u,\eta_{D/E,*}\boldsymbol{\alpha}} & \longleftarrow & \mathcal{E}^{\mathrm{loc}}(E/F)_{D,\eta_{D/E,*}\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_u} & \subset & \mathcal{E}^{\mathrm{loc}}(E/F)_{D,\eta_{D/E,*}\boldsymbol{\alpha}} & \hookrightarrow & \mathcal{E}_2(E/F)_{D,\eta_{D/E,*}\boldsymbol{\alpha}} \end{array}$$

(of course we may also replace $W_{E_w/F_v,D_u,\eta_{D/E,*}\boldsymbol{\alpha}}$ with the isomorphic

$$\left. W_{E_w/F_v,D,\eta_{D/E,*}\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_u} / \prod_{u\neq u'|w} D_{u'}^\times \right),$$

and

$$\begin{array}{ccccc} \mathcal{E}_3(D/F)_{\boldsymbol{\alpha}} & \longleftarrow & \mathcal{E}^{\mathrm{glob}}(D/F)_{\boldsymbol{\alpha}} & \twoheadrightarrow & W_{D/F,\boldsymbol{\alpha}} \\ \zeta_{D/E}\downarrow & & \zeta_{D/E}\downarrow & & \zeta_{D/E}\downarrow \\ \mathcal{E}_3(E/F)_{D,\eta_{D/E,*}\boldsymbol{\alpha}} & \longleftarrow & \mathcal{E}^{\mathrm{glob}}(E/F)_{D,\eta_{D/E,*}\boldsymbol{\alpha}} & \twoheadrightarrow & W_{E/F,D,\eta_{D/E,*}\boldsymbol{\alpha}}. \end{array}$$

Moreover they commute with the maps $\mathrm{loc}_{\boldsymbol{\alpha}}$ and $\mathrm{loc}_{\eta_{D/E,*}\boldsymbol{\alpha}}$.

We have $\eta_{D/E,*}{}^{\gamma}\boldsymbol{\alpha} = {}^{\eta_{D/E,*}\gamma}\eta_{D/E,*}\boldsymbol{\alpha}$ and so we get a map

$$\eta_{D/E,*} : \mathcal{H}(D/F) \longrightarrow \mathcal{H}(E/F)_D.$$

Moreover $i_{\eta_{D/E,*}\gamma} = (\eta_{D/E}, i_{\gamma})$. In particular $\zeta_{D/E}$ for the different $\boldsymbol{\alpha} \in \mathfrak{a}$ become identified. We also have

$$\eta_{D/E,*}{}^{t}\boldsymbol{\alpha} = {}^{\eta_{D/E}(t)}\eta_{D/E,*}\boldsymbol{\alpha}.$$

The map $\mathfrak{z}_{\eta_{D/E}(t)}$ for $\eta_{D/E,*}\boldsymbol{\alpha}$ is identified with $(1, \mathfrak{z}_t)$, and so

$$\zeta_{D/E} \circ \mathfrak{z}_t = \mathfrak{z}_{\eta_{D/E}(t)} \circ \zeta_{D/E}.$$

The compatibility of $\iota_w^{\boldsymbol{\alpha}}$ and $\iota_w^{\eta_{D/E,*}\boldsymbol{\alpha}}$ is harder to describe:

Lemma 5.5. *There is an isomorphism*

$$W_{E_w/F_v,D,\eta_{D/E,*}\boldsymbol{\alpha}} \cong ((D_w^\times \rtimes \mathrm{Gal}\,(D/E)) \rtimes W_{D_u/F_v,\boldsymbol{\alpha}})/W_{D_u/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/E)_u},$$

where

$$\begin{array}{rcl} W_{D_u/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/E)_u} & \longrightarrow & ((D_w^\times \rtimes \mathrm{Gal}\,(D/E)) \rtimes W_{D_u/F_v,\boldsymbol{\alpha}}) \\ \sigma & \longmapsto & ((\epsilon(\sigma)^{-1}, \sigma^{-1}), \sigma), \end{array}$$

and

$$\epsilon : W_{D_u/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/E)_u} \longrightarrow E_w^\times,$$

which we embed diagonally in $D_w^\times$, is the homomorphism sending

$$\sigma \longmapsto \prod_{\eta \in \mathrm{Gal}\,(D_u/E_w)} s_\eta \sigma s_{\eta\sigma}^{-1}$$

for any section $s : \mathrm{Gal}\,(D_u/E_w) \to W_{D_u/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/E)_u}$. The isomorphism sends $((a,\sigma), be^{\mathrm{loc}}(\tau)) \in ((D_w^\times \rtimes \mathrm{Gal}\,(D/E)) \rtimes W_{D_u/F_v,\boldsymbol{\alpha}})$ to

$$\Big[\Big(a\textstyle\prod_{\eta\in\mathrm{Gal}\,(D/E)} \alpha^{\mathrm{loc}}(\sigma,\eta)|_{\sigma\eta u}^{-1}, e^{\mathrm{loc}}(\sigma)\Big)\Big(\prod_{\eta\in\mathrm{Gal}\,(D/E)}(\eta(b)\alpha^{\mathrm{loc}}(\eta,\tau)/b)|_{\eta u}\alpha^{\mathrm{loc}}(\tau,\eta)|_{\tau\eta u}^{-1}, be^{\mathrm{loc}}(\tau)\Big)\Big]$$
$$\in (D_w^\times \rtimes \mathcal{E}^{\mathrm{loc}}(D/F)_{\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/F)_w})/\textstyle\prod_x D_x^\times,$$

where if $c \in \prod_x D_x^\times$ we write $c|_x \in \prod_x D_x^\times$ for the element that is c_x at x and 1 elsewhere.

Moreover the map

$$\iota_w^{\eta_{D/E,*}\boldsymbol{\alpha}} : ((D_w^\times \rtimes \mathrm{Gal}\,(D/E)) \rtimes W_{D_u/F_v,\boldsymbol{\alpha}})/W_{D_u/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/E)_u} \longrightarrow (\mathbb{A}_D^\times/D^\times \rtimes W_{D/F})/(\mathbb{A}_D^\times/D^\times)$$

sends

$$[((a,1),1)] \longmapsto [(a,1)]$$

and

$$[((1,\sigma),1)] \longmapsto \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\eta)_{\eta u}}[(\prod_{\eta\in\mathrm{Gal}\,(D/E)} e_{\boldsymbol{\alpha}}^{\mathrm{glob}}(\sigma\eta)e_{\boldsymbol{\alpha}}^{\mathrm{glob}}(\eta)^{-1}\widetilde{\sigma}^{-1}, \widetilde{\sigma})],$$

where $\widetilde{\sigma}$ is any lift of σ to $W_{D/F,\boldsymbol{\alpha}}$, and $[((1,1),\tau)]$ to

$$\mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\eta)_{\eta u}}[(\prod_{\eta\in\mathrm{Gal}\,(D/E)} (e_{\boldsymbol{\alpha}}^{\mathrm{glob}}(\eta)\iota_u^{\boldsymbol{\alpha}}(\tau)e_{\boldsymbol{\alpha}}^{\mathrm{glob}}(\eta\tau)^{-1})/(\iota_u^{\boldsymbol{\alpha}}(\tau)e_{\boldsymbol{\alpha}}^{\mathrm{glob}}(\eta)e_{\boldsymbol{\alpha}}^{\mathrm{glob}}(\tau\eta)^{-1}), \iota_u^{\boldsymbol{\alpha}}(\tau))].$$

Proof: The map ϵ is independent of the choice of section s, it is a homomorphism and it is valued in $E_w^\times \subset D_u^\times$. The map

$$W_{D_u/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/E)_u} \longrightarrow ((D_w^\times \rtimes \mathrm{Gal}\,(D/E)) \rtimes W_{D_u/F_v,\boldsymbol{\alpha}})$$

is a homomorphism (because ϵ is) and has normal image (again because ϵ is a homomorphism valued in $E_w^\times$). There is an exact sequence

$$(0) \longrightarrow D_w^\times \longrightarrow ((D_w^\times \rtimes \mathrm{Gal}\,(D/E)) \rtimes W_{D_u/F_v,\boldsymbol{\alpha}})/W_{D_u/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/E)_u} \longrightarrow \mathrm{Gal}\,(D/F)_w \longrightarrow (0)$$

because

$$(\mathrm{Gal}\,(D/E) \rtimes \mathrm{Gal}\,(D/F)_u)/\mathrm{Gal}\,(D/E)_u \xrightarrow{\sim} \mathrm{Gal}\,(D/F)_w.$$

The given map

$$((D_w^\times \rtimes \mathrm{Gal}\,(D/E)) \rtimes W_{D_u/F_v,\boldsymbol{\alpha}}) \longrightarrow W_{E_w/F_v,D,\eta_{D/E,*}\boldsymbol{\alpha}}$$

is compatible with the inclusion of $D_w^\times$ and the projection to $\mathrm{Gal}\,(D/F)_w$. Thus it suffices to show that it is a homomorphism that is trivial when restricted to $W_{D_u/F_v,\boldsymbol{\alpha}}|_{\mathrm{Gal}\,(D/E)_u}$. If $\sigma \in \mathrm{Gal}\,(D/E)$, write

$$\gamma(\sigma) = \prod_{\eta \in \mathrm{Gal}\,(D/E)} \alpha^{\mathrm{loc}}(\sigma,\eta)|_{\sigma\eta u}^{-1} \in D_w^\times.$$

If $\tau \in \mathrm{Gal}\,(D/F)_u$ write

$$\delta(\tau) = \prod_{\eta \in \mathrm{Gal}\,(D/E)} \alpha^{\mathrm{loc}}(\eta,\tau)|_{\eta u}/\alpha^{\mathrm{loc}}(\tau,\eta)|_{\tau\eta u}.$$

Then what we must check is the following

- $\gamma(\sigma_1)^{\sigma_1}\gamma(\sigma_2)\alpha^{\mathrm{loc}}(\sigma_1,\sigma_2)_w^{[D_u:E_w]} = \gamma(\sigma_1\sigma_2)$ for all $\sigma_1,\sigma_2 \in \mathrm{Gal}\,(D/E)$.
- $\prod_{\eta \in \mathrm{Gal}\,(D/E)}({}^{\eta}b_1/b_1)|_{\eta u}{}^{\tau_1}\left(\prod_{\eta \in \mathrm{Gal}\,(D/E)}({}^{\eta}b_2/b_2)|_{\eta u}\right) = \prod_{\eta \in \mathrm{Gal}\,(D/E)}({}^{\eta}(b_1{}^{\tau_1}b_2)/b_1{}^{\tau_1}b_2)|_{\eta u}$ for all $b_1,b_2 \in D_u^\times$ and $\tau_1 \in \mathrm{Gal}\,(D/F)_u$.
- $\gamma(\tau_1)^{\tau_1}\gamma(\tau_2)\alpha^{\mathrm{loc}}(\tau_1,\tau_2)_w^{[D_u:E_w]} = \prod_{\eta \in \mathrm{Gal}\,(D/E)}({}^{\eta}(\alpha^{\mathrm{loc}}(\tau_1,\tau_2)|_u)/\alpha^{\mathrm{loc}}(\tau_1,\tau_2)|_u)|_{\eta u}$ $\gamma(\tau_1\tau_2)\alpha^{\mathrm{loc}}(\tau_1,\tau_2)|_u^{[D_u:F_w]}$ for all $\tau_1,\tau_2 \in \mathrm{Gal}\,(D/F)_u$.
- $\epsilon(b)^{-1}\prod_{\eta \in \mathrm{Gal}\,(D/E)}({}^{\eta}b/b)|_{\eta u} b^{[D_u:E_w]} = 1$ for all $b \in D_u^\times$.
- $\epsilon(e_{\boldsymbol{\alpha}}^{\mathrm{loc}}(\sigma))^{-1}\gamma(\sigma^{-1})^{\sigma^{-1}}\delta(\sigma)\alpha^{\mathrm{loc}}(\sigma^{-1},\sigma)_w^{[D_u:F_w]}\alpha^{\mathrm{loc}}(1,1)_w^{[D_u:F_w]} = 1$ for all $\sigma \in \mathrm{Gal}\,(D/E)_u$.

These are equivalent to:

- $\gamma(\sigma_1)^{\sigma_1}\gamma(\sigma_2)\alpha^{\mathrm{loc}}(\sigma_1,\sigma_2)_w^{[D_u:E_w]} = \gamma(\sigma_1\sigma_2)$ for all $\sigma_1,\sigma_2 \in \mathrm{Gal}\,(D/E)$.
- $\prod_{\eta \in \mathrm{Gal}\,(D/E)}({}^{\tau_1\eta}b_2/{}^{\tau_1}b_2)|_{\tau_1\eta u} = \prod_{\eta \in \mathrm{Gal}\,(D/E)}({}^{\eta\tau_1}b_2)/{}^{\tau_1}b_2)|_{\eta u}$ for all $b_2 \in D_u^\times$ and $\tau_1 \in \mathrm{Gal}\,(D/F)_u$.
- $\gamma(\tau_1)^{\tau_1}\gamma(\tau_2) = \prod_{\eta \in \mathrm{Gal}\,(D/E)}({}^{\eta}\alpha^{\mathrm{loc}}(\tau_1,\tau_2)/\alpha^{\mathrm{loc}}(\tau_1,\tau_2))|_{\eta u}\gamma(\tau_1\tau_2)$ for all $\tau_1,\tau_2 \in \mathrm{Gal}\,(D/F)_u$.
- $\epsilon(b) = \prod_{\eta \in \mathrm{Gal}\,(D/E)}({}^{\eta}b)|_{\eta u}$ for all $b \in D_u^\times$.
- $\epsilon(e_{\boldsymbol{\alpha}}^{\mathrm{loc}}(\sigma))^{-1}\gamma(\sigma^{-1})^{\sigma^{-1}}\delta(\sigma)\alpha^{\mathrm{loc}}(\sigma^{-1},\sigma)_w^{[D_u:F_w]}\alpha^{\mathrm{loc}}(1,1)_w^{[D_u:F_w]} = 1$ for all $\sigma \in \mathrm{Gal}\,(D/E)_u$.

For the first of these note that

$$\begin{array}{rl} & \gamma(\sigma_1)^{\sigma_1}\gamma(\sigma_2)\alpha^{\rm loc}(\sigma_1,\sigma_2)_w^{[D_u:E_w]}\gamma(\sigma_1\sigma_2)^{-1} \\ = & \prod_{\eta\in{\rm Gal}\,(D/E)}(\alpha^{\rm loc}(\sigma_1,\sigma_2)|_{\eta u}\alpha^{\rm loc}(\sigma_1\sigma_2,\eta)_{\sigma_1\sigma_2\eta u}/\alpha^{\rm loc}(\sigma_1,\eta)|_{\sigma_1\eta u}({}^{\sigma_1}\alpha^{\rm loc}(\sigma_2,\eta))|_{\sigma_1\sigma_2\eta u}) \\ = & \prod_{\eta\in{\rm Gal}\,(D/E)}(\alpha^{\rm loc}(\sigma_1,\sigma_2)|_{\eta u}\alpha^{\rm loc}(\sigma_1,\sigma_2\eta)_{\sigma_1\sigma_2\eta u}/\alpha^{\rm loc}(\sigma_1,\eta)|_{\sigma_1\eta u}\alpha^{\rm loc}(\sigma_1,\sigma_2)|_{\sigma_1\sigma_2\eta u}) \\ = & \prod_{\eta\in{\rm Gal}\,(D/E)}(\alpha^{\rm loc}(\sigma_1,\sigma_2)|_{\eta u}\alpha^{\rm loc}(\sigma_1,\eta)_{\sigma_1\eta u}/\alpha^{\rm loc}(\sigma_1,\eta)|_{\sigma_1\eta u}\alpha^{\rm loc}(\sigma_1,\sigma_2)|_{\eta u}) \\ = & 1. \end{array}$$

For the second replace η on the left hand side by $\tau_1^{-1}\eta\tau_1$ and note that $\eta\tau_1 u=\eta u$.
For the third note that

$$\begin{array}{rl} & \prod_{\eta\in{\rm Gal}\,(D/E)}\alpha^{\rm loc}(\eta,\tau_1)|_{\eta u}({}^{\tau_1}\alpha^{\rm loc}(\eta,\tau_2))|_{\tau_1\eta u}\alpha^{\rm loc}(\tau_1\tau_2,\eta)|_{\tau_1\tau_2\eta u}/ \\ & \alpha^{\rm loc}(\tau_1,\eta)|_{\tau_1\eta u}({}^{\tau_1}\alpha^{\rm loc}(\tau_2,\eta))|_{\tau_1\tau_2\eta u}\alpha^{\rm loc}(\eta,\tau_1\tau_2)|_{\eta u} \\ = & \prod_{\eta\in{\rm Gal}\,(D/E)}\alpha^{\rm loc}(\eta,\tau_1)|_{\eta u}\alpha^{\rm loc}(\tau_1\eta,\tau_2)|_{\tau_1\eta u}\alpha^{\rm loc}(\tau_1,\eta)|_{\tau_1\eta u}\alpha^{\rm loc}(\tau_1,\tau_2\eta)|_{\tau_1\tau_2\eta u}\alpha^{\rm loc}(\tau_1\tau_2,\eta)|_{\tau_1\tau_2\eta u}/ \\ & \alpha^{\rm loc}(\tau_1,\eta)|_{\tau_1\eta u}\alpha^{\rm loc}(\tau_1,\eta\tau_2)|_{\tau_1\eta u}\alpha^{\rm loc}(\tau_1\tau_2,\eta)|_{\tau_1\tau_2\eta u}\alpha^{\rm loc}(\tau_1,\tau_2)|_{\tau_1\tau_2\eta u}\alpha^{\rm loc}(\eta,\tau_1\tau_2)|_{\eta u} \\ = & \prod_{\eta\in{\rm Gal}\,(D/E)}\alpha^{\rm loc}(\eta,\tau_1)|_{\eta u}\alpha^{\rm loc}(\eta\tau_1,\tau_2)|_{\eta u}\alpha^{\rm loc}(\tau_1,\eta\tau_2)_{\tau_1\eta u}/ \\ & \alpha^{\rm loc}(\tau_1,\eta\tau_2)|_{\tau_1\eta u}\alpha^{\rm loc}(\tau_1,\tau_2)|_{\tau_1\tau_2\eta u}\alpha^{\rm loc}(\eta,\tau_1\tau_2)|_{\eta u} \\ = & \prod_{\eta\in{\rm Gal}\,(D/E)}({}^{\eta}\alpha^{\rm loc}(\tau_1,\tau_2))|_{\eta u}/\alpha^{\rm loc}(\tau_1,\tau_2)|_{\eta u}. \end{array}$$

The fourth is true by the definition of ϵ. For the fifth first note that if $\sigma\in{\rm Gal}\,(D/E)_u$ then

$$\epsilon(e^{\rm loc}_{\boldsymbol\alpha}(\sigma))=\prod_{\eta\in{\rm Gal}\,(D/E)_u}e^{\rm loc}_{\boldsymbol\alpha}(\eta)e^{\rm loc}_{\boldsymbol\alpha}(\sigma)e^{\rm loc}_{\boldsymbol\alpha}(\eta\sigma)^{-1}=\prod_{\eta\in{\rm Gal}\,(D/E)_u}\alpha^{\rm loc}(\eta,\sigma)_u\in E_w^\times\subset D_u^\times.$$

Thus $\epsilon(e^{\rm loc}_{\boldsymbol\alpha}(\sigma))\in D_w^\times$ can be thought of as

$$\begin{array}{rl} & \prod_{\zeta\in{\rm Gal}\,(D/E)/{\rm Gal}\,(D/E)_u}\prod_{\eta\in{\rm Gal}\,(D/E)_u}{}^{\zeta}(\alpha^{\rm loc}(\eta,\sigma)|_u) \\ = & \prod_{\zeta\in{\rm Gal}\,(D/E)/{\rm Gal}\,(D/E)_u}\prod_{\eta\in{\rm Gal}\,(D/E)_u}({}^{\zeta}\alpha^{\rm loc}(\eta,\sigma))|_{\zeta u} \\ = & \prod_{\zeta\in{\rm Gal}\,(D/E)/{\rm Gal}\,(D/E)_u}\prod_{\eta\in{\rm Gal}\,(D/E)_u}(\alpha^{\rm loc}(\zeta\eta,\sigma)\alpha^{\rm loc}(\zeta,\eta)/\alpha^{\rm loc}(\zeta,\eta\sigma))|_{\zeta u} \\ = & \prod_{\zeta\in{\rm Gal}\,(D/E)/{\rm Gal}\,(D/E)_u}\prod_{\eta\in{\rm Gal}\,(D/E)_u}\alpha^{\rm loc}(\zeta\eta,\sigma)|_{\zeta\eta u}\alpha^{\rm loc}(\zeta,\eta)|_{\zeta u}/\alpha^{\rm loc}(\zeta,\eta\sigma)|_{\zeta\sigma u} \\ = & \prod_{\eta\in{\rm Gal}\,(D/E)}\alpha^{\rm loc}(\eta,\sigma)|_{\eta u}. \end{array}$$

Thus for the fifth part we need to show that

$$\begin{array}{rcl} 1 & = & \prod_{\eta\in{\rm Gal}\,(D/E)}({}^{\sigma^{-1}}\alpha^{\rm loc}(\eta,\sigma))|_{\sigma^{-1}\eta u}\alpha^{\rm loc}(\sigma^{-1},\sigma)|_{\eta u}\alpha^{\rm loc}(1,1)|_{\eta u}/ \\ & & \alpha^{\rm loc}(\eta,\sigma)|_{\eta u}\alpha^{\rm loc}(\sigma^{-1},\eta)|_{\sigma^{-1}\eta u}({}^{\sigma^{-1}}\alpha^{\rm loc}(\sigma,\eta))|_{\eta u} \end{array}$$

for all $\sigma\in{\rm Gal}\,(D/E)_u$. However

$$\begin{array}{rl} & \prod_{\eta\in{\rm Gal}\,(D/E)}({}^{\sigma^{-1}}\alpha^{\rm loc}(\eta,\sigma))|_{\sigma^{-1}\eta u}\alpha^{\rm loc}(\sigma^{-1},\sigma)|_{\eta u}\alpha^{\rm loc}(1,1)|_{\eta u}/ \\ & \alpha^{\rm loc}(\eta,\sigma)|_{\eta u}\alpha^{\rm loc}(\sigma^{-1},\eta)|_{\sigma^{-1}\eta u}({}^{\sigma^{-1}}\alpha^{\rm loc}(\sigma,\eta))|_{\eta u} \\ = & \prod_{\eta\in{\rm Gal}\,(D/E)}\alpha^{\rm loc}(\sigma^{-1}\eta,\sigma)|_{\sigma^{-1}\eta u}\alpha^{\rm loc}(\sigma^{-1},\eta)|_{\sigma^{-1}\eta u}\alpha^{\rm loc}(\sigma^{-1},\sigma)|_{\eta u}\alpha^{\rm loc}(1,1)|_{\eta u}\alpha^{\rm loc}(\sigma^{-1},\sigma\eta)_{\eta u}/ \\ & \alpha^{\rm loc}(\sigma^{-1},\eta\sigma)|_{\sigma^{-1}\eta u}\alpha^{\rm loc}(\eta,\sigma)|_{\eta u}\alpha^{\rm loc}(\sigma^{-1},\eta)|_{\sigma^{-1}\eta u}\alpha^{\rm loc}(1,\eta)|_{\eta u}\alpha^{\rm loc}(\sigma^{-1},\sigma)|_{\eta u} \\ = & \prod_{\eta\in{\rm Gal}\,(D/E)}\alpha^{\rm loc}(\eta,\sigma)|_{\eta u}\alpha^{\rm loc}(\sigma^{-1},\eta)|_{\sigma^{-1}\eta u}\alpha^{\rm loc}(\sigma^{-1},\sigma)|_{\eta u}\alpha^{\rm loc}(1,1)|_{\eta u}\alpha^{\rm loc}(\sigma^{-1},\eta)_{\sigma^{-1}\eta u}/ \\ & \alpha^{\rm loc}(\sigma^{-1},\eta)|_{\sigma^{-1}\eta u}\alpha^{\rm loc}(\eta,\sigma)|_{\eta u}\alpha^{\rm loc}(\sigma^{-1},\eta)|_{\sigma^{-1}\eta u}\alpha^{\rm loc}(1,1)|_{\eta u}\alpha^{\rm loc}(\sigma^{-1},\sigma)|_{\eta u} \\ = & 1. \end{array}$$

For the final part of the lemma, we have that $[(a,1)] \in (D_w^\times \rtimes \mathcal{E}^{\rm loc}(D/F)_{\boldsymbol\alpha})/\prod_x D_x^\times$ is mapped under $\iota_w^{\eta_{D/E,*}\boldsymbol\alpha}$ to $[(a,1)] \in (\mathbb{A}_D^\times/D^\times \rtimes W_{D/F,\boldsymbol\alpha})/(\mathbb{A}_D^\times/D^\times)$. Moreover

$$\left[\left(\prod_{\eta\in{\rm Gal}\,(D/E)} \alpha^{\rm loc}(\sigma,\eta)|_{\sigma\eta u}^{-1}, e^{\rm loc}_{\boldsymbol\alpha}(\sigma)\right)\right] \in (D_w^\times \rtimes \mathcal{E}^{\rm loc}(D/F)_{\boldsymbol\alpha})/\prod_x D_x^\times$$

is mapped under $\iota_w^{\eta_{D/E,*}\boldsymbol\alpha}$ to

$$\begin{aligned}
& \left[\left(\textstyle\prod_{\eta\in{\rm Gal}\,(D/E)} \alpha^{\rm loc}(\sigma,\eta)|_{\sigma\eta u}^{-1}\eta_{D/E}(\beta(\sigma))_w, e^{\rm glob}(\sigma)\right)\right]\\
= & \left[\left(\textstyle\prod_{\eta\in{\rm Gal}\,(D/E)}(\alpha^{\rm glob}(\sigma,\eta)\beta(\sigma)^\sigma\beta(\eta)/\beta(\sigma\eta))|_{\sigma\eta u}^{-1}\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\sigma)|_{\eta u}, e^{\rm glob}(\sigma)\right)\right]\\
= & \left[\left(\textstyle\prod_{\eta\in{\rm Gal}\,(D/E)} \alpha^{\rm glob}(\sigma,\eta)^{-1}\beta(\sigma)|_{\sigma\eta u}^{-1}({}^\sigma\beta(\eta))|_{\sigma\eta u}^{-1}\beta(\sigma\eta)|_{\sigma\eta u}\beta(\sigma)|_{\eta u}, e^{\rm glob}(\sigma)\right)\right]\\
= & \left[\left(\textstyle\prod_{\eta\in{\rm Gal}\,(D/E)} \alpha^{\rm glob}(\sigma,\eta)^{-1}\beta(\sigma)|_{\eta u}^{-1\sigma}(\beta(\eta)|_{\eta u}^{-1})\beta(\eta)|_{\eta u}\beta(\sigma)|_{\eta u}, e^{\rm glob}(\sigma)\right)\right]\\
= & \ {\rm conj}_{\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)|_{\eta u}}\left[\left(\textstyle\prod_{\eta\in{\rm Gal}\,(D/E)} e^{\rm glob}(\sigma\eta)e^{\rm glob}(\eta)^{-1}e^{\rm glob}(\sigma)^{-1}, e^{\rm glob}(\sigma)\right)\right]\\
= & \ {\rm conj}_{\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)|_{\eta u}}\left[\left(\textstyle\prod_{\eta\in{\rm Gal}\,(D/E)} e^{\rm glob}(\sigma\eta)e^{\rm glob}(\eta)^{-1}\widetilde\sigma^{-1}, \widetilde\sigma\right)\right]
\end{aligned}$$

for any lift $\widetilde\sigma \in W_{D/F,\boldsymbol\alpha}$ of σ. Finally

$$\left[\left(\prod_{\eta\in{\rm Gal}\,(D/E)} (\eta(b)\alpha^{\rm loc}(\eta,\tau)/b)|_{\eta u}\alpha^{\rm loc}(\tau,\eta)|_{\tau\eta u}^{-1}, be^{\rm loc}_{\boldsymbol\alpha}(\tau)\right)\right] \in (D_w^\times \rtimes \mathcal{E}^{\rm loc}(D/F)_{\boldsymbol\alpha})/\prod_x D_x^\times$$

is mapped under $\iota_w^{\eta_{D/E,*}\boldsymbol\alpha}$ to

$$\begin{aligned}
& \left[\left(\eta_{D/E}(\beta(\tau))|_w\textstyle\prod_{\eta\in{\rm Gal}\,(D/E)}(\eta(b)\alpha^{\rm loc}(\eta,\tau)/b)|_{\eta u}\alpha^{\rm loc}(\tau,\eta)|_{\tau\eta u}^{-1}, be^{\rm glob}(\tau)\right)\right]\\
= & \left[\left(\eta_{D/E}(\beta(\tau)\beta(\tau)|_u^{-1})|_w\textstyle\prod_{\eta\in{\rm Gal}\,(D/E)}(\eta(b)\alpha^{\rm loc}(\eta,\tau)/b)|_{\eta u}\alpha^{\rm loc}(\tau,\eta)|_{\tau\eta u}^{-1}, \beta(\tau)_u be^{\rm glob}(\tau)\right)\right]\\
= & \left[\left(\textstyle\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\tau)|_{\eta u}(\eta(b)/b\beta(\tau)|_u)|_{\eta u}(\alpha^{\rm glob}(\eta,\tau)\beta(\eta)^\eta\beta(\tau)\beta(\eta\tau)^{-1})|_{\eta u}\right.\right.\\
& \left.\left.(\alpha^{\rm glob}(\tau,\eta)\beta(\tau)^\tau\beta(\eta)\beta(\tau\eta)^{-1})|_{\tau\eta u}^{-1}, \iota_u^{\boldsymbol\alpha}(be^{\rm loc}_{\boldsymbol\alpha}(\tau))\right)\right]\\
= & \left[\left(\textstyle\prod_{\eta\in{\rm Gal}\,(D/E)}(\eta(b\beta(\tau)|_u)/b\beta(\tau)|_u)|_{\eta u}(\alpha^{\rm glob}(\eta,\tau)/\alpha^{\rm glob}(\tau,\eta))\right.\right.\\
& \left.\left.\beta(\eta)|_{\eta u}\beta(\eta\tau)|_{\eta u}^{-1}\beta(\tau)|_{\tau\eta u}^{-1}{}^\tau(\beta(\eta)|_{\eta u}^{-1})\beta(\tau\eta)|_{\tau\eta u}\beta(\tau)|_{\eta u}, \iota_u^{\boldsymbol\alpha}(be^{\rm loc}_{\boldsymbol\alpha}(\tau))\right)\right]\\
= & \ {\rm conj}_{\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)|_{\eta u}}\left[\left(\textstyle\prod_{\eta\in{\rm Gal}\,(D/E)}((e^{\rm glob}(\eta)e^{\rm glob}(\tau)e^{\rm glob}(\eta\tau)^{-1})/(e^{\rm glob}(\tau)e^{\rm glob}(\eta)e^{\rm glob}(\tau\eta)^{-1}))\right.\right.\\
& \left.\left.(\eta(b\beta(\tau)|_u)/b\beta(\tau)|_u)|_{\eta u}\beta(\eta\tau)|_{\eta u}^{-1}\beta(\tau)|_{\eta\tau u}^{-1}\beta(\eta\tau)|_{\eta\tau u}\beta(\tau)|_{\eta u}, \iota_u^{\boldsymbol\alpha}(be^{\rm loc}_{\boldsymbol\alpha}(\tau))\right)\right]\\
= & \ {\rm conj}_{\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)|_{\eta u}}\left[\left(\textstyle\prod_{\eta\in{\rm Gal}\,(D/E)}(e^{\rm glob}(\eta)\iota_u^{\boldsymbol\alpha}(be^{\rm loc}_{\boldsymbol\alpha}(\tau))e^{\rm glob}(\eta\tau)^{-1})/\right.\right.\\
& \left.\left.(\iota_u^{\boldsymbol\alpha}(be^{\rm loc}_{\boldsymbol\alpha}(\tau))e^{\rm glob}(\eta)e^{\rm glob}(\tau\eta)^{-1}), \iota_u^{\boldsymbol\alpha}(be^{\rm loc}_{\boldsymbol\alpha}(\tau))\right)\right].
\end{aligned}$$

□

The following lemma is straight forward to verify.

Lemma 5.6. *Suppose that $D' \supset D \supset E \supset F$ are finite Galois extensions of F. Suppose also that $\boldsymbol\alpha_{D'} \in \mathcal{Z}(D'/F)$ and $\boldsymbol\alpha_D \in \mathcal{Z}(D/F)$ and $\boldsymbol\alpha_E \in \mathcal{Z}(E/F)$ satisfy*

$\eta_{D'/D,*}\boldsymbol{\alpha}_{D'} = {}^{t'}\inf_{D'/D}\boldsymbol{\alpha}_D$ *and* $\eta_{D/E,*}\boldsymbol{\alpha}_D = {}^{t}\inf_{D/E}\boldsymbol{\alpha}_E$ *with* $t' \in T_{2,D}(\mathbb{A}_{D'})$ *and* $t \in T_{2,E}(\mathbb{A}_D)$. *Then*

$$\eta_{D'/E,*}\boldsymbol{\alpha}_{D'} = {}^{t\eta_{D/E}(t')}\inf_{D'/E}\boldsymbol{\alpha}_E.$$

5.3. **Global algebraic cohomology.** We will define the algebraic cohomology of $\mathcal{E}^{\mathrm{glob}}(E/F)_{\mathfrak{a}}$ using the algebraicity conditions

$$\mathcal{N} = \{\nu \in \mathrm{Hom}\,(T_{2,E}, G)(\mathbb{A}_E) :\ \nu \text{ is } G(\mathbb{A}_E)\text{−conjugate to an element of } \mathrm{Hom}\,(T_{2,E}, G)(E)\}$$

and

$$\mathcal{N}_{\mathrm{basic}} = \mathrm{Hom}\,(T_{2,E}, Z(G))(E).$$

Note that in this case an element $\nu \in \mathcal{N}$ may not be determined by $\overline{\nu} : \mathcal{E}^{\mathrm{glob}}(E/F) \to G(\mathbb{A}_E)$. We will denote the corresponding pointed sets of cycles and cohomology classes $Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_{\mathfrak{a}}, G(\mathbb{A}_E))$ and $Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_{\mathfrak{a}}, G(\mathbb{A}_E))_{\mathrm{basic}}$ and $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F), G(\mathbb{A}_E))$ and $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F), G(\mathbb{A}_E))_{\mathrm{basic}}$. The map $\mathrm{loc}_{\mathfrak{a}}$ induces an isomorphism

$$\mathrm{loc}_{\mathfrak{a}}^* : Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_{\mathfrak{a}}, G(\mathbb{A}_E)) \xrightarrow{\sim} Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_{\mathfrak{a}}, G(\mathbb{A}_E))$$

which preserves basic subsets and passes to cohomology. We will denote its inverse simply

$$\mathrm{loc}_{\mathfrak{a}} : Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_{\mathfrak{a}}, G(\mathbb{A}_E)) \xrightarrow{\sim} Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_{\mathfrak{a}}, G(\mathbb{A}_E)).$$

Following Kottwitz, to define the algebraic cohomology of $\mathcal{E}_3(E/F)$ we will use the algebraicity conditions

$$\mathcal{N} = \mathrm{Hom}\,(T_{3,E}, G)(E)$$

and

$$\mathcal{N}_{\mathrm{basic}} = \mathrm{Hom}\,(T_{3,E}, Z(G))(E).$$

We will denote the corresponding pointed sets of cycles and cohomology classes $Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_{\mathfrak{a}}, G(E))$ and $Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$ and $H^1_{\mathrm{alg}}(\mathcal{E}_3(E/F), G(E))$ and $H^1_{\mathrm{alg}}(\mathcal{E}_3(E/F), G(E))_{\mathrm{basic}}$. Note that there is a natural map

$$Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_{\mathfrak{a}}, G(E)) \longrightarrow Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_{\mathfrak{a}}, G(\mathbb{A}_E))$$

which preserves basic subsets and passes to cohomology. If E_v^0/F_v is a finite extension abstractly isomorphic to E_w/F_v for some $w|v$, then the composite

$$\mathrm{res}_{E_v^0} \circ \mathrm{loc} : H^1_{\mathrm{alg}}(\mathcal{E}_3(E/F), G(E)) \longrightarrow H^1_{\mathrm{alg}}(W_{E_v^0/F_v}, G(E_v^0))$$

coincides with the localization map defined by Kottwitz in [K3].

If $\mathfrak{a}, \mathfrak{a}' \in \mathcal{H}(E/F)$ and if $t \in T_{2,E}(\mathbb{A}_E)$ with $\mathfrak{a}' = {}^t\mathfrak{a}$, then we get an isomorphism

$$z_t = ((\mathfrak{z}_t)^{-1})^* : Z^1_{\mathrm{alg}}(\mathcal{E}_{\mathfrak{a}}, G(A_E)) \xrightarrow{\sim} Z^1_{\mathrm{alg}}(\mathcal{E}_{\mathfrak{a}'}, G(A_E))$$

for $\mathcal{E} = \mathcal{E}_3(E/F)$ or $\mathcal{E}^{\mathrm{glob}}(E/F)$ or $\mathcal{E}_2(E/F)$ or $\mathcal{E}^{\mathrm{loc}}(E/F)_S$ or $\mathcal{E}(E_w/F_v)$ and $A_E = E$ or $\mathbb{A}_E$ or $\mathbb{A}_E$ or $\mathbb{A}_{E,S}$ or $E_w^\times$ respectively. This takes basic elements to basic elements and passes to cohomology (as ${}^g z_t(\phi) = z_t({}^g\phi)$). We have $z_t(\phi_1\phi_2) = z_t(\phi_1)z_t(\phi_2)$ and if $f : G_1 \to G_2$, then $z_t(f_*\phi) = f_*(z_t(\phi))$. Moreover z_t commutes with the maps res_S and res_w and $\mathrm{res}_{E_v^0}$, while

$$\mathrm{loc}_{t_{\mathfrak{a}}} z_t(\phi) = {}^{(\mathrm{loc}_{\mathfrak{a}}\phi)(t)} z_t(\mathrm{loc}_{\mathfrak{a}}\phi).$$

Finally if $a \in T_{2,E}(\mathbb{A}_F)$ and $b \in \mathcal{E}^{\mathrm{glob}}(E/F)^0$ and $c \in \prod_w E_w^\times$, then

$$z_{abct}(\phi) = {}^{\phi(b)} z_t(\phi)$$

if $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_{\mathfrak{a}}, G(E))$ or $Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_{\mathfrak{a}}, G(\mathbb{A}_E))$, while

$$z_{abct}(\phi) = {}^{\phi(c^{-1})} z_t(\phi)$$

if $\phi \in Z^1_{\mathrm{alg}}(W_{E_w/F_v,\mathfrak{a}}, G(E_w))$ or $Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{S,\mathfrak{a}}, G(\mathbb{A}_{E,S}))$ or $Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_{\mathfrak{a}}, G(\mathbb{A}_E))$. Thus on the level of cohomology z_t is independent of t and only depends on $\mathfrak{a}$ and $\mathfrak{a}'$.

Suppose that $D \supset E$ is another finite Galois extension of F. We will define the algebraic cohomology of $\mathcal{E}^{\mathrm{glob}}(E/F)_{D,\mathfrak{a}}$ using the algebraicity conditions

$$\mathcal{N} = \{\nu \in \mathrm{Hom}\,(T_{2,E}, G)(\mathbb{A}_D) : \ \nu \text{ is } G(\mathbb{A}_D)\text{−conjugate to an element of } \mathrm{Hom}\,(T_{2,E}, G)(D)\}$$

and

$$\mathcal{N}_{\mathrm{basic}} = \mathrm{Hom}\,(T_{2,E}, Z(G))(D).$$

We will denote the corresponding pointed sets of cycles and cohomology classes $Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_{D,\mathfrak{a}}, G(\mathbb{A}_D))$ and $Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_{D,\mathfrak{a}}, G(\mathbb{A}_D))_{\mathrm{basic}}$ and $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_D, G(\mathbb{A}_D))$ and $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_D, G(\mathbb{A}_D))_{\mathrm{basic}}$. We also define the algebraic cohomology of $\mathcal{E}_3(E/F)_D$ using the algebraicity conditions

$$\mathcal{N} = \mathrm{Hom}\,(T_{3,E}, G)(D)$$

and

$$\mathcal{N}_{\mathrm{basic}} = \mathrm{Hom}\,(T_{3,E}, Z(G))(D).$$

We will denote the corresponding pointed sets of cycles and cohomology classes $Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_{D,\mathfrak{a}}, G(D))$ and $Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_{D,\mathfrak{a}}, G(D))_{\mathrm{basic}}$ and $H^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_D, G(D))$ and $H^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_D, G(D))_{\mathrm{basic}}$. We define algebraic cocycles and cohomology for $\mathcal{E}^{\mathrm{glob}}(E/F)|_{\mathrm{Gal}\,(D/F)}$ and $\mathcal{E}_3(E/F)|_{\mathrm{Gal}\,(D/F)}$ using these same algebraicity conditions.

Let $\mathfrak{a}_E \in \mathcal{H}(E/F)$ and $\mathfrak{a}_D \in \mathcal{H}(D/F)$ and choose $t \in T_{2,E}(\mathbb{A}_D)$ with ${}^t(\inf_{D/E} \mathfrak{a}_E) = (\eta_{D/E,*}\mathfrak{a}_D) \in \mathcal{H}(E/F)_D$. Using the functorialities (B) then (C) then $(\mathfrak{z}_t^*)^{-1}$ then (B) again from the end of section 3.1, we get morphisms

$$\begin{array}{rcl} \inf^{\mathrm{loc}}_{D/E,t} : Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{\mathfrak{a}_E}, G(\mathbb{A}_E)) & \longrightarrow & Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{\mathfrak{a}_E}|_{\mathrm{Gal}\,(D/F)}, G(\mathbb{A}_D)) \\ & \longrightarrow & Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{D,\inf \mathfrak{a}_E}, G(\mathbb{A}_D)) \\ & \stackrel{(\mathfrak{z}_t^*)^{-1}}{\longrightarrow} & Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/F)_{D,\eta_{D/E,*}\mathfrak{a}_D}, G(\mathbb{A}_D)) \\ & \longrightarrow & Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(D/F)_{\mathfrak{a}_D}, G(\mathbb{A}_D)) \end{array}$$

and

$$\begin{array}{rcl}\mathrm{inf}_{2,D/E,t}: Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_{\mathfrak{a}_E}, G(\mathbb{A}_E)) & \longrightarrow & Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_{\mathfrak{a}_E}|_{\mathrm{Gal}\,(D/F)}, G(\mathbb{A}_D)) \\ & \longrightarrow & Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_{D,\mathrm{inf}\,\mathfrak{a}_E}, G(\mathbb{A}_D)) \\ & \stackrel{(\mathfrak{z}_t^*)^{-1}}{\longrightarrow} & Z^1_{\mathrm{alg}}(\mathcal{E}_2(E/F)_{D,\eta_{D/E,*}\mathfrak{a}_D}, G(\mathbb{A}_D)) \\ & \longrightarrow & Z^1_{\mathrm{alg}}(\mathcal{E}_2(D/F)_{\mathfrak{a}_D}, G(\mathbb{A}_D))\end{array}$$

and

$$\begin{array}{rcl}\mathrm{inf}^{\mathrm{glob}}_{D/E,t}: Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_{\mathfrak{a}_E}, G(\mathbb{A}_E)) & \longrightarrow & Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_{\mathfrak{a}_E}|_{\mathrm{Gal}\,(D/F)}, G(\mathbb{A}_D)) \\ & \longrightarrow & Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_{D,\mathrm{inf}\,\mathfrak{a}_E}, G(\mathbb{A}_D)) \\ & \stackrel{(\mathfrak{z}_t^*)^{-1}}{\longrightarrow} & Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(E/F)_{D,\eta_{D/E,*}\mathfrak{a}_D}, G(\mathbb{A}_D)) \\ & \longrightarrow & Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{glob}}(D/F)_{\mathfrak{a}_D}, G(\mathbb{A}_D))\end{array}$$

and

$$\begin{array}{rcl}\mathrm{inf}_{3,D/E,t}: Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_{\mathfrak{a}_E}, G(E)) & \longrightarrow & Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_{\mathfrak{a}_E}|_{\mathrm{Gal}\,(D/F)}, G(D)) \\ & \longrightarrow & Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_{D,\mathrm{inf}\,\mathfrak{a}_E}, G(D)) \\ & \stackrel{(\mathfrak{z}_t^*)^{-1}}{\longrightarrow} & Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/F)_{D,\eta_{D/E,*}\mathfrak{a}_D}, G(D)) \\ & \longrightarrow & Z^1_{\mathrm{alg}}(\mathcal{E}_3(D/F)_{\mathfrak{a}_D}, G(D)).\end{array}$$

These maps are all injective and induce maps on the basic subsets. We have

$$\mathrm{loc}_{\mathfrak{a}_D}(\inf_{D/E,t}(\phi)) = {}^{(\mathrm{loc}_{\mathfrak{a}_E}\phi)(t)}\inf_{D/E,t}(\mathrm{loc}_{\mathfrak{a}_E}\phi)$$

and so

$$(\mathrm{loc}_{\mathfrak{a}_D}(\inf_{D/E,t}(\phi)))|_{T_{2,D}(\mathbb{A}_E)} = (\mathrm{loc}_{\mathfrak{a}_E}\phi)|_{T_{2,E}(\mathbb{A}_E)} \circ \eta_{D/E}.$$

If $a \in \mathcal{E}^{\mathrm{glob}}(E/F)^0_D$ and $b \in \prod_u D_u^\times$ and $c \in T_{2,E}(\mathbb{A}_F)$, then

$$\overset{\mathrm{loc}}{\inf_{D/E,abct}} = \overset{\mathrm{loc}}{\inf_{D/E,t}} \circ \mathrm{conj}_b^* \quad \text{and} \quad \inf_{2,D/E,abct} = \inf_{2,D/E,t} \circ \mathrm{conj}_b^*.$$

Moreover

$$z_{t_D} \circ \inf_{D/E,t} = \inf_{D/E,t_E t\eta^0_{D/E}(t_D)^{-1}} \circ z_{t_E}$$

for each of the four types of cocycles. Moreover if $D' \supset D$ is another finite Galois extension of F and if $\mathfrak{a}_{D'}\mathcal{H}(D'/F)$ and if $t' \in T_{2,D}(\mathbb{A}_{D'})$ with $\eta_{D'/D,*}\mathfrak{a}_{D'} = {}^{t'}\inf_{D'/D}\mathfrak{a}_D$, then

$$\inf_{D'/D,t'} \circ \inf_{D/E,t} = \inf_{D'/E,t\eta_{D/E}(t')}.$$

Each of the maps $\mathrm{inf}^{\mathrm{loc}}_{D/E,t}$, $\mathrm{inf}_{2,D/E,t}$, $\mathrm{inf}^{\mathrm{glob}}_{D/E,t}$, and $\mathrm{inf}_{3,D/E,t}$ induces a map in cohomology, which does not depend on the choice of t, and so we will denote the induced cohomological maps simply $\mathrm{inf}^{\mathrm{loc}}_{D/E}$, $\mathrm{inf}_{2,D/E}$, $\mathrm{inf}^{\mathrm{glob}}_{D/E}$, and $\mathrm{inf}_{3,D/E}$. These cohomological maps are also injective and commute with localization.

We define

$$B^{\text{glob}}(F,G) = \lim_{\to, E/F} H^1_{\text{alg}}(\mathcal{E}^{\text{glob}}(E/F), G(\mathbb{A}_E)) \cong B(\mathbb{A}_F, G)$$

and

$$B(F,G) = \lim_{\to, E/F} H^1_{\text{alg}}(\mathcal{E}_3(E/F), G(E)),$$

and similarly for the basic subsets. We have a map

$$\text{loc} : B(F,G) \longrightarrow B^{\text{glob}}(F,G) \cong B(\mathbb{A}_F, G) \longrightarrow B^{\text{loc}}(F,G)$$

and hence maps

$$\text{res}_v \circ \text{loc} : B(F,G) \longrightarrow B(F_v, G)$$

for any place v of F. These maps preserve the basic subsets.

5.4. **Global algebraic cohomology of reductive groups.** *Now suppose that G is reductive.*

If $\boldsymbol{\alpha} = (\alpha^{\text{glob}}, \alpha^{\text{loc}}, \beta) \in \mathcal{Z}(E/F)$ and T over F is a torus split over E then

$$\text{cor}_{\boldsymbol{\alpha}} = \text{cor}_{\alpha^{\text{glob}}} : \mathbb{Z}[V_E]_0 \otimes X_*(T) \longrightarrow Z^1_{\text{alg}}(\mathcal{E}_3(E/F)_{\mathfrak{a}}, T(E))$$

induces a bijection

$$\text{cor}^{\text{glob}} : (\mathbb{Z}[V_E]_0 \otimes X_*(T))_{\text{Gal}(E/F)} \xrightarrow{\sim} H^1_{\text{alg}}(\mathcal{E}_3(E/F), T(E))$$

which is independent of $(\alpha^{\text{glob}}, \alpha^{\text{loc}}, \beta)$. (Note that $\text{cor}_{\alpha^{\text{glob}}}$ depends on $(\alpha^{\text{glob}}, \alpha^{\text{loc}}, \beta)$ and not just its image in $\mathcal{H}(E/F)$.)

We have the following special case of the general observation made in item E of section 3.1:

Lemma 5.7. *Suppose that $(\alpha^{\text{glob}}, \alpha^{\text{loc}}, \beta) \in \mathcal{Z}(E/F)$. Suppose also that T is a torus split by E and that $\chi : T_{3,E} \to T$ is a homomorphism (which must then be defined over E). Set*

$$b = \prod_{\eta \in \text{Gal}(E/F)} \eta^{-1} \chi(\beta(\eta))^{-1}.$$

Then

$$\text{loc}_{\mathfrak{a}} \text{cor}_{\alpha^{\text{glob}}}(\chi) = {}^b \text{cor}_{\alpha^{\text{loc}}}(\chi)$$

(where $\mathfrak{a} = [(\alpha^{\text{glob}}, \alpha^{\text{loc}}, \beta)]$).

Kottwitz shows that for any reductive G split by E there is a map

$$\kappa_G : H^1_{\text{alg}}(\mathcal{E}_3(E/F), G(E)) \longrightarrow (\mathbb{Z}[V_E]_0 \otimes_{\mathbb{Z}} \Lambda_G)_{\text{Gal}(E/F)}$$

which is functorial in G and gives $(\text{cor}^{\text{glob}})^{-1}$ in the case that G is a torus. The image of $\kappa_G(\boldsymbol{\phi})$ in $(\mathbb{Z}[V_E] \otimes_{\mathbb{Z}} \Lambda_G)_{\text{Gal}(E/F)}$ equals $\kappa_G(\text{loc}\boldsymbol{\phi})$. Moreover κ_G is compatible with $\inf_{D/E}$ and the natural isomorphism

$$(\mathbb{Z}[V_D]_0 \otimes_{\mathbb{Z}} \Lambda_G)_{\text{Gal}(D/F)} \xrightarrow{\sim} (\mathbb{Z}[V_E]_0 \otimes_{\mathbb{Z}} \Lambda_G)_{\text{Gal}(E/F)}$$

induced by the map $\mathbb{Z}[V_D] \to \mathbb{Z}[V_E]$ sending u to $u|_E$. (To see the map is an isomorphism it suffices to show that $\mathbb{Z}[V_D]_{0,\mathrm{Gal}\,(D/E)} \stackrel{\sim}{\to} \mathbb{Z}[V_E]_0$. This follows from the long exact sequence associates to $(0) \to \mathbb{Z}[V_D]_0 \to \mathbb{Z}[V_D] \to \mathbb{Z} \to (0)$ and the surjectivity of $\bigoplus_{w\in V_E} \mathrm{Gal}\,(D_u/E_w)^{\mathrm{ab}} \to \mathrm{Gal}\,(D/E)^{\mathrm{ab}}$.)

If E/F splits G then Kottwitz proves that there is a cartesian square

$$\begin{array}{ccc} B(F,G)_{\mathrm{basic}} & \stackrel{\oplus_{v|\infty}\mathrm{res}_v \circ \mathrm{loc}}{\longrightarrow} & \prod_{v|\infty} B(F_v,G)_{\mathrm{basic}} \\ \kappa_G \downarrow & & \prod_{v|\infty} \kappa_G \downarrow \\ (\mathbb{Z}[V_E]_0 \otimes \Lambda_G)_{\mathrm{Gal}\,(E/F)} & \longrightarrow & \prod_{v|\infty} \Lambda_{G,\mathrm{Gal}\,(E_w/F_v)} \\ \sum_w w \otimes \lambda_w & \longmapsto & (\sum_{\sigma \in \mathrm{Gal}\,(E_{\widetilde{v}}/F_v)\backslash \mathrm{Gal}\,(E/F)} \sigma \lambda_{\sigma^{-1}\widetilde{v}})_{v|\infty}, \end{array}$$

where $\widetilde{v}|v$. (See proposition 15.1 of [K3].) In particular the fibres of κ_F are finite.

If S is a finite set of places of F, we will write $B(F,G)_{S,\mathrm{basic}}$ for the inverse image in $B(F,G)_{\mathrm{basic}}$ under κ_G of the image of $\mathbb{Z}[V_{E,S}]_0 \otimes_{\mathbb{Z}} \Lambda_G$ in $(\mathbb{Z}[V_E]_0 \otimes_{\mathbb{Z}} \Lambda_G)_{\mathrm{Gal}\,(E/F)}$. (Here E/F is any finite Galois extension that splits G.)

Lemma 5.8. *If S is a finite set of places of F, then there is a finite Galois extension D/F such that $B(F,G)_{S,\mathrm{basic}}$ is contained in the image of $H^1_{\mathrm{alg}}(\mathcal{E}_3(D/F), G(D))$.*

Proof: Let E/F be a finite Galois extension which splits G. Note that $\Lambda_G/X_*(Z(G)^0)$ is finite and hence $(\mathbb{Z}[S_E]_0 \otimes_{\mathbb{Z}} \Lambda_G)_{\mathrm{Gal}\,(E/F)}/(\mathbb{Z}[V_{E,S}]_0 \otimes_{\mathbb{Z}} \Lambda_{Z(G)^0})_{\mathrm{Gal}\,(E/F)}$ is finite. We conclude that $B(F,Z(G)^0)_S$ has only finitely many orbits on $B(F,G)_{S,\mathrm{basic}}$. Moreover $B(F,Z(G)^0)_S$ is finitely generated (being isomorphic to the image of $\mathbb{Z}[V_{E,S}] \otimes \Lambda_{Z(G)^0}$ in $(\mathbb{Z}[V_E]_0 \otimes_{\mathbb{Z}} \Lambda_{Z(G)^0})_{\mathrm{Gal}\,(E/F)})$. Thus there is a finite Galois extension D/F containing E such that $B(F,G)_{S,\mathrm{basic}}$ is contained in the image of $H^1_{\mathrm{alg}}(\mathcal{E}_3(D/F), G(D))$.$\Box$

Kottwitz also shows that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \ker^1(F,G) & \longrightarrow & B(F,G)_{\mathrm{basic}} & \longrightarrow & B(\mathbb{A}_F,G) & \stackrel{\overline{\kappa}_F}{\longrightarrow} & (\Lambda_G)_{\mathrm{Gal}\,(\overline{F}/F)} \\ & & || & & \uparrow & & \uparrow & & || \\ (0) & \longrightarrow & \ker^1(F,G) & \longrightarrow & H^1(F,G) & \longrightarrow & H^1(\mathbb{A}_F,G) & \stackrel{\overline{\kappa}_F}{\longrightarrow} & (\Lambda_G)_{\mathrm{Gal}\,(\overline{F}/F)}. \end{array}$$

(See proposition 15.6 of [K3].)

Now suppose that $F = \mathbb{Q}$ and that E is sufficiently large that

- E splits G;
- E is totally imaginary;
- $B(\mathbb{Q},G)_{\{\infty\},\mathrm{basic}}$ is contained in the image of $H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))$.

Suppose moreover that Y is a compactifying $G(\mathbb{R})$-conjugacy class of cocharacters of G defined over $\mathbb{C}$, and that $\tau \in \mathrm{Aut}\,(\mathbb{C})$. Then there is a unique class $\boldsymbol{\phi}_{G,Y,\tau} \in H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}}$ such that

- $\kappa_G(\boldsymbol{\phi}_{G,Y,\tau}) = (v(\rho) - v(\tau\rho)) \otimes {}^{\rho^{-1}}\lambda_G(Y)$, where $\rho : K \hookrightarrow \mathbb{C}$ and $v(\rho)$ denotes the corresponding infinite place of K (this is independent of the choice of ρ);
- and $\mathrm{res}_{\mathbb{C}/\mathbb{R}}\mathrm{loc}\boldsymbol{\phi}_{G,Y,\tau} = \widehat{\boldsymbol{\lambda}}_G(Y - {}^{\tau}[Y]_{G(\mathbb{C})})$.

In this case $\mathrm{res}^{\infty}\mathrm{loc}\boldsymbol{\phi}_{G,Y,\tau} = 1$. If $\phi \in \boldsymbol{\phi}_{G,Y,\tau}$, then $Y({}^{\tau}[Y]_{G(\mathbb{C})}, Y)$ is a compactifying conjugacy class of cocharacters of ${}^{\phi}G$ over $\mathbb{C}$, which we will denote ${}^{\tau,\phi}Y$.

If $f : G_1 \to G_2$ and Y_i is a compactifying $G_i(\mathbb{R})$-conjugacy class of cocharacters of G over $\mathbb{C}$ with $fY_1 \subset Y_2$, then $f_*\boldsymbol{\phi}_{G_1,Y_1,\tau} = \boldsymbol{\phi}_{G_2,Y_2,\tau}$. Hence, if $\phi \in \boldsymbol{\phi}_{G_1,Y_1,\tau}$, then $f : {}^{\phi}G_1 \to {}^{f\circ\phi}G_2$ and $f({}^{\tau,\phi}Y_1) \subset {}^{\tau,f\circ\phi}Y_2$.

If $G = T$ is a torus and $Y = \{\mu\} \subset X_*(T)(\mathbb{C})$, then

$$\boldsymbol{\phi}_{G,\{\mu\},\tau} = \mathrm{cor}\left((v(\rho) - v(\tau\rho)) \otimes {}^{\rho^{-1}}\mu.\right.$$

6. Rigidification data.

6.1. **Rigidification data.** Suppose that E/F is a finite Galois extension of number fields, that $\mathfrak{a} \in \mathcal{Z}(E/F)$.

By a *rigidification* of $\mathfrak{a}$ we shall mean an isomorphism of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathbb{A}_E^\times/E^\times & \stackrel{r_E}{\longrightarrow} & W_{E^{\mathrm{ab}}/F} & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & (0) \\ && || && \widetilde{\Gamma}\downarrow \wr && || && \\ (0) & \longrightarrow & \mathbb{A}_E^\times/E^\times & \stackrel{r_E}{\longrightarrow} & W_{E/F,\mathfrak{a}} & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & (0). \end{array}$$

Such a rigidifcation always exists and it is unique up to composition with conjugation by an element of $\mathbb{A}_E^\times/E^\times$.

Suppose that $\rho : E^{\mathrm{ab}} \to \overline{F}_v$ is F-linear. Write $w(\rho)$ (resp. $u(\rho)$) for the place of E (resp. E^{ab}) induced by ρ. There is a morphism of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & (\rho(E)F_v)^\times & \longrightarrow & W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v} & \longrightarrow & \mathrm{Gal}\,(\rho(E)F_v/F_v) & \longrightarrow & (0) \\ && \rho^{-1}\downarrow && \theta_\rho\downarrow && \uparrow\wr && \\ (0) & \longrightarrow & \mathbb{A}_E^\times/E^\times & \longrightarrow & W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(E/F)_{w(\rho)}} & \longrightarrow & \mathrm{Gal}\,(E/F)_{w(\rho)} & \longrightarrow & (0) \end{array}$$

which is determined up to conjugation by an element of $\overline{(E_\infty^\times)^0E^\times}/E^\times$. (Here the right hand vertical map is the one induced by ρ.) There is also an isomorphism of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & (\rho(E)F_v)^\times & \longrightarrow & W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v} & \longrightarrow & \mathrm{Gal}\,(\rho(E)F_v/F_v) & \longrightarrow & (0) \\ && \rho\uparrow\wr && \widetilde{\Theta}_{v,\rho}\downarrow\wr && \uparrow\wr && \\ (0) & \longrightarrow & E_{w(\rho)}^\times & \longrightarrow & W_{E_{w(\rho)}/F_v,\mathfrak{a}} & \longrightarrow & \mathrm{Gal}\,(E_{w(\rho)}/F_v) & \longrightarrow & (0), \end{array}$$

which is unique up to composition with conjugation by an element of $E_{w(\rho)}^\times$. The composites

$$\widetilde{\Gamma} \circ \theta_\rho : W_{\rho(E^{\mathrm{ab}})F_v/F_v} \longrightarrow W_{E/F,\mathfrak{a}}$$

and

$$\iota_{w(\rho)}^{\mathfrak{a}} \circ \widetilde{\Theta}_{v,\rho} : W_{\rho(E^{\mathrm{ab}})F_v/F_v} \longrightarrow W_{E/F,\mathfrak{a}}$$

must be equal up to multiplication by an element of $Z^1(\mathrm{Gal}\,(E/F)_{w(\rho)}, \mathbb{A}_E^\times/E^\times)$, and hence we must have

$$\mathrm{conj}_a \circ \widetilde{\Gamma} \circ \theta_\rho = \iota_{w(\rho)}^{\mathfrak{a}} \circ \widetilde{\Theta}_{v,\rho},$$

for some $a \in \mathbb{A}_E^\times/E^\times$. We will say that $\widetilde{\Gamma}$ is *adapted* to (ρ, θ_ρ) if for some choice of $\widetilde{\Theta}_{v,\rho}$ we may take $a = 1$. We see that a rigidification adapted to (ρ, θ_ρ) always exists and is unique up to composition with conjugation by an element of

$$(\mathbb{A}_E^\times/E^\times)^{\mathrm{Gal}\,(E/F)_{w(\rho)}} E_{w(\rho)}^\times.$$

Note that if $\widetilde{\Gamma}$ is adapted to (ρ, θ_ρ) and if $a \in \overline{(E_\infty^\times)^0E^\times}/E^\times$, then $\mathrm{conj}_a \circ \widetilde{\Gamma}$ is adapted to $(\rho, \mathrm{conj}_a \circ \theta_\rho)$. If $\widetilde{\Gamma}$ is a rigidification of $\mathfrak{a}$ adapted to (ρ, θ_ρ) then

$$\mathrm{conj}_{t_{w(\rho)}} \circ \widetilde{\Gamma}$$

is a rigidification of ${}^t\mathfrak{a}$ adapted to (ρ,θ_ρ) (because

$$\iota^{{}^t\mathfrak{a}}_{w(\rho)} = \mathrm{conj}_{t_{w(\rho)}} \circ \iota^{\mathfrak{a}}_{w(\rho)}).$$

If $\widetilde{\Gamma}$ is a rigidification of $\mathfrak{a}$ adapted to (ρ,θ_ρ), if $\boldsymbol{\alpha}\in\mathfrak{a}$ and if $\sigma\in W_{E^{\mathrm{ab}}/F}$ then

$$\widetilde{\Gamma}^\sigma = \widetilde{\Gamma}^{\sigma,\boldsymbol{\alpha}} = \mathrm{conj}_{\beta(\sigma^{-1})_{w(\rho\sigma)}} \circ \mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})} \circ \widetilde{\Gamma} \circ \mathrm{conj}_\sigma$$

is a rigidification of $\mathfrak{a}$ adapted to $(\rho\sigma, \mathrm{conj}_{\sigma^{-1}}\circ\theta_\rho)$. Indeed

$$\widetilde{\Gamma}^\sigma\circ(\mathrm{conj}_{\sigma^{-1}}\circ\theta_\rho) = \mathrm{conj}_{\beta(\sigma^{-1})_{w(\rho\sigma)}}\circ\mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}\circ\iota^{\mathfrak{a}}_{w(\rho)}\circ\widetilde{\Theta}_{v,\rho} = \iota^{\mathfrak{a}}_{w(\rho\sigma)}\circ\mathrm{conj}_{e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma^{-1})}\circ\widetilde{\Theta}_{v,\rho},$$

while $\mathrm{conj}_{e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma^{-1})}\circ\widetilde{\Theta}_{v,\rho} : W_{((\rho\sigma)(E)F_v)^{\mathrm{ab}}/F_v} \xrightarrow{\sim} W_{E_{w(\rho\sigma)}/F_v,\boldsymbol{\alpha}}$ extends $\sigma^{-1}\circ\rho^{-1} : ((\rho\sigma)(E)F_v)^\times \xrightarrow{\sim} E^\times_{w(\rho\sigma)}$. We have

$$\widetilde{\Gamma}^{\sigma,{}^\gamma\boldsymbol{\alpha}} = \mathrm{conj}_{\gamma^{\mathrm{loc}}(\sigma^{-1})^{-1}_{w(\rho\sigma)}} \circ \widetilde{\Gamma}^{\sigma,\boldsymbol{\alpha}}.$$

Note that

$$\widetilde{\Gamma}^{\sigma_1\sigma_2,\boldsymbol{\alpha}} = \mathrm{conj}_{\alpha^{\mathrm{loc}}(\sigma_1^{-1},\sigma_2^{-1})_{w(\rho\sigma_1\sigma_2)}} \circ (\widetilde{\Gamma}^{\sigma_1,\boldsymbol{\alpha}})^{\sigma_2,\boldsymbol{\alpha}}.$$

By *a Galois rigidification* of $\mathfrak{a}$ adapted to ρ we mean an isomorphism of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathrm{Gal}\,(E^{\mathrm{ab}}/E) & \longrightarrow & \mathrm{Gal}\,(E^{\mathrm{ab}}/F) & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & (0) \\ & & \mathrm{Art}_E \uparrow \wr & & \Gamma \downarrow \wr & & || & & \\ (0) & \longrightarrow & \mathbb{A}_E^\times/\overline{(E_\infty^\times)^0E^\times} & \longrightarrow & W_{E/F,\mathfrak{a}}/\overline{(E_\infty^\times)^0E^\times} & \longrightarrow & \mathrm{Gal}\,(E/F) & \longrightarrow & (0) \end{array}$$

which lifts to a rigidification of $\mathfrak{a}$ adapted to (ρ,θ_ρ), for some, and hence any, θ_ρ. Such a Galois rigidification exists and it is unique up to composition with conjugation by an element of $(\mathbb{A}_E^\times/E^\times)^{\mathrm{Gal}\,(E/F)_{w(\rho)}}E^\times_{w(\rho)}$. If Γ is a Galois rigidification of $\mathfrak{a}$ adapted to ρ, then $\mathrm{conj}_{t_{w(\rho)}}\circ\Gamma$ is a Galois rigidification of ${}^t\mathfrak{a}$ adapted to ρ. Moreover

$$\Gamma^{\sigma,\mathfrak{a}} = \mathrm{conj}_{\beta(\sigma^{-1})_{w(\rho\sigma)}} \circ \mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})} \circ \Gamma \circ \mathrm{conj}_\sigma$$

is a Galois rigidification of $\mathfrak{a}$ adapted to $\rho\sigma$.

We call two Galois rigidifications Γ and Γ' both adapted to ρ *equivalent* if $\Gamma' = \mathrm{conj}_a\circ\Gamma$ for some $a\in E^\times_{w(\rho)}$. If $\Gamma\sim\Gamma'$, then $\mathrm{conj}_a\circ\Gamma\sim\mathrm{conj}_a\circ\Gamma'$ and

$$\Gamma^{\sigma,\boldsymbol{\alpha}} \sim (\Gamma')^{\sigma,\boldsymbol{\alpha}} \sim (\Gamma')^{\sigma,\boldsymbol{\alpha}'}$$

for $\boldsymbol{\alpha},\boldsymbol{\alpha}'\in\mathfrak{a}$. Thus if $[\Gamma]$ is an equivalence class of Galois rigidifcations of $\mathfrak{a}$ adapted to ρ and $t\in T_{2,E}(\mathbb{A}_E)$, then

$${}^t[\Gamma] = [\mathrm{conj}_{t_{w(\rho)}}\circ\Gamma]$$

is a well defined equivalence class of Galois rigidifcations of ${}^t\mathfrak{a}$ adapted to ρ. Moreover if $\sigma\in\mathrm{Gal}\,(E^{\mathrm{ab}}/F)$, then

$$[\Gamma]^\sigma = [\Gamma^{\sigma,\boldsymbol{\alpha}}]$$

is a well defined equivalence class of Galois rigidifcations of $\mathfrak{a}$ adapted to $\rho\sigma$. Moreover

$${}^{t_1t_2}[\Gamma] = {}^{t_1}({}^{t_2}[\Gamma])$$

and

$$[\Gamma]^{\sigma_1\sigma_2} = ([\Gamma]^{\sigma_1})^{\sigma_2}.$$

Lemma 6.1.

$$({}^t[\Gamma])^\sigma = {}^t([\Gamma]^\sigma).$$

Proof:

$$\begin{array}{rcl} (\mathrm{conj}_{t_{w(\rho)}} \circ \Gamma)^\sigma & = & \mathrm{conj}_{\sigma^{-1}t_{w(\rho)}} \circ \mathrm{conj}_{({}^t\beta)(\sigma^{-1})_{w(\rho\sigma)}/\beta(\sigma^{-1})_{w(\rho\sigma)}} \circ \Gamma^\sigma \\ & = & \mathrm{conj}_{(\sigma^{-1}t)_{w(\rho\sigma)}} \circ \mathrm{conj}_{(t/\sigma^{-1}(t))_{w(\rho\sigma)}} \circ \Gamma^\sigma \\ & = & \mathrm{conj}_{t_{w(\rho\sigma)}} \circ \Gamma^\sigma. \end{array}$$

□

By *complete rigidification data* for $\mathfrak{a}$ we mean the choice for each place v of F and each F-linear $\rho : E^{\mathrm{ab}} \to \overline{F_v}$ an equivalence class $[\Gamma_{v,\rho}]$ of Galois rigidifications of $\mathfrak{a}$ adapted to ρ, such that

- if $\sigma \in \mathrm{Gal}\,(E^{\mathrm{ab}}/F)$ then $[\Gamma_{\rho\sigma}] = [\Gamma_\rho]^\sigma$.

Note that it is equivalent to specify $[\Gamma_{v,\rho_v}]$ for one choice of F-linear embedding $\rho_v : E^{\mathrm{ab}} \hookrightarrow \overline{F}_v$ for each place v of F.

Lemma 6.2. *(1) Complete rigidification data for* $\mathfrak{a}$ *exists.*

(2) If $t \in T_{2,E}(\mathbb{A}_E)$ *and if* $\{[\Gamma_{v,\rho}]\}$ *is complete rigidification data for* $\mathfrak{a}$*, then* $\{{}^t[\Gamma_{v,\rho}]\}$ *is complete rigidification data for* ${}^t\mathfrak{a}$*.*

(3) If $\{[\Gamma_{v,\rho}]\}$ *and* $\{[\Gamma'_{v,\rho}]\}$ *are complete rigidification data for* $\mathfrak{a}$*, then there exists* $t \in T_{2,E}(\mathbb{A}_F)$ *such that* $[\Gamma'_{v,\rho}] = {}^t[\Gamma_{v,\rho}]$ *for all* v *and* ρ*.*

Proof: The first two parts follow immediately from the above discussion. For the third part choose for each place v of F an F-linear embedding $\rho_v : E^{\mathrm{ab}} \to \overline{F_v}$. Then we can find $t_{w(\rho_v)} \in (\mathbb{A}_E^\times/E^\times)^{\mathrm{Gal}\,(E/F)_{w(\rho_v)}}$ such that

$$[\Gamma'_{v,\rho_v}] = [\mathrm{conj}_{t_{w(\rho_v)}} \circ \Gamma_{v,\rho_v}].$$

Define $t \in T_{2,E}(\mathbb{A}_F)$ by

$$t_{\sigma w(\rho_v)} = \sigma t_{w(\rho_v)}.$$

Then

$$[\Gamma'_{v,\rho_v}] = {}^t[\Gamma_{v,\rho_v}]$$

and so

$$[\Gamma'_{v,\rho_v\sigma}] = [\Gamma'_{v,\rho_v}]^\sigma = ({}^t[\Gamma_{v,\rho_v}])^\sigma = {}^t([\Gamma_{v,\rho_v}]^\sigma) = {}^t[\Gamma_{v,\rho_v\sigma}]$$

for all $\sigma \in \mathrm{Gal}\,(E^{\mathrm{ab}}/F)$. The third part follows. □

We will write $\mathcal{H}(E/F)^+$ for the set of pairs $(\mathfrak{a}, \{[\Gamma_{v,\rho}]\})$, where $\mathfrak{a} \in \mathcal{H}(E/F)$ and $\{[\Gamma_{v,\rho}]\}$ is complete rigidification data for $\mathfrak{a}$. It comes with a transitive action of $T_{2,E}(\mathbb{A}_E)$ compatible with the action of $T_{2,E}(\mathbb{A}_E)$ on $\mathcal{H}(E/F)$.

We may define $\mathcal{Z}(E/F)^+$ as the set of 4-tuples $\boldsymbol{\alpha}^+ = (\alpha^{\mathrm{glob}}, \alpha^{\mathrm{loc}}, \beta, [\{\Gamma_{v,\rho}\}])$, where $\boldsymbol{\alpha} = (\alpha^{\mathrm{glob}}, \alpha^{\mathrm{loc}}, \beta) \in \mathcal{Z}(E/F)$ and where $\{[\Gamma_{v,\rho}]\}$ is complete rigidification data for

$[\boldsymbol{\alpha}]$. We call $\boldsymbol{\alpha}_1^+$ and $\boldsymbol{\alpha}_2^+$ *equivalent* if we can find $(\gamma^{\text{glob}}, \gamma^{\text{loc}})$ with $\boldsymbol{\alpha}_2 = {}^{(\gamma^{\text{glob}}, \gamma^{\text{loc}})}\boldsymbol{\alpha}_1$ and $[\Gamma_{2,v,\rho}] = [i_{\gamma^{\text{glob}}} \circ \Gamma_{1,v,\rho}]$ for all v and ρ. Then $\mathcal{H}(E/F)^+$ is just the set of equivalence classes of elements of $\mathcal{Z}(E/F)^+$. The actions of $T_{2,E}(\mathbb{A}_E)$ on $\mathcal{Z}(E/F)$ and $\mathcal{H}(E/F)^+$ lift to an action on $\mathcal{Z}(E/F)^+$.

If $\boldsymbol{\alpha}_0 \in \mathcal{Z}(E/\mathbb{Q})$ and $\rho_0 : E_{w(\infty)} \xrightarrow{\sim} \mathbb{C}$ are as described at the end of section 5.1, then we will extend $\boldsymbol{\alpha}_0$ to

$$\boldsymbol{\alpha}_0^+ = (\boldsymbol{\alpha}_0, \{[\Gamma_{v,\rho,0}]\}) \in \mathcal{Z}(E/\mathbb{Q})^+$$

where we choose $\Gamma_{\infty,\rho_0,0}$ so that it has a lift $\widetilde{\Gamma}_{\infty,\rho_0,0} : W_{E^{\text{ab}}/\mathbb{Q}} \xrightarrow{\sim} W_{E/\mathbb{Q},\boldsymbol{\alpha}_0}$ with

$$\widetilde{\Gamma}_{\infty,\rho_0,0} \circ \theta_{\rho_0} = \iota_{w(\infty)}^{\boldsymbol{\alpha}_0} \circ \widetilde{\Theta}_0.$$

6.2. **Comparing correstrictions.** In this section we will put ourselves in the following situation. E/F will be a finite Galois extension of number fields, v will be a place of F, τ an element of $\mathrm{Aut}\,(\overline{F_v}/F)$. We will fix $\boldsymbol{\alpha} = (\alpha^{\text{glob}}, \alpha^{\text{loc}}, \beta) \in \mathcal{Z}(E/F)$ and a lifting $\boldsymbol{\alpha}^+ = (\alpha^{\text{glob}}, \alpha^{\text{loc}}, \beta, \{[\Gamma_{v,\rho}]\}) \in \mathcal{Z}(E/F)^+$ of $\boldsymbol{\alpha}$. Set $\mathfrak{a} = [\boldsymbol{\alpha}] \in \mathcal{H}(E/F)$ and $\mathfrak{a}^+ = [\boldsymbol{\alpha}^+] \in \mathcal{H}(E/F)^+$. If $\rho : E^{\text{ab}} \to \overline{F_v}$ is F-linear we define

$$g_{\mathfrak{a}^+,v,\rho}(\tau) = \left(\left(\mathrm{loc}_{\mathfrak{a}} \widetilde{\Gamma_{v,\rho}(\tau^\rho)} \right)^{-1} e_{\boldsymbol{\alpha}}^{\text{loc}}(\tau^\rho) \right)_{w(\tau\rho)} \in \mathbb{A}_E^\times,$$

where $\widetilde{\Gamma_{v,\rho}(\tau^\rho)}$ is any lift of $\Gamma_{v,\rho}(\tau^\rho)$ to $\mathcal{E}_{\mathfrak{a}}^{\text{glob}}(E/F)$. This element is not independent of all choices, but its image

$$\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau) \in \mathbb{A}_E^\times / \overline{(E_\infty^\times)^0 E^\times} E_{w(\rho)}^\times E_{w(\tau\rho)}^\times$$

is well defined.

Lemma 6.3. *(1)* $\overline{g}_{\mathfrak{a}^+,v,\rho\sigma}(\tau) = {}^{\sigma^{-1}}\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau) \beta(\sigma^{-1})_{w(\rho\sigma)} / \beta(\sigma^{-1})_{w(\tau\rho\sigma)}$.

(2) $\overline{g}_{\mathfrak{a}^+,v,\tau_2\rho}(\tau_1) = {}^{(\tau_2^\rho)^{-1}}\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau_1) \beta((\tau_2^\rho)^{-1})_{w(\tau_2\rho)} / \beta((\tau_2^\rho)^{-1})_{w(\tau_1\tau_2\rho)}$.

(3) $\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau_1\tau_2) = \overline{g}_{\mathfrak{a}^+,v,\tau_2\rho}(\tau_1) \overline{g}_{\mathfrak{a}^+,v,\rho}(\tau_2) \in \mathbb{A}_E^\times / \overline{(E_\infty^\times)^0 E^\times} E_{w(\rho)}^\times E_{w(\tau_2\rho)}^\times E_{w(\tau_1\tau_2\rho)}^\times$.

(4) $\overline{g}_{{}^t\mathfrak{a}^+,v,\rho}(\tau) = (t_{w(\rho)}/t_{w(\tau\rho)}) \overline{g}_{\mathfrak{a}^+,v,\rho}(\tau)$.

(5) If τ *fixes the image* $\rho(E)$ *then* $\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau) = \mathrm{Art}_E^{-1}(\tau^\rho)^{-1}$.

Proof: For the first part

$$\begin{array}{rl}
& \overline{g}_{\mathfrak{a}^+,v,\rho\sigma}(\tau) \\
= & \left(\left(\mathrm{loc}_{\boldsymbol{\alpha}} \Gamma_{v,\rho\sigma} \widetilde{(\sigma^{-1}\tau^\rho\sigma)} \right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^{\rho\sigma}) \right)_{w(\tau\rho\sigma)} \\
= & \left(\mathrm{loc}_{\boldsymbol{\alpha}} \left(\beta(\sigma^{-1})_{w(\rho\sigma)} e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1}) \Gamma_{v,\rho} \widetilde{(\tau^\rho)} e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})^{-1} \beta(\sigma^{-1})^{-1}_{w(\rho\sigma)} \right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^{\rho\sigma}) \right)_{w(\tau\rho\sigma)} \\
= & \left(\beta(\sigma^{-1})_{w(\rho\sigma)} \beta(\sigma^{-1})^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma^{-1}) (\mathrm{loc}_{\boldsymbol{\alpha}} \widetilde{\Gamma_{v,\rho}(\tau^\rho)})^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma^{-1})^{-1} \beta(\sigma^{-1}) \beta(\sigma^{-1})^{-1}_{w(\rho\sigma)} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^{\rho\sigma}) \right)_{w(\tau\rho\sigma)} \\
= & \left(e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma^{-1}) (\mathrm{loc}_{\boldsymbol{\alpha}} \widetilde{\Gamma_{v,\rho}(\tau^\rho)})^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^\rho) e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma^{-1})^{-1} \right)_{w(\tau\rho\sigma)} \\
& \left(\beta(\sigma^{-1})_{w(\rho\sigma)} \beta(\sigma^{-1})^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma^{-1}) e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^\rho)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma^{-1})^{-1} \beta(\sigma^{-1}) \beta(\sigma^{-1})^{-1}_{w(\rho\sigma)} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma^{-1}\tau^\rho\sigma) \right)_{w(\tau\rho\sigma)} \\
= & {}^{\sigma^{-1}}\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau) \beta(\sigma^{-1})_{w(\rho\sigma)} \beta(\sigma^{-1})^{-1}_{w(\tau\rho\sigma)} ({}^{\sigma^{-1}(\tau^\rho)^{-1}\sigma}\beta(\sigma^{-1}))_{w(\tau\rho\sigma)} {}^{\sigma^{-1}(\tau^\rho)^{-1}\sigma}(\beta(\sigma^{-1})_{w(\rho\sigma)})^{-1} \\
= & {}^{\sigma^{-1}}\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau) \beta(\sigma^{-1})_{w(\rho\sigma)} \beta(\sigma^{-1})^{-1}_{w(\tau\rho\sigma)} {}^{(\tau^{\rho\sigma})^{-1}}(\beta(\sigma^{-1})_{w(\rho\sigma)} / \beta(\sigma^{-1})_{w(\rho\sigma)}) \\
= & {}^{\sigma^{-1}}\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau) \beta(\sigma^{-1})_{w(\rho\sigma)} \beta(\sigma^{-1})^{-1}_{w(\tau\rho\sigma)}.
\end{array}$$

The second part follows from the first.

For the third part we have

$$\begin{aligned}
&\overline{g}_{\mathfrak{a}^+,v,\tau_2\rho}(\tau_1)\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau_2)\\
= \ &\left(\left(\mathrm{loc}_{\boldsymbol{\alpha}}\widetilde{\Gamma_{v,\tau_2\rho}(\tau_1^{\tau_2\rho})}\right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau_1^{\tau_2\rho})\right)_{w(\tau_1\tau_2\rho)}\left(\left(\mathrm{loc}_{\boldsymbol{\alpha}}\widetilde{\Gamma_{v,\rho}(\tau_2^{\rho})}\right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau_2^{\rho})\right)_{w(\tau_2\rho)}\\
= \ &\left(\mathrm{loc}_{\boldsymbol{\alpha}}\left(\mathrm{conj}_{\beta((\tau_2^\rho)^{-1})_{w(\tau_2\rho)}e^{\mathrm{glob}}_{\boldsymbol{\alpha}}((\tau_2^\rho)^{-1})}(\widetilde{\Gamma_{v,\rho}(\tau_1^{\rho})})\right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau_1^{\tau_2\rho})\right)_{w(\tau_1\tau_2\rho)}\\
&\left(\left(\mathrm{loc}_{\boldsymbol{\alpha}}\widetilde{\Gamma_{v,\rho}(\tau_2^{\rho})}\right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau_2^{\rho})\right)_{w(\tau_1\tau_2\rho)}({}^{(\tau_2^\rho)^{-1}}\beta(\tau_2^\rho))_{w(\tau_2\rho)}/({}^{(\tau_2^\rho)^{-1}}\beta(\tau_2^\rho))_{w(\tau_1\tau_2\rho)}\\
= \ &\left(\left(\mathrm{conj}_{\beta((\tau_2^\rho)^{-1})_{w(\tau_2\rho)}\beta((\tau_2^\rho)^{-1})^{-1}e^{\mathrm{loc}}_{\boldsymbol{\alpha}}((\tau_2^\rho)^{-1})}\mathrm{loc}_{\boldsymbol{\alpha}}(\widetilde{\Gamma_{v,\rho}(\tau_1^{\rho})})^{-1}\right) e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau_1^{\tau_2\rho})\right)_{w(\tau_1\tau_2\rho)}\\
&\left(\left(\mathrm{loc}_{\boldsymbol{\alpha}}\widetilde{\Gamma_{v,\rho}(\tau_2^{\rho})}\right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau_2^{\rho})\right)_{w(\tau_1\tau_2\rho)}({}^{(\tau_2^\rho)^{-1}}\beta(\tau_2^\rho))_{w(\tau_2\rho)}/({}^{(\tau_2^\rho)^{-1}}\beta(\tau_2^\rho))_{w(\tau_1\tau_2\rho)}\\
= \ &\Bigg(\beta((\tau_2^\rho)^{-1})_{w(\tau_2\rho)}\beta((\tau_2^\rho)^{-1})^{-1}\left(\mathrm{loc}_{\boldsymbol{\alpha}}\widetilde{\Gamma_{v,\rho}(\tau_2^{\rho})}\right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau_2^{\rho})e^{\mathrm{loc}}_{\boldsymbol{\alpha}}((\tau_2^\rho)^{-1})\mathrm{loc}_{\boldsymbol{\alpha}}(\widetilde{\Gamma_{v,\rho}(\tau_1^{\rho})})^{-1}e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau_1^\rho\tau_2^\rho)\\
&e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau_1^\rho\tau_2^\rho)^{-1}e^{\mathrm{loc}}_{\boldsymbol{\alpha}}((\tau_2^\rho)^{-1})^{-1}\beta((\tau_2^\rho)^{-1})\beta((\tau_2^\rho)^{-1})^{-1}_{w(\tau_2\rho)}e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau_1^{\tau_2\rho})\Bigg)_{w(\tau_1\tau_2\rho)}\\
&({}^{(\tau_2^\rho)^{-1}}\beta(\tau_2^\rho))_{w(\tau_2\rho)}/({}^{(\tau_2^\rho)^{-1}}\beta(\tau_2^\rho))_{w(\tau_1\tau_2\rho)}\\
= \ &\beta((\tau_2^\rho)^{-1})_{w(\tau_2\rho)}\beta((\tau_2^\rho)^{-1})^{-1}_{w(\tau_1\tau_2\rho)}({}^{(\tau_2^\rho)^{-1}}\alpha^{\mathrm{loc}}(\tau_2^\rho,(\tau_2^\rho)^{-1}))_{w(\tau_1\tau_2\rho)}\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau_1\tau_2)\\
&\left(e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau_1^\rho\tau_2^\rho)^{-1}e^{\mathrm{loc}}_{\boldsymbol{\alpha}}((\tau_2^\rho)^{-1})^{-1}\beta((\tau_2^\rho)^{-1})\beta((\tau_2^\rho)^{-1})^{-1}_{w(\tau_2\rho)}e^{\mathrm{loc}}_{\boldsymbol{\alpha}}((\tau_2^\rho)^{-1}\tau_1^\rho\tau_2^\rho)\right)_{w(\tau_1\tau_2\rho)}\\
&({}^{(\tau_2^\rho)^{-1}}\beta(\tau_2^\rho))_{w(\tau_2\rho)}/({}^{(\tau_2^\rho)^{-1}}\beta(\tau_2^\rho))_{w(\tau_1\tau_2\rho)}\\
= \ &\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau_1\tau_2)(\beta((\tau_2^\rho)^{-1})^{(\tau_2^\rho)^{-1}}\beta(\tau_2^\rho))_{w(\tau_2\rho)}(\beta((\tau_2^\rho)^{-1})^{(\tau_2^\rho)^{-1}}\beta(\tau_2^\rho))^{-1}_{w(\tau_1\tau_2\rho)}({}^{(\tau_2^\rho)^{-1}}\alpha^{\mathrm{loc}}(\tau_2^\rho,(\tau_2^\rho)^{-1}))_{w(\tau_1\tau_2\rho)}\\
&{}^{(\tau_2^\rho)^{-1}(\tau_1^\rho)^{-1}\tau_2^\rho}(\beta((\tau_2^\rho)^{-1})_{w(\tau_1\tau_2\rho(\tau_2^\rho)^{-1}(\tau_1^\rho)^{-1}\tau_2^\rho)}/\beta((\tau_2^\rho)^{-1})_{w(\tau_2\rho)})({}^{(\tau_2^\rho)^{-1}(\tau_1^\rho)^{-1}\tau_2^\rho}\alpha^{\mathrm{loc}}((\tau_2^\rho)^{-1},\tau_1^\rho\tau_2^\rho))^{-1}_{w(\tau_1\tau_2\rho)}\\
= \ &\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau_1\tau_2)(\pi_{w(\tau_2\rho)}/\pi_{w(\tau_1\tau_2\rho)})(\alpha^{\mathrm{loc}}((\tau_2^\rho)^{-1},\tau_2^\rho)\beta(1)/\alpha^{\mathrm{glob}}((\tau_2^\rho)^{-1},\tau_2^\rho))\\
&({}^{(\tau_2^\rho)^{-1}}\alpha^{\mathrm{loc}}(\tau_2^\rho,(\tau_2^\rho)^{-1}))_{w(\tau_1\tau_2\rho)}({}^{(\tau_2^\rho)^{-1}(\tau_1^\rho)^{-1}\tau_2^\rho}\alpha^{\mathrm{loc}}((\tau_2^\rho)^{-1},\tau_1^\rho\tau_2^\rho))^{-1}_{w(\tau_1\tau_2\rho)}\\
= \ &\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau_1\tau_2)(\pi_{w(\tau_2\rho)}/\pi_{w(\tau_1\tau_2\rho)})(\alpha^{\mathrm{loc}}((\tau_2^\rho)^{-1},\tau_2^\rho)\alpha^{\mathrm{loc}}(1,1)/\alpha^{\mathrm{glob}}((\tau_2^\rho)^{-1},\tau_2^\rho)\alpha^{\mathrm{glob}}(1,1))\\
= \ &\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau_1\tau_2)\pi_{w(\tau_2\rho)}(\alpha^{\mathrm{loc}}((\tau_2^\rho)^{-1},\tau_2^\rho)\alpha^{\mathrm{loc}}(1,1)).
\end{aligned}$$

For the fourth part we have

$$\begin{aligned}
&\overline{g}_{{}^t\mathfrak{a}^+,v,\rho}(\tau)\\
= \ &\left(\left(\mathrm{loc}_{{}^t\boldsymbol{\alpha}}\widetilde{\Gamma_{v,\rho}(\tau^{\rho})}\right)^{-1} e^{\mathrm{loc}}_{{}^t\boldsymbol{\alpha}}(\tau^{\rho})\right)_{w(\tau\rho)}\\
= \ &\left(\left(t^{-1}(\mathrm{loc}_{\boldsymbol{\alpha}}t_{w(\rho)}\widetilde{\Gamma_{v,\rho}(\tau^{\rho})}t^{-1}_{w(\rho)})t\right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^{\rho})\right)_{w(\tau\rho)}\\
= \ &\left((({}^{(\tau^\rho)^{-1}}(t^{-1}_{w(\rho)}t))/(t^{-1}_{w(\rho)}t))\left(\mathrm{loc}_{\boldsymbol{\alpha}}\widetilde{\Gamma_{v,\rho}(\tau^{\rho})}\right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^{\rho})\right)_{w(\tau\rho)}\\
= \ &(({}^{(\tau^\rho)^{-1}}(t^{-1}_{w(\rho)}t_{w(\rho)}))/(t^{-1}_{w(\rho)}t_{w(\tau\rho)}))\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau).
\end{aligned}$$

For the fifth part we have

$$\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau) = \mathrm{Art}_E^{-1}(\tau^\rho)^{-1}\alpha^{\mathrm{loc}}(1,1)_{w(\rho)} = \mathrm{Art}_E^{-1}(\tau^\rho)^{-1}.$$

□

Now suppose further that T/F is a torus which splits over E and that μ a cocharacter of T defined over $\overline{F_v}$.

If $\rho : E \hookrightarrow \overline{F_v}$ is F-linear then we saw in lemma 5.7 how to find an element $b_\rho \in T(\mathbb{A}_E)$ with

$$\mathrm{loc}_{\mathfrak{a}} \mathrm{cor}_{\alpha^{\mathrm{glob}}}({}^{\rho^{-1}}\mu) \circ (\pi_{w(\rho)}/\pi_{w(\tau\rho)}) = {}^{b_\rho}\mathrm{cor}_{\alpha^{\mathrm{loc}}}({}^{\rho^{-1}}\mu) \circ (\pi_{w(\rho)}/\pi_{w(\tau\rho)}).$$

However it turns out that the choice of the extension $\boldsymbol{\alpha}^+$ allows us to choose b_ρ more canonically, so that it is largely independent of the choice of ρ. That is what we will explain next.

Define

$$b_{\boldsymbol{\alpha}^+,v,\mu,\tau,\rho} = \prod_{\eta \in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)((\beta(\eta)_{w(\tau\rho)}/\beta(\eta)_{w(\rho)})\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau)) \in T(\mathbb{A}_E)/\overline{T(F)T(F_\infty)^0}T(F_v).$$

Also write $\overline{b}_{\mathfrak{a}^+,v,\mu,\tau}$ for its image in $T(\mathbb{A}_E)/\overline{T(F)T(F_\infty)^0}T(E)T(E_v)$, which is independent of the representative $\boldsymbol{\alpha}^+$ of $\mathfrak{a}^+$ which one chooses.

Lemma 6.4. *(1)*

$$\mathrm{loc}_{\boldsymbol{\alpha}} \mathrm{cor}_{\alpha^{\mathrm{glob}}}({}^{\rho^{-1}}\mu) \circ (\pi_{w(\rho)}/\pi_{w(\tau\rho)}) = {}^{b_{\boldsymbol{\alpha}^+,v,\mu,\tau,\rho}}\mathrm{cor}_{\alpha^{\mathrm{loc}}}({}^{\rho^{-1}}\mu) \circ (\pi_{w(\rho)}/\pi_{w(\tau\rho)}).$$

(2) $\overline{b}_{\mathfrak{a}^+,v,\mu,\tau}$ does not depend on the choice of F-linear $\rho : E^{\mathrm{ab}} \hookrightarrow \overline{F}_v$ used in its definition. (And hence justifying the notation.)

(3) If τ fixes the image $\rho(E)$ (this being independent of the choice of ρ), then

$$\overline{b}_{\mathfrak{a}^+,v,\mu,\tau} = \prod_{\rho:E\hookrightarrow\overline{F_v}} ({}^{\rho^{-1}}\mu)(\mathrm{Art}_E^{-1}\tau^{\widetilde{\rho}})^{-1},$$

where ρ runs over F-linear embeddings $E \hookrightarrow \overline{F_v}$ and where $\widetilde{\rho}$ is any extension of ρ to E^{ab}.

(4) $\overline{b}_{\mathfrak{a}^+,v,\mu,\tau_1\tau_2} = \overline{b}_{\mathfrak{a}^+,v,{}^{\tau_2}\mu,\tau_1}\overline{b}_{\mathfrak{a}^+,v,\mu,\tau_2}$.

(5) If $\sigma \in \mathrm{Gal}\,(E/F)$, then

$$\sigma\overline{b}_{\mathfrak{a}^+,v,\mu,\tau} = \overline{b}_{\mathfrak{a}^+,v,\mu,\tau} \prod_{\rho} ({}^{\rho^{-1}}({}^{\tau}\mu/\mu))(\beta(\sigma)_{w(\rho)})$$

where ρ runs over F-linear embeddings $\rho : E \hookrightarrow \overline{F_v}$.

(6) If $\chi : T \to T'$ over F, then $\overline{b}_{\mathfrak{a}^+,v,\chi\circ\mu,\tau} = \chi(\overline{b}_{\mathfrak{a}^+,v,\mu,\tau})$.

(7) If $t \in T_{2,E}(\mathbb{A}_E)$, then

$$\begin{aligned} \overline{b}_{{}^{t}\mathfrak{a}^+,v,\mu,\tau} &= \overline{b}_{\mathfrak{a}^+,v,\mu,\tau} \textstyle\prod_\rho ({}^{\rho^{-1}}\mu) \circ (\pi_{w(\rho)}/\pi_{w(\tau\rho)})(t) \\ &= \overline{b}_{\mathfrak{a}^+,v,\mu,\tau} \textstyle\prod_\rho ({}^{\rho^{-1}}(\mu/{}^{\tau}\mu))(t_{w(\rho)}) \end{aligned}$$

where ρ runs over F-linear embeddings $\rho : E \hookrightarrow \overline{F_v}$.

Proof: The first part follows immediately from lemma 5.7 because

$$\prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}(\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau)) \in T(\mathbb{A}_F).$$

For the second part, if $\sigma \in \mathrm{Gal}\,(E^{\mathrm{ab}}/F)$ then

$$\begin{array}{ll} & b_{\boldsymbol{\alpha}^+,v,\mu,\tau,\rho\circ\sigma} \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}\sigma^{-1}({}^{\rho^{-1}}\mu)\left({}^{\sigma}(\beta(\eta)_{w(\tau\rho\sigma)}/\beta(\eta)_{w(\rho\sigma)})^{\sigma}\overline{g}_{\mathfrak{a}^+,v,\rho\sigma}(\tau)\right) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)\left(({}^{\sigma}\beta(\sigma^{-1}\eta)_{w(\tau\rho\sigma)}/{}^{\sigma}\beta(\sigma^{-1}\eta)_{w(\rho\sigma)})\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau)^{\sigma}(\beta(\sigma^{-1})_{w(\rho\sigma)}/\beta(\sigma^{-1})_{w(\tau\rho\sigma)})\right) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)\Big((\pi_{\tau\rho}/\pi_{\rho})({}^{\sigma}(\alpha^{\mathrm{glob}}(\sigma^{-1},\eta)\beta(\sigma^{-1})^{\sigma^{-1}}\beta(\eta)/\alpha^{\mathrm{loc}}(\sigma^{-1},\eta))) \\ & (\pi_{\rho}/\pi_{\tau\rho})({}^{\sigma}\beta(\sigma^{-1}))\overline{g}_{\mathfrak{a}^+,v,\rho}\Big) \\ = & b_{\boldsymbol{\alpha}^+,v,\mu,\tau,\rho}\prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)\left((\pi_{\tau\rho}/\pi_{\rho})({}^{\sigma}\alpha^{\mathrm{glob}}(\sigma^{-1},\eta)/{}^{\sigma}\alpha^{\mathrm{loc}}(\sigma^{-1},\eta))\right) \\ \in & b_{\boldsymbol{\alpha}^+,v,\mu,\tau,\rho}T(E)T(E_v), \end{array}$$

as desired.

For the third part note that

$$\begin{array}{rcl} \overline{b}_{\mathfrak{a}^+,v,\mu,\tau} & = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)((\beta(\eta)_{w(\rho)}/\beta(\eta)_{w(\rho)})\mathrm{Art}_{E}^{-1}(\tau^{\rho})^{-1}) \\ & = & \prod_{\eta\in \mathrm{Gal}\,(E/F)}({}^{\eta^{-1}\rho^{-1}}\mu)(\eta^{-1}\mathrm{Art}_{E}^{-1}(\tau^{\rho})^{-1}) \\ & = & \prod_{\eta\in \mathrm{Gal}\,(E/F)}({}^{(\rho\eta)^{-1}}\mu)(\mathrm{Art}_{E}^{-1}(\tau^{\rho\eta})^{-1}). \end{array}$$

For the fourth part note that

$$\begin{array}{ll} & \overline{b}_{\mathfrak{a}^+,v,\mu,\tau_1\tau_2} \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)((\beta(\eta)_{w(\tau_1\tau_2\rho)}/\beta(\eta)_{w(\rho)})\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau_1\tau_2)) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)((\beta(\eta)_{w(\tau_1\tau_2\rho)}/\beta(\eta)_{w(\tau_2\rho)})\overline{g}_{\mathfrak{a}^+,v,\tau_2\rho}(\tau_1)(\beta(\eta)_{w(\tau_2\rho)}/\beta(\eta)_{w(\rho)})\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau_2)) \\ = & \overline{b}_{\mathfrak{a}^+,v,\mu,\tau_2}\prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{(\tau_2\rho)^{-1}\tau_2}\mu)((\beta(\eta)_{w(\tau_1\tau_2\rho)}/\beta(\eta)_{w(\tau_2\rho)})\overline{g}_{\mathfrak{a}^+,v,\tau_2\rho}(\tau_1)) \\ = & \overline{b}_{\mathfrak{a}^+,v,\mu,\tau_2}\overline{b}_{\mathfrak{a}^+,v,{}^{\tau_2}\mu,\tau_1}. \end{array}$$

For the fifth part we have

$$\begin{array}{ll} & {}^{\sigma}\overline{b}_{\mathfrak{a}^+,v,\mu,\tau}/\overline{b}_{\mathfrak{a}^+,v,\mu,\tau} \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \sigma\eta^{-1}({}^{\rho^{-1}}\mu)(\beta(\eta)_{w(\tau\rho)}/\beta(\eta)_{w(\rho)})/\prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)(\beta(\eta)_{w(\tau\rho)}/\beta(\eta)_{w(\rho)}) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)(\beta(\eta\sigma)_{w(\tau\rho)}/\beta(\eta\sigma)_{w(\rho)})/\prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)(\beta(\eta)_{w(\tau\rho)}/\beta(\eta)_{w(\rho)}) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)(\beta(\eta\sigma)_{w(\tau\rho)}\beta(\eta)_{w(\rho)}{}^{\eta}\beta(\sigma)_{w(\rho\eta)}/\beta(\eta)_{w(\tau\rho)}{}^{\eta}\beta(\sigma)_{w(\tau\rho\eta)}\beta(\eta\sigma)_{w(\rho)}) \\ & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)^{\eta}(\beta(\sigma)_{w(\tau\rho\eta)}/\beta(\sigma)_{w(\rho\eta)}) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)(\alpha^{\mathrm{glob}}(\eta,\sigma)\alpha^{\mathrm{loc}}(\eta,\sigma)_{w(\rho)}/\alpha^{\mathrm{loc}}(\eta,\sigma)_{w(\tau\rho)}\alpha^{\mathrm{glob}}(\eta,\sigma)) \\ & \prod_{\rho}({}^{\rho^{-1}}\mu)(\beta(\sigma)_{w(\tau\rho)}/\beta(\sigma)_{w(\rho)}) \\ = & \prod_{\rho}({}^{\rho^{-1}}\mu)(\beta(\sigma)_{w(\tau\rho)})/\prod_{\rho}({}^{\rho^{-1}}\mu)(\beta(\sigma)_{w(\rho)}) \\ = & \prod_{\rho}({}^{\rho^{-1}\tau}\mu)(\beta(\sigma)_{w(\rho)})/\prod_{\rho}({}^{\rho^{-1}}\mu)(\beta(\sigma)_{w(\rho)}) \\ = & \prod_{\rho}({}^{\rho^{-1}}({}^{\tau}\mu/\mu))(\beta(\sigma)_{w(\rho)}). \end{array}$$

The sixth part is clear. For the seventh we have

$$\begin{array}{rl} & \overline{b}_{t_{\mathfrak{a}^+},v,\mu,\tau}/\overline{b}_{\mathfrak{a}^+,v,\mu,\tau} \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)((t/{}^{\eta}t)_{w(\tau\rho)}({}^{\eta}t/t)_{w(\rho)}(t_{w(\rho)}/t_{w(\tau\rho)})) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)((t_{w(\tau\rho)}/{}^{\eta}t_{w(\tau\rho\eta)})({}^{\eta}t_{w(\rho\eta)}/t_{w(\rho)})(t_{w(\rho)}/t_{w(\tau\rho)})) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \eta^{-1}({}^{\rho^{-1}}\mu)^{\eta}(t_{w(\rho\eta)}/t_{w(\tau\rho\eta)}) \\ = & \prod_{\rho}({}^{\rho^{-1}}\mu)(t_{w(\rho)}/t_{w(\tau\rho)}) \\ = & \prod_{\rho}({}^{\rho^{-1}}\mu)(t_{w(\rho)})/\prod_{\rho}({}^{\rho^{-1}}\mu)(t_{w(\tau\rho)}) \\ = & \prod_{\rho}({}^{\rho^{-1}}\mu)(t_{w(\rho)})/\prod_{\rho}({}^{\rho^{-1}\tau}\mu)(t_{w(\rho)}) \\ = & \prod_{\rho}({}^{\rho^{-1}}(\mu/{}^{\tau}\mu))(t_{w(\rho)}). \end{array}$$

□

Finally consider the case $F=\mathbb{Q}$ and $v=\infty$. We will write

$$\phi_{\boldsymbol{\alpha},\infty,\mu,\tau,\rho}=\mathrm{cor}\,_{\alpha^{\mathrm{glob}}}({}^{\rho^{-1}}\mu)\circ(\pi_{w(\rho)}/\pi_{w(\tau\rho)})\in \boldsymbol{\phi}_{T,m\{\mu\},\tau}\subset Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}),T(E)),$$

so that

$$\mathrm{res}^{\infty}\mathrm{loc}_{\mathfrak{a}}\phi_{\boldsymbol{\alpha},\infty,\mu,\tau,\rho}={}^{b_{\boldsymbol{\alpha}^+,\infty,\mu,\tau,\rho}}1.$$

We can calculate the element $\overline{b}_{[\boldsymbol{\alpha}_0^+],\infty,\mu,\tau}$ (with the notation of the end of sections 5.1 and 6.1). We have:

$$\begin{array}{rl} & \overline{b}_{[\boldsymbol{\alpha}^+_{0,\varphi}],\infty,\mu,\tau} \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/\mathbb{Q})} \eta^{-1}({}^{\rho_0^{-1}}\mu)((\beta(\eta)_{w(\tau\rho_0)}/\beta(\eta)_{w(\rho_0)})((\mathrm{loc}_{\boldsymbol{\alpha}_0}\Gamma_{\infty,\rho_0,0}(\tau^{\rho_0}))^{-1}e^{\mathrm{loc}}_{\boldsymbol{\alpha}_0}(\tau^{\rho_0}))_{w(\tau\rho_0)}) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/\mathbb{Q})} \eta^{-1}({}^{\rho_0^{-1}}\mu)((\beta(\eta)_{w(\tau\rho_0)}/\beta(\eta)_{w(\rho_0)})(\Gamma_{\infty,\rho_0,0}(\tau^{\rho_0})^{-1}\beta(\tau^{\rho_0})e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\tau^{\rho_0}))_{w(\tau\rho_0)}) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/\mathbb{Q})} \eta^{-1}({}^{\rho_0^{-1}}\mu)((\Gamma_{\infty,\rho_0,0}(\tau^{\rho_0})^{-1}e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\tau^{\rho_0}))^{\tau^{\rho_0,-1}}(\beta(\tau^{\rho_0})_{w(\rho_0)})\beta(\eta)_{w(\tau\rho_0)}/\beta(\eta)_{w(\rho_0)}) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/\mathbb{Q})} \eta^{-1}({}^{\rho_0^{-1}}\mu)((\Gamma_{\infty,\rho_0,0}(\tau^{\rho_0})^{-1}e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\tau^{\rho_0}))^{\tau^{\rho_0,-1}}\alpha(\tau^{\rho_0},\tau^{\rho_0,-1})^{-1}\alpha(\eta,\eta^{-1}\tau^{\rho_0,-1})^{-1}\alpha(\eta,\eta^{-1})) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/\mathbb{Q})} \eta^{-1}({}^{\rho_0^{-1}}\mu)(\Gamma_{\infty,\rho_0,0}(\tau^{\rho_0})^{-1}e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\tau^{\rho_0})e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\tau^{\rho_0})^{-1}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\tau^{\rho_0,-1})^{-1}e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\tau^{\rho_0})^{-1})e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\tau^{\rho_0}) \\ & (e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\tau^{\rho_0,-1})e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\eta^{-1}\tau^{\rho_0,-1})^{-1}e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\eta)^{-1})(e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\eta)e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\eta^{-1}))) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/\mathbb{Q})} \eta({}^{\rho^{-1}}\mu)(\Gamma_{\infty,\rho_0,0}(\tau^{\rho})^{-1}e^{\mathrm{glob}}(\eta(\tau^{\rho})^{-1})^{-1}e^{\mathrm{glob}}(\eta)). \end{array}$$

6.3. **Change of field.** Again in order to compare $\mathcal{H}(E/F)^+$ with $\mathcal{H}(D/F)^+$ it will be convenient to to describe an intermediate theory. So suppose that $D\supset E\supset F$ are finite Galois extensions of a number field F, and that $\mathfrak{a}\in\mathcal{H}(E/F)_D$.

Suppose that v is a place of F. We will write $R_{D,E,F,v}$ for the set of pairs (ρ',ρ), where $\rho':E^{\mathrm{ab}}\to\overline{F_v}$ and $\rho:D\to\overline{F_v}$ are F-linear maps with $\rho'|_E=\rho|_E$. We will write $w(\rho')$, (resp. $u(\rho')$, resp. $u(\rho)$) for the place of E (resp. E^{ab}, resp. D) induced by ρ' (resp. ρ', resp. ρ). We have a morphism

$$\theta_{(\rho',\rho)}:W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v,\rho,D}\longrightarrow W_{E^{\mathrm{ab}}/F,D}$$

well defined up to Δ_E-conjugation. (See section 2.8.) By a *rigidification* of $\mathfrak{a}$ *adapted to* $\theta_{(\rho',\rho)}$ we shall mean an isomorphism of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathbb{A}_D^\times/D^\times & \longrightarrow & W_{E^{\mathrm{ab}}/F,D} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \\ & & \| & & \widetilde{\Gamma}\downarrow\wr & & \| & & \\ (0) & \longrightarrow & \mathbb{A}_D^\times/D^\times & \longrightarrow & W_{E/F,D,\boldsymbol{\alpha}} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \end{array}$$

such that there exists an isomorphism of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & D_{w(\rho)}^\times & \longrightarrow & W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v,\rho,D} & \longrightarrow & \mathrm{Gal}\,(D/F)_{w(\rho)} & \longrightarrow & (0) \\ & & \| & & \widetilde{\Theta}_{v,(\rho',\rho)}\downarrow\wr & & \| & & \\ (0) & \longrightarrow & D_{w(\rho)}^\times & \longrightarrow & W_{E_{w(\rho)}/F_v,D,\boldsymbol{\alpha}} & \longrightarrow & \mathrm{Gal}\,(D/F)_{w(\rho)} & \longrightarrow & (0), \end{array}$$

(such an isomorphism being unique up to composition with conjugation by an element of $D_{w(\rho)}^\times$) with

$$\widetilde{\Gamma}\circ\theta_{(\rho',\rho)} = \iota^{\mathfrak{a}}_{w(\rho')}\circ\widetilde{\Theta}_{v,(\rho',\rho)}.$$

As above, such a rigidification adapted to $\theta_{(\rho',\rho)}$ exists. (The main point being that any two such maps $W_{(\rho(E)F_v)^{\mathrm{ab}}/F_v,\rho,D}\to W_{E/F,D,\mathfrak{a}}$ of extensions must differ by an element of $Z^1(\mathrm{Gal}\,(D/F)_{w(\rho)},\mathbb{A}_D^\times/D^\times)$ and hence by conjugation by an element of $\mathbb{A}_D^\times/D^\times$. Altering an initial choice of $\widetilde{\Gamma}$ by such a conjugation, we get a $\widetilde{\Gamma}$ adapted to $\theta_{(\rho',\rho)}$.) Moreover a rigidification adapted to $\theta_{(\rho',\rho)}$ is unique up to composition with conjugation by an element of

$$(\mathbb{A}_D^\times/D^\times)^{\mathrm{Gal}\,(D/F)_{w(\rho)}}D_{w(\rho)}^\times.$$

Note that if $\widetilde{\Gamma}$ is adapted to $\theta_{(\rho',\rho)}$ and if $a\in\Delta_E$, then $\mathrm{conj}_a\circ\widetilde{\Gamma}$ is adapted to $\mathrm{conj}_a\circ\theta_{(\rho',\rho)}$. If $\widetilde{\Gamma}$ is a rigidification of $\mathfrak{a}$ adapted to $\theta_{(\rho',\rho)}$ then

$$\mathrm{conj}_{t_{w(\rho)}}\circ\widetilde{\Gamma}$$

is a rigidification of ${}^t\mathfrak{a}$ adapted to $\theta_{(\rho',\rho)}$ (because

$$\iota^{{}^t\mathfrak{a}}_{w(\rho)} = \mathrm{conj}_{t_{w(\rho)}}\circ\iota^{\mathfrak{a}}_{w(\rho)}).$$

If $\widetilde{\Gamma}$ is a rigidification of $\mathfrak{a}$ adapted to $\theta_{(\rho',\rho)}$, if $\boldsymbol{\alpha}\in\mathfrak{a}$ and if $\sigma\in\mathrm{Gal}\,(E^{\mathrm{ab}}/F)|_{\mathrm{Gal}\,(D/F)}$ then

$$\widetilde{\Gamma}^\sigma = \widetilde{\Gamma}^{\sigma,\boldsymbol{\alpha}} = \mathrm{conj}_{\beta(\sigma^{-1})_{w(\rho\sigma)}}\circ\mathrm{conj}_{e^{\mathrm{glob}}(\sigma^{-1})}\circ\widetilde{\Gamma}\circ\mathrm{conj}_\sigma$$

is rigidification data adapted to $\mathrm{conj}_{\sigma^{-1}}\circ\theta_{(\rho',\rho)}$ (which is a possible choice of $\theta_{(\rho',\rho)\circ\sigma}$). Indeed

$$\widetilde{\Gamma}^\sigma\circ(\mathrm{conj}_{\sigma^{-1}}\circ\theta_{(\rho',\rho)}) = \mathrm{conj}_{\beta(\sigma^{-1})_{w(\rho\sigma)}}\circ\mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}\circ\iota^{\mathfrak{a}}_{w(\rho)}\circ\widetilde{\Theta}_{v,(\rho',\rho)} = \iota^{\mathfrak{a}}_{w(\rho\sigma)}\circ\mathrm{conj}_{e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma^{-1})}\circ\widetilde{\Theta}_{v,(\rho',\rho)},$$

while $\mathrm{conj}_{e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma^{-1})}\circ\widetilde{\Theta}_{v,(\rho',\rho)}: W_{((\rho\sigma)(E)F_v)^{\mathrm{ab}}/F_v,\rho,D}\xrightarrow{\sim} W_{E_{w(\rho\sigma)}/F_v,D,\boldsymbol{\alpha}}$. Note that

$$\widetilde{\Gamma}^{\sigma,{}^\gamma\boldsymbol{\alpha}} = \mathrm{conj}_{\gamma^{\mathrm{loc}}(\sigma^{-1})^{-1}_{w(\rho\sigma)}}\circ\widetilde{\Gamma}^{\sigma,\boldsymbol{\alpha}}$$

and

$$\widetilde{\Gamma}^{\sigma_1\sigma_2,\boldsymbol{\alpha}} = \mathrm{conj}_{\alpha^{\mathrm{loc}}(\sigma_1^{-1},\sigma_2^{-1})_{w(\rho\sigma_1\sigma_2)}} \circ (\widetilde{\Gamma}^{\sigma_1,\boldsymbol{\alpha}})^{\sigma_2,\boldsymbol{\alpha}}.$$

Suppose that $\widetilde{\Gamma} : W_{E^{\mathrm{ab}}/F} \to W_{E/F,\mathfrak{a}}$ is rigidification of $\mathfrak{a} \in \mathcal{H}(E/F)$ adapted to $\theta_{\rho'} : W_{(\rho'(E)F_v)^{\mathrm{ab}}/F_v} \to W_{E^{\mathrm{ab}}/F}$. Recall that $\theta_{(\rho',\rho)} : W_{(\rho'(E)F_v)^{\mathrm{ab}}/F_v,\rho,D} \to W_{E^{\mathrm{ab}}/F,D}$ is defined as $1 \times (\theta_{\rho'} \times 1)$. We define

$$\widetilde{\Gamma}^D : W_{E/F,D} = (\mathbb{A}_D^\times/D^\times \rtimes W_{E^{\mathrm{ab}}/F}|_{\mathrm{Gal}\,(D/F)})/\mathbb{A}_E^\times \xrightarrow{\sim} W_{E/F,D,\mathrm{inf}_{D/E}\,\mathfrak{a}} = (\mathbb{A}_D^\times/D^\times \rtimes W_{E^{\mathrm{ab}}/E,\mathfrak{a}}|_{\mathrm{Gal}\,(D/F)})/(\mathbb{A}_E^\times/E^\times)$$

by

$$(a,(\sigma,\tau)) \longmapsto (a,(\widetilde{\Gamma}(\sigma),\tau)).$$

If $\widetilde{\Theta}_{v,\rho'} : W_{(\rho'(E)F_v)^{\mathrm{ab}}/F_v} \xrightarrow{\sim} W_{E_{w(\rho')}/F_v,\mathfrak{a}}$ with

$$\widetilde{\Gamma} \circ \theta_{\rho'} = \iota^{\mathfrak{a}}_{w(\rho')} \circ \widetilde{\Theta}_{v,\rho'},$$

then we have

$$\widetilde{\Gamma}^D \circ \theta_{(\rho',\rho)} = \iota^{\mathrm{inf}_{D/E}\,\mathfrak{a}}_{w(\rho')} \circ \widetilde{\Theta}^D_{v,(\rho',\rho)},$$

where $\widetilde{\Theta}^D_{v,(\rho',\rho)}$ is the map

$$\begin{array}{ccc} W_{(\rho'(E)F_v)^{\mathrm{ab}}/F_v,\rho,D} & \xrightarrow{\sim} & W_{E_{w(\rho')}/F_v,D,\mathrm{inf}_{D/E}\,\mathfrak{a}} \\ \| & & \| \\ (D^\times_{w(\rho')} \rtimes W_{(\rho'(E)F_v)^{\mathrm{ab}}/F_v}|_{\mathrm{Gal}\,(D/F)_{w(\rho')}})/E^\times_{w(\rho')} & \xrightarrow{\sim} & (D^\times_{w(\rho')} \rtimes W_{E_{w(\rho')}/F_v,\mathfrak{a}}|_{\mathrm{Gal}\,(D/F)_{w(\rho')}})/E^\times_{w(\rho')} \\ (a,(\sigma,\tau)) & \longmapsto & (a,(\widetilde{\Theta}_{v,\rho}(\sigma),\tau)). \end{array}$$

Thus $\widetilde{\Gamma}^D$ is a rigidification of $\mathrm{inf}_{D/E}\,\mathfrak{a}$ adapted to $\theta_{(\rho',\rho)}$. Note that

$$\widetilde{\Gamma}^{D,(\sigma',\sigma),\mathrm{inf}_{D/E}\,\boldsymbol{\alpha}} = \widetilde{\Gamma}^{\sigma',\boldsymbol{\alpha},D}$$

and

$$(\mathrm{conj}_{t_{w(\rho')}} \circ \widetilde{\Gamma})^D = \mathrm{conj}_{t_{w(\rho')}} \circ \widetilde{\Gamma}^D.$$

Suppose now that $\widetilde{\Gamma} : W_{D^{\mathrm{ab}}/F} \to W_{D/F,\mathfrak{a}}$ is rigidification data for $\mathfrak{a} \in \mathcal{H}(D/F)$ adapted to $\theta_{\widetilde{\rho}} : W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/F_v} \to W_{D^{\mathrm{ab}}/F}$. If $\boldsymbol{\alpha} \in \mathfrak{a}$ we define $\widetilde{\Gamma}^{E,\widetilde{\rho},\boldsymbol{\alpha}} : W_{E^{\mathrm{ab}}/F,D} \to W_{E/F,D,\eta_{D/E,*}\mathfrak{a}}$ to be the map

$$\begin{array}{rcl} \widetilde{\Gamma}^{E,\widetilde{\rho},\boldsymbol{\alpha}} : (\mathbb{A}_D^\times/D^\times \rtimes W_{D^{\mathrm{ab}}/F})/\mathbb{A}_D^\times & \longrightarrow & (\mathbb{A}_D^\times/D^\times \rtimes W_{D/F,\mathfrak{a}})/\mathbb{A}_D^\times \\ [(a,\sigma)] & \longmapsto & \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\eta)_{\eta u}}[(a\gamma_{\boldsymbol{\alpha},E}(\widetilde{\Gamma}(\sigma)),\widetilde{\Gamma}(\sigma))], \end{array}$$

where

$$\begin{array}{rcl} \gamma_{\boldsymbol{\alpha},E} : W_{D/F,\boldsymbol{\alpha}} & \longrightarrow & \mathbb{A}_D^\times/D^\times \\ ae^{\mathrm{glob}}(\sigma) & \longmapsto & (N_{D/E}(a)/a^{[D:E]}) \prod_{\eta\in\mathrm{Gal}\,(D/E)} \alpha^{\mathrm{glob}}(\eta,\sigma)/\alpha^{\mathrm{glob}}(\sigma,\eta) \\ & = & \prod_{\eta\in\mathrm{Gal}\,(D/E)} e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta)(ae^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma))e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta\sigma)^{-1})/((ae^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma))e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta)e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma\eta)^{-1}). \end{array}$$

This is well defined because, if $a \in \mathbb{A}_D^\times/D^\times$, then

$$\begin{array}{rcl} \gamma_{\boldsymbol{\alpha},E}(a) & = & (N_{D/E}(a\alpha^{\mathrm{glob}}(1,1)^{-1})/(a\alpha^{\mathrm{glob}}(1,1)^{-1})^{[D:E]}) \prod_{\eta\in\mathrm{Gal}\,(D/E)} \alpha^{\mathrm{glob}}(\eta,1)/\alpha^{\mathrm{glob}}(1,\eta) \\ & = & N_{D/E}(a)/a^{[D:E]}. \end{array}$$

(Note that on the left hand side of the definition of $\widetilde{\Gamma}^{E,\widetilde{\rho},\boldsymbol{\alpha}}$ the map $\mathbb{A}_D^\times \to \mathbb{A}_D^\times/D^\times \rtimes W_{D^{\mathrm{ab}}/F}$ sends a to $((N_{D/E}a)^{-1}, r_D(a))$; while on the right hand side the map $\mathbb{A}_D^\times \to \mathbb{A}_D^\times/D^\times \rtimes W_{D/F,\boldsymbol{\alpha}}$ sends a to $(a^{-[D:E]}, a)$.) It is a homomorphism because

$$\begin{array}{rl} & \gamma_{\boldsymbol{\alpha},E}(a_1 e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma_1) a_2 e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma_2)) \\ = & \gamma_{\boldsymbol{\alpha},E}(a_1{}^{\sigma_1} a_2 \alpha^{\mathrm{glob}}(\sigma_1,\sigma_2) e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma_1\sigma_2)) \\ = & (N_{D/E}(a_1{}^{\sigma_1}a_2)/(a_1{}^{\sigma_1}a_2)^{[D:E]}) \prod_{\eta\in \mathrm{Gal}\,(D/E)} {}^{\eta}\alpha^{\mathrm{glob}}(\sigma_1,\sigma_2)\alpha^{\mathrm{glob}}(\eta,\sigma_1\sigma_2)/\alpha^{\mathrm{glob}}(\sigma_1,\sigma_2)\alpha^{\mathrm{glob}}(\sigma_1\sigma_2,\eta) \\ = & (N_{D/E}(a_1{}^{\sigma_1}a_2)/(a_1{}^{\sigma_1}a_2)^{[D:E]}) \prod_{\eta\in \mathrm{Gal}\,(D/E)} \alpha^{\mathrm{glob}}(\eta\sigma_1,\sigma_2)\alpha^{\mathrm{glob}}(\eta,\sigma_1)/{}^{\sigma_1}\alpha^{\mathrm{glob}}(\sigma_2,\eta)\alpha^{\mathrm{glob}}(\sigma_1,\sigma_2\eta) \\ = & (N_{D/E}(a_1{}^{\sigma_1}a_2)/(a_1{}^{\sigma_1}a_2)^{[D:E]}) \prod_{\eta\in \mathrm{Gal}\,(D/E)} \alpha^{\mathrm{glob}}(\sigma_1\eta,\sigma_2)\alpha^{\mathrm{glob}}(\eta,\sigma_1)/{}^{\sigma_1}\alpha^{\mathrm{glob}}(\sigma_2,\eta)\alpha^{\mathrm{glob}}(\sigma_1,\eta\sigma_2) \\ = & (N_{D/E}(a_1)^{\sigma_1}N_{D/E}(a_2))/(a_1{}^{\sigma_1}a_2)^{[D:E]}) \prod_{\eta\in \mathrm{Gal}\,(D/E)} {}^{\sigma_1}\alpha^{\mathrm{glob}}(\eta,\sigma_2)\alpha^{\mathrm{glob}}(\eta,\sigma_1)/{}^{\sigma_1}\alpha^{\mathrm{glob}}(\sigma_2,\eta)\alpha^{\mathrm{glob}}(\sigma_1,\eta) \\ = & \gamma_{\boldsymbol{\alpha},E}(a_1 e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma_1))^{\sigma_1}\gamma_{\boldsymbol{\alpha},E}(a_2 e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma_2)). \end{array}$$

Also note that

$$\gamma_{\gamma_{\boldsymbol{\alpha}},E}(i_{\gamma^{\mathrm{glob}}}(\sigma)) = \gamma_{\boldsymbol{\alpha},E}(\sigma)^{(\sigma-1)} \prod_{\eta\in \mathrm{Gal}\,(D/E)} \gamma^{\mathrm{glob}}(\eta)$$

and

$$\gamma_{t_{\boldsymbol{\alpha}},E} = \gamma_{\boldsymbol{\alpha},E}$$

and

$$\gamma_{\boldsymbol{\alpha},E}(\mathrm{conj}_a(\sigma)) = (N_{D/E}(a/{}^{\sigma}a)/(a/{}^{\sigma}a)^{[D:E]})\gamma_{\boldsymbol{\alpha},E}(\sigma)$$

for $a \in \mathbb{A}_D^\times/D^\times$, and

$$\begin{array}{rcl} {}^{\sigma}\gamma_{E,\boldsymbol{\alpha}}(\tau) & = & \gamma_{E,\boldsymbol{\alpha}}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma))^{-1}\gamma_{E,\boldsymbol{\alpha}}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma)\tau) \\ & = & \gamma_{E,\boldsymbol{\alpha}}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma))^{-1}\gamma_{E,\boldsymbol{\alpha}}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma)\tau e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma)^{-1})^{\sigma\tau\sigma^{-1}}\gamma_{E,\boldsymbol{\alpha}}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma)) \\ & = & {}^{(\sigma\tau\sigma^{-1}-1)}\gamma_{E,\boldsymbol{\alpha}}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma))\gamma_{E,\boldsymbol{\alpha}}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma)\tau e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma)^{-1}). \end{array}$$

Lemma 6.5. *Suppose that $\widetilde{\Gamma} : W_{D^{\mathrm{ab}}/F} \to W_{D/F,\mathfrak{a}}$ is rigidification data for $\mathfrak{a} \in \mathcal{H}(D/F)$ adapted to $\theta_{\widetilde{\rho}} : W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/F_v} \to W_{D^{\mathrm{ab}}/F}$, and that $\boldsymbol{\alpha} \in \mathfrak{a}$. Then $\widetilde{\Gamma}^{E,\widetilde{\rho},\boldsymbol{\alpha}}$ is rigidification data for $\eta_{D/E,*}\mathfrak{a}$ adapted to $\theta_{(\widetilde{\rho},\widetilde{\rho})}$.*

Proof: Suppose that $\widetilde{\Theta} : W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/F_v} \xrightarrow{\sim} W_{D_{u(\widetilde{\rho})}/F_v,\mathfrak{a}}$ is an isomorphism of extensions such that $\iota^{\mathfrak{a}}_{u(\widetilde{\rho})} \circ \widetilde{\Theta} = \widetilde{\Gamma} \circ \theta_{\widetilde{\rho}}$. Then we define an isomorphism

$$\widetilde{\Theta}^E : W_{(\widetilde{\rho}(E)F_v)^{\mathrm{ab}}/F_v,\widetilde{\rho}|_D,D} \xrightarrow{\sim} W_{E_{w(\widetilde{\rho})}/F_v,D,\eta_{D/E,*}\mathfrak{a}}$$

to be the map

$$\begin{array}{rcl} ((D^\times_{w(\widetilde{\rho})} \rtimes \mathrm{Gal}\,(D/E)) \rtimes W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/F_v})/W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/(\widetilde{\rho}(E)F_v)} & \longrightarrow & ((D^\times_{w(\widetilde{\rho})} \rtimes \mathrm{Gal}\,(D/E)) \rtimes W_{D_{u(\widetilde{\rho})}/F_v,\boldsymbol{\alpha}})/(W_{D_{u(\widetilde{\rho})}/F_v} \\ {[((a,\sigma),\tau)]} & \longmapsto & [((a,\sigma),\widetilde{\Theta}(\tau))]. \end{array}$$

It will suffice to show the equality of maps

$$\widetilde{\Gamma}^{E,\widetilde{\rho}} \circ \theta_{(\widetilde{\rho},\widetilde{\rho})} = \iota^{\eta_{D/E,*}\mathfrak{a}}_{w(\widetilde{\rho})} \circ \widetilde{\Theta}^E : W_{(\widetilde{\rho}(E)F_v)^{\mathrm{ab}}/F_v,\widetilde{\rho}|_D,D} \longrightarrow W_{E/F,D,\eta_{D/E,*}\mathfrak{a}},$$

which we will realize more concretely as maps

$$((D^\times_{w(\widetilde{\rho})} \rtimes \mathrm{Gal}\,(D/E)) \rtimes W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/F_v})/W_{(\widetilde{\rho}(D)F_v)^{\mathrm{ab}}/(\widetilde{\rho}(E)F_v)} \longrightarrow (\mathbb{A}_D^\times/D^\times \rtimes W_{D/F,\boldsymbol{\alpha}})/(\mathbb{A}_D^\times/.D^\times).$$

The first map, $\widetilde{\Gamma}^{E,\widetilde{\rho}} \circ \theta_{(\widetilde{\rho},\widetilde{\rho})}$, sends $a \in D^\times_{w(\widetilde{\rho})}$ to $[(a,1)]$. It sends $\sigma \in \mathrm{Gal}\,(D/E)$ first to

$$[(r_D^{-1}(\mathrm{tr}_{W^{\mathrm{ab}}_{D^{\mathrm{ab}}/E}/W^{\mathrm{ab}}_{D^{\mathrm{ab}}/D}} \widetilde{\sigma})^{-1}, \widetilde{\sigma})] \in (\mathbb{A}_D^\times/D^\times \rtimes W_{D^{\mathrm{ab}}/F})/(\mathbb{A}_D^\times/D^\times)$$

and then to

$$\begin{array}{rl} & \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)} \beta(\eta)_{\eta u}}[((\prod_{\eta\in\mathrm{Gal}\,(D/E)} e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta)\widetilde{\Gamma}(\widetilde{\sigma})e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta\sigma)^{-1})^{-1} \\ & \prod_{\eta\in\mathrm{Gal}\,(D/E)}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta)\widetilde{\Gamma}(\widetilde{\sigma})e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta\sigma)^{-1})/(\widetilde{\Gamma}(\widetilde{\sigma})e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta)e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma\eta)^{-1}), \widetilde{\Gamma}(\widetilde{\sigma}))] \\ = & \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)} \beta(\eta)_{\eta u}}[((\prod_{\eta\in\mathrm{Gal}\,(D/E)} \widetilde{\Gamma}(\widetilde{\sigma})e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta)e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma\eta)^{-1})^{-1}, \widetilde{\Gamma}(\widetilde{\sigma}))] \\ \in & (\mathbb{A}_D^\times/D^\times \rtimes W_{D/F,\boldsymbol{\alpha}})/(\mathbb{A}_D^\times/D^\times), \end{array}$$

where $\widetilde{\sigma} \in W_{D^{\mathrm{ab}}/E}$ is any lift of σ. It also sends $\tau \in W_{(\rho(D)F_v)^{\mathrm{ab}}/F_v}$ to

$$\begin{array}{l} \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)} \beta(\eta)_{\eta u}}[(\prod_{\eta\in\mathrm{Gal}\,(D/E)}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta)(\widetilde{\Gamma}\circ\theta_{\widetilde{\rho}})(\tau)e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta\tau)^{-1})/ \\ ((\widetilde{\Gamma}\circ\theta_{\widetilde{\rho}})(\tau)e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta)e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\tau\eta)^{-1}), (\widetilde{\Gamma}\circ\theta_{\widetilde{\rho}})(\tau))] \in (\mathbb{A}_D^\times/D^\times \rtimes W_{D/F,\mathfrak{a}})/(\mathbb{A}_D^\times/D^\times). \end{array}$$

On the other hand, using lemma 5.5, the second map sends $a \in D^\times_{w(\widetilde{\rho})}$ to $[(a,1)]$. It sends $\sigma \in \mathrm{Gal}\,(D/E)$ to

$$\begin{array}{rl} & \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)} \beta(\eta)_{\eta u}}[((\prod_{\eta\in\mathrm{Gal}\,(D/E)} \widetilde{\sigma}e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta)e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma\eta)^{-1})^{-1}, \widetilde{\sigma})] \\ \in & (\mathbb{A}_D^\times/D^\times \rtimes W_{D/F,\mathfrak{a}})/(\mathbb{A}_D^\times/D^\times), \end{array}$$

where $\widetilde{\sigma} \in W_{D/E,\mathfrak{a}}$ is any lift of σ. It also sends $\tau \in W_{(\rho(D)F_v)^{\mathrm{ab}}/F_v}$ to

$$\begin{array}{l} \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)} \beta(\eta)_{\eta u}}[(\prod_{\eta\in\mathrm{Gal}\,(D/E)}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta)((\iota^{\mathfrak{a}}_{u(\widetilde{\rho})}\circ\widetilde{\Theta})(\tau)e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta\tau)^{-1})/ \\ ((\iota^{\mathfrak{a}}_{u(\widetilde{\rho})}\circ\widetilde{\Theta})(\tau)e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\eta)e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\tau\eta)^{-1}), (\iota^{\mathfrak{a}}_{u(\widetilde{\rho})}\circ\widetilde{\Theta})(\tau))] \in (\mathbb{A}_D^\times/D^\times \rtimes W_{D/F,\mathfrak{a}})/(\mathbb{A}_D^\times/D^\times). \end{array}$$

The lemma follows. □

It is straightforward to verify that

$$(i_{\gamma^{\mathrm{glob}}} \circ \widetilde{\Gamma})^{E,\widetilde{\rho},{}^\gamma\boldsymbol{\alpha}} = \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)} \gamma^{\mathrm{loc}}(\eta)^{-1}_{\eta u(\widetilde{\rho})}} \circ i_{\eta_{D/E}\circ\gamma^{\mathrm{glob}}} \circ \widetilde{\Gamma}^{E,\widetilde{\rho},\boldsymbol{\alpha}}$$

and

$$(\mathrm{conj}_{t_{u(\widetilde{\rho})}} \circ \widetilde{\Gamma})^{E,\widetilde{\rho},{}^t\boldsymbol{\alpha}} = \mathrm{conj}_{\eta_{D/E}(t)_{w(\widetilde{\rho})}} \circ \widetilde{\Gamma}^{E,\widetilde{\rho},\boldsymbol{\alpha}}.$$

If $\sigma \in \mathrm{Gal}\,(D^{\mathrm{ab}}/F)$, then we have that

$$\begin{aligned}
& (\widetilde{\Gamma}^{\sigma,\boldsymbol{\alpha}})^{E,\widetilde{\rho}\sigma,\boldsymbol{\alpha}}(a,\tau) \\
= \; & \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho}\sigma)}}(a\gamma_{\boldsymbol{\alpha},E}(\widetilde{\Gamma}^{\sigma,\boldsymbol{\alpha}}(\tau)),\widetilde{\Gamma}^{\sigma,\boldsymbol{\alpha}}(\tau)) \\
= \; & \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho}\sigma)}}(a\gamma_{\boldsymbol{\alpha},E}({}^{(1-\tau)}\beta(\sigma^{-1})_{u(\widetilde{\rho}\sigma)}\mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}(\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau)))), \\
& {}^{(1-\tau)}\beta(\sigma^{-1})_{u(\widetilde{\rho}\sigma)}\mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}(\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau)))) \\
= \; & \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho}\sigma)}}(a^{(1-\tau)}(N_{D/E}\beta(\sigma^{-1})_{u(\widetilde{\rho}\sigma)}/\beta(\sigma^{-1})^{[D:E]}_{u(\widetilde{\rho}\sigma)}) \\
& \gamma_{\boldsymbol{\alpha},E}(\mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}(\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau))))^{(1-\tau)}\beta(\sigma^{-1})^{[D:E]}_{u(\widetilde{\rho}\sigma)},\mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}(\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau)))) \\
= \; & \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho}\sigma)}}\mathrm{conj}_{N_{D/E}\beta(\sigma^{-1})_{u(\widetilde{\rho}\sigma)}}(a\gamma_{\boldsymbol{\alpha},E}(\mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}(\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau)))), \\
& \mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}(\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau)))).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
& (\widetilde{\Gamma}^{E,\widetilde{\rho},\boldsymbol{\alpha}})^{(\sigma|_{E^{\mathrm{ab}}},\sigma|_D),\eta_{D/E,*}\boldsymbol{\alpha}}(a,\tau) \\
= \; & \mathrm{conj}_{(\eta_{D/E}\circ\beta)(\sigma^{-1})_{w(\widetilde{\rho}\sigma)}}\circ\mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}(\widetilde{\Gamma}^{E,\widetilde{\rho},\boldsymbol{\alpha}}({}^\sigma a,\mathrm{conj}_\sigma(\tau))) \\
= \; & \mathrm{conj}_{(\eta_{D/E}\circ\beta)(\sigma^{-1})_{w(\widetilde{\rho}\sigma)}}\circ\mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}\circ\mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}}({}^\sigma a\gamma_{\boldsymbol{\alpha},E}(\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau))),\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau))) \\
= \; & \mathrm{conj}_{(\eta_{D/E}\circ\beta)(\sigma^{-1})_{w(\widetilde{\rho}\sigma)}}\circ\mathrm{conj}_{\sigma^{-1}\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}}(a^{\sigma^{-1}}\gamma_{\boldsymbol{\alpha},E}(\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau))), \\
& \mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}(\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau)))) \\
= \; & \mathrm{conj}_{(\eta_{D/E}\circ\beta)(\sigma^{-1})_{w(\widetilde{\rho}\sigma)}}\circ\mathrm{conj}_{\sigma^{-1}\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}}(a^{(\tau-1)}\gamma_{E,\boldsymbol{\alpha}}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})) \\
& \gamma_{E,\boldsymbol{\alpha}}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau))e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})^{-1}),\mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}(\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau)))) \\
= \; & \mathrm{conj}_{(\eta_{D/E}\circ\beta)(\sigma^{-1})_{w(\widetilde{\rho}\sigma)}}\circ\mathrm{conj}_{\sigma^{-1}\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}}\circ\mathrm{conj}^{-1}_{\gamma_{E,\boldsymbol{\alpha}}(e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1}))} \\
& (a\gamma_{E,\boldsymbol{\alpha}}(\mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}(\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau)))),\mathrm{conj}_{e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma^{-1})}(\widetilde{\Gamma}(\mathrm{conj}_\sigma(\tau)))) \\
= \; & \mathrm{conj}_A((\widetilde{\Gamma}^{\sigma,\boldsymbol{\alpha}})^{E,\widetilde{\rho}\sigma,\boldsymbol{\alpha}}(a,\tau)),
\end{aligned}$$

where

$$\begin{aligned}
& A \\
= \; & \textstyle\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\sigma^{-1})_{\eta u(\widetilde{\rho}\sigma)}{}^{\sigma^{-1}}\beta(\eta)_{\eta u(\widetilde{\rho})}(\alpha^{\mathrm{glob}}(\sigma^{-1},\eta)/\alpha^{\mathrm{glob}}(\eta,\sigma^{-1}))\beta(\eta)^{-1}_{\eta u(\widetilde{\rho}\sigma)}{}^\eta\beta(\sigma^{-1})^{-1}_{u(\widetilde{\rho}\sigma)} \\
= \; & \textstyle\prod_{\eta\in\mathrm{Gal}\,(D/E)}\beta(\sigma^{-1})_{\eta\sigma^{-1}u(\widetilde{\rho})}({}^{\sigma^{-1}}\beta(\eta))_{\sigma^{-1}\eta u(\widetilde{\rho})}(\alpha^{\mathrm{glob}}(\sigma^{-1},\eta)/\alpha^{\mathrm{glob}}(\eta,\sigma^{-1}))\beta(\eta)^{-1}_{\eta u(\widetilde{\rho}\sigma)}{}^\eta\beta(\sigma^{-1})^{-1}_{u(\widetilde{\rho}\sigma)} \\
= \; & \textstyle\prod_{\eta\in\mathrm{Gal}\,(D/E)}(\beta(\sigma^{-1})^{\sigma^{-1}}\beta(\eta)\beta(\sigma^{-1}\eta)^{-1}\alpha^{\mathrm{glob}}(\sigma^{-1},\eta))_{\sigma^{-1}\eta u(\widetilde{\rho})}\beta(\sigma^{-1}\eta)_{\sigma^{-1}\eta u(\widetilde{\rho})}/ \\
& ((\alpha^{\mathrm{glob}}(\eta,\sigma^{-1})\beta(\eta)^\eta\beta(\sigma^{-1})\beta(\eta\sigma^{-1})^{-1})_{\eta u(\widetilde{\rho}\sigma)}\beta(\eta\sigma^{-1})_{\eta u(\widetilde{\rho}\sigma)}) \\
= \; & \textstyle\prod_{\eta\in\mathrm{Gal}\,(D/E)}\alpha^{\mathrm{loc}}(\sigma^{-1},\eta)_{\sigma^{-1}\eta u(\widetilde{\rho})}/\alpha^{\mathrm{loc}}(\eta,\sigma^{-1})_{\eta u(\widetilde{\rho}\sigma)}.
\end{aligned}$$

Thus we have shown that

$$(\widetilde{\Gamma}^{E,\widetilde{\rho},\boldsymbol{\alpha}})^{(\sigma|_{E^{\mathrm{ab}}},\sigma|_D),\eta_{D/E,*}\boldsymbol{\alpha}} = \mathrm{conj}_{\prod_{\eta\in\mathrm{Gal}\,(D/E)}\alpha^{\mathrm{loc}}(\sigma^{-1},\eta)_{\sigma^{-1}\eta u(\widetilde{\rho})}/\alpha^{\mathrm{loc}}(\eta,\sigma^{-1})_{\eta u(\widetilde{\rho}\sigma)}}\circ(\widetilde{\Gamma}^{\sigma,\boldsymbol{\alpha}})^{E,\widetilde{\rho}\sigma,\boldsymbol{\alpha}}.$$

By *a Galois rigidification* of $\mathfrak{a}$ adapted to (ρ',ρ) we mean an isomorphism of extensions

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathrm{Gal}\,(D^{\mathrm{ab}}/D) & \longrightarrow & \mathrm{Gal}\,(E^{\mathrm{ab}}/F)_D & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \\ && \mathrm{Art}_D \uparrow \wr && \Gamma \downarrow \wr && || && \\ (0) & \longrightarrow & \mathbb{A}_D^\times/\overline{(D_\infty^\times)^0 D^\times} & \longrightarrow & W_{E/F,D,\mathfrak{a}}/\overline{(D_\infty^\times)^0 D^\times} & \longrightarrow & \mathrm{Gal}\,(D/F) & \longrightarrow & (0) \end{array}$$

which lifts to a rigidification of $\mathfrak{a}$ adapted to $\theta_{(\rho',\rho)}$, for some $\theta_{(\rho',\rho)}$. Such a Galois rigidification exists and it is unique up to composition with conjugation by an element of $r_D((\mathbb{A}_D^\times/D^\times)^{\mathrm{Gal}\,(D/F)_{w(\rho)}} D_{w(\rho)}^\times)$. If Γ is a Galois rigidification of $\mathfrak{a}$ adapted to ρ, then $\mathrm{conj}_{t_{w(\rho)}} \circ \Gamma$ is a Galois rigidification of ${}^t\mathfrak{a}$ adapted to ρ. Moreover, if $\sigma \in \mathrm{Gal}\,(E^{\mathrm{ab}}/F)_D$ then

$$\Gamma^\sigma = \Gamma^{\sigma,\boldsymbol{\alpha}} = \mathrm{conj}_{(\beta(\sigma^{-1})_{w(\rho\sigma)})} \circ \mathrm{conj}_{e_{\boldsymbol{\alpha}}^{\mathrm{glob}}(\sigma^{-1})} \circ \Gamma \circ \mathrm{conj}_\sigma$$

is Galois rigidification of $\mathfrak{a}$ adapted to $\rho\sigma$.

We call two Galois rigidifications Γ and Γ' of $\mathfrak{a}$ both adapted to ρ *equivalent* if $\Gamma' = \mathrm{conj}_a \circ \Gamma$ for some $a \in D_{w(\rho)}^\times$. If $\Gamma \sim \Gamma'$, then $\mathrm{conj}_a \circ \Gamma \sim \mathrm{conj}_a \circ \Gamma'$ and

$$\Gamma^{\sigma,\boldsymbol{\alpha}} \sim (\Gamma')^{\sigma,\boldsymbol{\alpha}} \sim (\Gamma')^{\sigma,\boldsymbol{\alpha}'}$$

if $\boldsymbol{\alpha},\boldsymbol{\alpha}' \in \mathfrak{a}$. Thus if $[\Gamma]$ is an equivalence class of Galois rigidifications of $\mathfrak{a}$ adapted to (ρ',ρ) and $t \in T_{2,E}(\mathbb{A}_D)$, then

$${}^t[\Gamma] = [\mathrm{conj}_{t_{w(\rho)}} \circ \Gamma]$$

is a well defined equivalence class of Galois rigidifications of ${}^t\mathfrak{a}$ adapted to (ρ',ρ). Moreover if $\sigma \in \mathrm{Gal}\,(E^{\mathrm{ab}}/F)|_{\mathrm{Gal}\,(D/F)}$, then

$$[\Gamma]^\sigma = [\Gamma^{\sigma,\boldsymbol{\alpha}}]$$

is a well defined equivalence class of Galois rigidifications of $\mathfrak{a}$ adapted to $(\rho',\rho)\sigma$. Moreover

$${}^{t_1t_2}[\Gamma] = {}^{t_1}({}^{t_2}[\Gamma])$$

and

$$[\Gamma]^{\sigma_1\sigma_2} = ([\Gamma]^{\sigma_1})^{\sigma_2}$$

and

$$({}^t[\Gamma])^\sigma = {}^t([\Gamma]^\sigma).$$

The latter is proved in the same way as lemma 6.1.

If Γ is a Galois rigidification of $\mathfrak{a} \in \mathcal{H}(E/F)$ adapted to ρ', then we obtain Galois rigidification data Γ^D for $\inf_{D/E}\mathfrak{a}$ adapted to (ρ',ρ). If $\Gamma \sim \Gamma'$, then $\Gamma^D \sim (\Gamma')^D$. Thus to an equivalence class $[\Gamma]$ we may associate an equivalence class

$$[\Gamma]^D = [\Gamma^D].$$

$$({}^t[\Gamma])^D = {}^t([\Gamma]^D)$$

and

$$([\Gamma]^D)^{(\sigma',\sigma)} = ([\Gamma]^{\sigma'})^D.$$

If Γ is a Galois rigidification of $\mathfrak{a} \in \mathcal{H}(D/F)$ adapted to $\widetilde{\rho}$, then we get a Galois rigidification $\Gamma^{E,\widetilde{\rho},\boldsymbol{\alpha}}$ of $\eta_{D/E,*}\mathfrak{a}$ adapted to $(\widetilde{\rho},\widetilde{\rho})$. If $\Gamma \sim \Gamma'$, then

$$\Gamma^{E,\widetilde{\rho},\boldsymbol{\alpha}} \sim (\Gamma')^{E,\widetilde{\rho},\boldsymbol{\alpha}} \sim (\Gamma')^{E,\widetilde{\rho},\boldsymbol{\alpha}}$$

if $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathfrak{a}$. Thus we can define

$$[\Gamma]^{E,\widetilde{\rho}} = [\Gamma^{E,\widetilde{\rho},\boldsymbol{\alpha}}].$$

We have

$$({}^t[\Gamma])^{E,\widetilde{\rho}} = {}^{\eta_{D/E}(t)}([\Gamma]^{E,\widetilde{\rho}})$$

and

$$([\Gamma]^{E,\widetilde{\rho}})^{(\widetilde{\sigma},\widetilde{\sigma})} = ([\Gamma]^{\widetilde{\sigma}})^{E,\widetilde{\rho}\widetilde{\sigma}}.$$

By *complete rigidification data* for $\mathfrak{a}$ we mean the choice for each place v of F and each $(\rho',\rho) \in R_{D,E,F,v}$ an equivalence class $[\Gamma_{(\rho',\rho)}]$ of Galois rigidifications of $\mathfrak{a}$ adapted to (ρ',ρ), such that

- if $\sigma \in \mathrm{Gal}\,(E^{\mathrm{ab}}/F)|_{\mathrm{Gal}\,(D/F)}$ then $[\Gamma_{v,(\rho',\rho)\sigma}] = [\Gamma_{v,(\rho',\rho)}]^\sigma$.

Note that it suffices to prescribe $[\Gamma_{v,(\rho',\rho)}]$ for one (ρ',ρ) in $R_{D,E,F,v}$ for each v. Such a choice extends uniquely to complete rigidification data by the above formula. (That such an extension indeed defines complete rigidification data follows because $[\Gamma]^{\sigma_1\sigma_2} = ([\Gamma]^{\sigma_1})^{\sigma_2}$.) Note that in particular complete rigidification exists for $\boldsymbol{\alpha}$.

Lemma 6.6. *(1) If $t \in T_{2,E}(\mathbb{A}_D)$ and if $\{[\Gamma_{v,(\rho',\rho)}]\}$ is complete rigidification data for $\mathfrak{a}$, then $\{{}^t[\Gamma_{v,(\rho',\rho)}]\}$ is complete rigidification data for ${}^t\mathfrak{a}$.*
(2) If $\{[\Gamma_{v,(\rho',\rho)}]\}$ and $\{[\Gamma'_{v,(\rho',\rho)}]\}$ are complete rigidification data for $\mathfrak{a}$, then there exists $t \in T_{2,E}(\mathbb{A}_F)$ such that $[\Gamma'_{v,(\rho',\rho)}] = {}^t[\Gamma_{v,(\rho',\rho)}]$ for all v and (ρ',ρ).

Proof: The first part is clear. For the second part choose for each place v of F a pair $(\rho'_v,\rho_v) \in R_{D,E,F,v}$. Then we can find $t_v \in (\mathbb{A}_D^\times/D^\times)^{\mathrm{Gal}\,(D/F)_{w(\rho'_v)}}$ such that

$$\Gamma'_{v,(\rho'_v,\rho_v)} \sim \mathrm{conj}_{t_v} \circ \Gamma_{v,(\rho'_v,\rho_v)}.$$

Define $t \in T_{2,E}(\mathbb{A}_F)$ by

$$t_{\sigma w(\rho'_v)} = \sigma t_{w(\rho'_v)}.$$

Then

$$\begin{array}{rcl} [\Gamma'_{v,(\rho'_v,\rho_v)\sigma}] & = & ({}^t[\Gamma_{v,(\rho'_v,\rho_v)}])^\sigma \\ & = & {}^t([\Gamma_{v,(\rho'_v,\rho_v)}]^\sigma) \\ & = & {}^t[\Gamma_{v,(\rho'_v,\rho_v)\sigma}]. \end{array}$$

□

We will write $\mathcal{H}(E/F)_D^+$ for the set of pairs $(\mathfrak{a},\{[\Gamma_{v,(\rho',\rho)}]\})$, where $\mathfrak{a} \in \mathcal{H}(E/F)_D$ and $\{[\Gamma_{v,(\rho',\rho)}]\}$ is complete rigidification data for $\mathfrak{a}$. It comes with a transitive action of $T_{2,E}(\mathbb{A}_D)$ compatible with the action of $T_{2,E}(\mathbb{A}_D)$ on $\mathcal{H}(E/F)_D$.

We may define $\mathcal{Z}(E/F)_D^+$ as the set of 4-tuples $\boldsymbol{\alpha}^+ = (\alpha^{\mathrm{glob}},\alpha^{\mathrm{loc}},\beta,[\{\Gamma_{v,(\rho',\rho)}\}])$, where $\boldsymbol{\alpha} = (\alpha^{\mathrm{glob}},\alpha^{\mathrm{loc}},\beta) \in \mathcal{Z}(E/F)_D$ and where $\{[\Gamma_{v,(\rho',\rho)}]\}$ is complete rigidification data for $[\boldsymbol{\alpha}]$. We call $\boldsymbol{\alpha}_1^+$ and $\boldsymbol{\alpha}_2^+$ *equivalent* if we can find $(\gamma^{\mathrm{glob}},\gamma^{\mathrm{loc}})$ with $\boldsymbol{\alpha}_2 =$

${}^{(\gamma^{\mathrm{glob}},\gamma^{\mathrm{loc}})}\boldsymbol{\alpha}_1$ and $[\Gamma_{2,v,(\rho,\rho')}] = [i_{\gamma^{\mathrm{glob}}} \circ \Gamma_{1,v,(\rho',\rho)}]$ for all v and (ρ',ρ). Then $\mathcal{H}(E/F)^+_D$ is just the set of equivalence classes of elements of $\mathcal{Z}(E/F)^+_D$. The actions of $T_{2,E}(\mathbb{A}_D)$ on $\mathcal{Z}(E/F)_D$ and $\mathcal{H}(E/F)^+_D$ lift to an action on $\mathcal{Z}(E/F)^+_D$.

Suppose that $\boldsymbol{\alpha}^+ = (\alpha^{\mathrm{glob}}, \alpha^{\mathrm{loc}}, \beta, \{[\Gamma_{v,\rho'}]\}) \in \mathcal{Z}(E/F)^+$. Then $\{[\Gamma^D_{v,(\rho',\rho)} = \Gamma^D_{v,\rho'}]\}$ is complete rigidification data for $\inf_{D/E} \boldsymbol{\alpha}$. We will write

$$\inf_{D/E} \boldsymbol{\alpha}^+ = (\inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)} \alpha^{\mathrm{glob}}, \inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)} \alpha^{\mathrm{loc}}, \inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)} \beta, \{[\Gamma^D_{v,(\rho',\rho)}]\}) \in \mathcal{Z}(E/F)^+_D.$$

Note that

$$\inf_{D/E} {}^{\gamma}\boldsymbol{\alpha}^+ = {}^{\gamma} \inf_{D/E} \boldsymbol{\alpha}^+,$$

so that $\inf_{D/E}$ descends to a map $\inf_{D/E} : \mathcal{H}(E/F)^+ \to \mathcal{H}(E/F)^+_D$. Moreover, if $t \in T_{2,E}(\mathbb{A}_E)$, then

$$\inf_{D/E} {}^{t}\boldsymbol{\alpha}^+ = {}^{t} \inf_{D/E} \boldsymbol{\alpha}^+.$$

Now suppose that $\boldsymbol{\alpha}^+ = (\alpha^{\mathrm{glob}}, \alpha^{\mathrm{loc}}, \beta, \{[\Gamma_{v,\widetilde{\rho}}]\}) \in \mathcal{Z}(D/F)^+$. If $\sigma \in \mathrm{Gal}\,(E^{\mathrm{ab}}/F)_D$ and $\widetilde{\rho}\sigma = (\rho',\rho)$, then we define

$$[\Gamma^E_{v,(\rho',\rho)}] = [(\Gamma^{E,\widetilde{\rho},\boldsymbol{\alpha}})^{\sigma,\eta_{D/E,*}\boldsymbol{\alpha}}].$$

This does not depend on the choice of $\widetilde{\rho}$ or σ. Moreover $\{[\Gamma^E_{v,(\rho',\rho)}]\}$ is complete rigidification data for $\eta_{D/E,*}\boldsymbol{\alpha}$. We will write

$$\eta_{D/E,*}\boldsymbol{\alpha}^+ = (\eta_{D/E,*}\alpha^{\mathrm{glob}}, \eta_{D/E,*}\alpha^{\mathrm{loc}}, \eta_{D/E,*}\beta, \{[\Gamma^E_{v,(\rho',\rho)}]\}) \in \mathcal{Z}(E/F)^+_D.$$

Note that

$$\eta_{D/E,*}{}^{\gamma}\boldsymbol{\alpha}^+ = {}^{\eta_{D/E,*}\gamma}\eta_{D/E,*}\boldsymbol{\alpha}^+$$

and

$$\eta_{D/E,*}{}^{t}\boldsymbol{\alpha}^+ = {}^{\eta_{D/E}(t)}\eta_{D/E,*}\boldsymbol{\alpha}^+.$$

The following lemma is straight forward to verify.

Lemma 6.7. *Suppose that $D' \supset D \supset E \supset F$ are finite Galois extensions of F. Suppose also that $\boldsymbol{\alpha}^+_{D'} \in \mathcal{Z}(D'/F)$ and $\boldsymbol{\alpha}^+_D \in \mathcal{Z}(D/F)$ and $\boldsymbol{\alpha}^+_E \in \mathcal{Z}(E/F)$ satisfy $\eta_{D'/D,*}\boldsymbol{\alpha}^+_{D'} = {}^{t'}\inf_{D'/D} \boldsymbol{\alpha}^+_D$ and $\eta_{D/E,*}\boldsymbol{\alpha}^+_D = {}^{t}\inf_{D/E} \boldsymbol{\alpha}^+_E$ with $t' \in T_{2,D}(\mathbb{A}_{D'})$ and $t \in T_{2,E}(\mathbb{A}_D)$. Then*

$$\eta_{D'/E,*}\boldsymbol{\alpha}^+_{D'} = {}^{t\eta_{D/E}(t')} \inf_{D'/E} \boldsymbol{\alpha}^+_E.$$

If $\rho : E^{\mathrm{ab}}D \to \overline{F_v}$ is F-linear and $\tau \in \mathrm{Aut}\,(\overline{F}_v/F)$, we define

$$\overline{g}_{\boldsymbol{\alpha}^+,v,\rho}(\tau) = \left(\left(\mathrm{loc}_{\boldsymbol{\alpha}} \widetilde{\Gamma_{v,(\rho,\rho)}(\tau^\rho)} \right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^\rho) \right)_{w(\tau\rho)} \in \mathbb{A}^\times_D / \overline{(D^\times_\infty)^0 D^\times} D^\times_{w(\rho)} D^\times_{w(\tau\rho)},$$

where $\widetilde{\Gamma_{v,(\rho,\rho)}(\tau^\rho)}$ is any lift of $\Gamma_{v,(\rho,\rho)}(\tau^\rho)$ to $\mathcal{E}^{\mathrm{glob}}_{\mathfrak{a}}(E/F)_D$. This element is independent of the choice of $\Gamma_{v,(\rho,\rho)} \in [\Gamma_{v,(\rho,\rho)}]$ and of the lift $\widetilde{\Gamma_{v,(\rho,\rho)}(\tau^\rho)}$ of $\Gamma_{v,(\rho,\rho)}(\tau^\rho)$.

Lemma 6.8. *(1)* $\overline{g}_{\gamma\boldsymbol{\alpha}^+,v,\rho}(\tau) = \overline{g}_{\boldsymbol{\alpha}^+,v,\rho}(\tau)$.
(2) $\overline{g}_{t_{\boldsymbol{\alpha}^+},v,\rho}(\tau) = (t_{w(\rho)}/t_{w(\tau\rho)})\overline{g}_{\boldsymbol{\alpha}^+,v,\rho}(\tau)$.
(3) $\overline{g}_{\mathrm{ind}_{D/E}\boldsymbol{\alpha}^+,v,\rho}(\tau) = \overline{g}_{\boldsymbol{\alpha}^+,v,\rho}(\tau)$.
(4) $\overline{g}_{\eta_{D/E,*}\boldsymbol{\alpha}^+,v,\rho}(\tau) = (N_{D/E}\overline{g}_{\boldsymbol{\alpha}^+,v,\widetilde{\rho}}(\tau)) \prod_{\eta \in \mathrm{Gal}\,(D/E)} \beta(\eta)_{\eta u(\widetilde{\rho})}/\beta(\eta)_{\eta u(\tau\widetilde{\rho})}$.

Proof: For the first part we have

$$\begin{aligned}
&\overline{g}_{\boldsymbol{\alpha}^+,v,\rho}(\tau) \\
= &\left(i_{\gamma^{\mathrm{loc}}}\left(\left(\mathrm{loc}_{\boldsymbol{\alpha}}\widetilde{\Gamma_{v,(\rho,\rho)}(\tau^\rho)}\right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^\rho)\right)\right)_{w(\tau\rho)} \\
= &\left(\left(\mathrm{loc}_{\gamma\boldsymbol{\alpha}} i_{\gamma^{\mathrm{glob}}}\widetilde{\Gamma_{v,(\rho,\rho)}(\tau^\rho)}\right)^{-1} \gamma^{\mathrm{loc}}(\tau^\rho) e^{\mathrm{loc}}_{\gamma\boldsymbol{\alpha}}(\tau^\rho)\right)_{w(\tau\rho)} \\
= &\overline{g}_{\gamma\boldsymbol{\alpha}^+,v,\rho}(\tau).
\end{aligned}$$

For the second we have

$$\begin{aligned}
&\overline{g}_{t_{\alpha^+},v,\rho}(\tau) \\
= &\mathfrak{z}_t^{-1}\left(\left(\mathrm{loc}_{t\boldsymbol{\alpha}}\mathfrak{z}_t\left(\widetilde{\mathrm{conj}_{t_{w(\rho)}}\Gamma_{v,(\rho,\rho)}(\tau^\rho)}\right)\right)^{-1} e^{\mathrm{loc}}_{t\boldsymbol{\alpha}}(\tau^\rho)\right)_{w(\tau\rho)} \\
= &\left(\left(\mathrm{conj}_t^{-1}\mathrm{loc}_{\boldsymbol{\alpha}}\left(\widetilde{\mathrm{conj}_{t_{w(\rho)}}\Gamma_{v,(\rho,\rho)}(\tau^\rho)}\right)\right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^\rho)\right)_{w(\tau\rho)} \\
= &\left(({}^{(\tau^\rho)^{-1}}t/t)\left(\mathrm{loc}_{\boldsymbol{\alpha}}\left(\widetilde{\mathrm{conj}_{t_{w(\rho)}}\Gamma_{v,(\rho,\rho)}(\tau^\rho)}\right)\right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^\rho)\right)_{w(\tau\rho)} \\
= &({}^{(\tau^\rho)^{-1}}t/t)_{w(\tau\rho)}(t_{w(\rho)}/{}^{(\tau^\rho)^{-1}}(t_{w(\rho)}))\left(\left(\mathrm{loc}_{\boldsymbol{\alpha}}\left(\widetilde{\Gamma_{v,(\rho,\rho)}(\tau^\rho)}\right)\right)^{-1} e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^\rho)\right)_{w(\tau\rho)} \\
= &(t_{w(\rho)}/t_{w(\tau\rho)})\overline{g}_{\alpha^+,v,\rho}(\tau).
\end{aligned}$$

For the third part, under the identification

$$\mathcal{E}_2(E/F)_{D,\mathrm{inf}_{D/E}\boldsymbol{\alpha}} \cong (T_{2,E}(\mathbb{A}_D) \rtimes (\mathcal{E}_2(E/F)_{\boldsymbol{\alpha}} \times_{\mathrm{Gal}\,(E/F)} \mathrm{Gal}\,(D/F)))/T_{2,E}(\mathbb{A}_E)$$

$e^{\mathrm{loc}}_{\mathrm{inf}_{D/E}\boldsymbol{\alpha}}(\tau^\rho)$ is identified with $[(1,(e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\tau^\rho),\tau^\rho))]$. Moreover, under the identification

$$W_{E/F,D,\mathrm{inf}_{D/E}\boldsymbol{\alpha}} \cong (\mathbb{A}_D^\times/D^\times \rtimes (W_{E/F,\boldsymbol{\alpha}} \times_{\mathrm{Gal}\,(E/F)} \mathrm{Gal}\,(D/F)))/\mathbb{A}_E^\times$$

$\Gamma^D_{v,\rho}(\tau^\rho)$ is identified with $[(1,(\Gamma_{v,\rho}(\tau^\rho),\tau^\rho))]$. Moreover $\mathrm{loc}_{\mathrm{inf}_{D/E}\boldsymbol{\alpha}}\widetilde{\Gamma^D_{v,\rho}(\tau^\rho)}$ can be chosen to be $[(1,(\mathrm{loc}_{\boldsymbol{\alpha}}\widetilde{\Gamma_{v,\rho}(\tau^\rho)},\tau^\rho))]$. the third part follows.

For the fourth part we will make use of the identifications

$$W_{E/F,D,\eta_{D/E,*}\boldsymbol{\alpha}} \cong (\mathbb{A}_D^\times/D^\times \rtimes W_{D/F,\boldsymbol{\alpha}})/\mathbb{A}_D^\times$$

and

$$\mathcal{E}^{\rm glob}(E/F)_{D,\eta_{D/E,*}\boldsymbol{\alpha}} \cong (\mathcal{E}^{\rm glob}(E/F)_D^0 \rtimes \mathcal{E}^{\rm glob}(D/F)_{\boldsymbol{\alpha}})/\mathcal{E}^{\rm glob}(D/F)^0$$

and

$$\mathcal{E}_2(E/F)_{D,\eta_{D/E,*}\boldsymbol{\alpha}} \cong (T_{2,E}(\mathbb{A}_D) \rtimes \mathcal{E}_2(D/F)_{\boldsymbol{\alpha}})/T_{2,D}(\mathbb{A}_D).$$

Under these identifications $\Gamma^{E,\widetilde{\rho}}_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})$ corresponds to

$$\begin{aligned}
& \mathrm{conj}_{\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}}[(\gamma_{\boldsymbol{\alpha},E}(\widetilde{\Gamma_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})}), \widetilde{\Gamma_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})})] \\
= \; & \mathrm{conj}_{\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}}[(N_{D/E}(\widetilde{\Gamma_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})}e^{\rm glob}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}})^{-1})\gamma_{\boldsymbol{\alpha},E}(e^{\rm glob}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}}))/(\widetilde{\Gamma_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})}e^{\rm glob}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}})^{-1})^{[D:E]}, \\
& \widetilde{\Gamma_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})})] \\
= \; & \mathrm{conj}_{\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}}[(N_{D/E}(\widetilde{\Gamma_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})}e^{\rm glob}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}})^{-1})\prod_{\eta\in{\rm Gal}\,(D/E)}(\alpha^{\rm glob}(\eta,\tau^{\widetilde{\rho}})/\alpha^{\rm glob}(\tau^{\widetilde{\rho}},\eta)), \\
& e^{\rm glob}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}}))].
\end{aligned}$$

Applying $\mathrm{loc}_{\eta_{D/E,*}\boldsymbol{\alpha}}$ we get

$$\begin{aligned}
& \mathrm{conj}_{\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}}[(N_{D/E}(\widetilde{\Gamma_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})}e^{\rm glob}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}})^{-1})\prod_{\eta\in{\rm Gal}\,(D/E)}(\alpha^{\rm glob}(\eta,\tau^{\widetilde{\rho}})/\alpha^{\rm glob}(\tau^{\widetilde{\rho}},\eta)) \\
& (\eta_{D/E}\beta(\tau^{\widetilde{\rho}}))^{-1}, e^{\rm loc}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}}))]
\end{aligned}$$

and then taking the inverse gives

$$\begin{aligned}
& \mathrm{conj}_{\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}}[({}^{(\tau^{\widetilde{\rho}})^{-1}}N_{D/E}(e^{\rm glob}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}})\widetilde{\Gamma_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})}^{-1})\prod_{\eta\in{\rm Gal}\,(D/E)}{}^{(\tau^{\widetilde{\rho}})^{-1}}(\alpha^{\rm glob}(\tau^{\widetilde{\rho}},\eta)/\alpha^{\rm glob}(\eta,\tau^{\widetilde{\rho}})) \\
& {}^{(\tau^{\widetilde{\rho}})^{-1}}(\eta_{D/E}\beta(\tau^{\widetilde{\rho}})), e^{\rm loc}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}})^{-1})] \\
= \; & \mathrm{conj}_{\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}}[(N_{D/E}(\widetilde{\Gamma_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})}^{-1}e^{\rm glob}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}}))\prod_{\eta\in{\rm Gal}\,(D/E)}{}^{(\tau^{\widetilde{\rho}})^{-1}}(\alpha^{\rm glob}(\tau^{\widetilde{\rho}},\eta)/\alpha^{\rm glob}(\eta,\tau^{\widetilde{\rho}})) \\
& {}^{(\tau^{\widetilde{\rho}})^{-1}}(\eta_{D/E}\beta(\tau^{\widetilde{\rho}})), e^{\rm loc}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}})^{-1})].
\end{aligned}$$

Multiplying on the right by $e^{\rm loc}_{\eta_{D/E,*}\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}}) = [(1, e^{\rm loc}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}}))]$ and taking the $w(\tau\rho)$-component gives

$$\begin{aligned}
& \left(\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}\right)N_{D/E}(\widetilde{\Gamma_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})}^{-1}e^{\rm glob}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}}))\left(\prod_{\eta\in{\rm Gal}\,(D/E)}{}^{(\tau^{\widetilde{\rho}})^{-1}}(\alpha^{\rm glob}(\tau^{\widetilde{\rho}},\eta)/\alpha^{\rm glob}(\eta,\tau^{\widetilde{\rho}}))\right) \\
& {}^{(\tau^{\widetilde{\rho}})^{-1}}\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\tau^{\widetilde{\rho}})_{\eta u(\widetilde{\rho})}{}^{(\tau^{\widetilde{\rho}})^{-1}}\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)^{-1}_{\eta u(\widetilde{\rho})}.
\end{aligned}$$

However

$$\widetilde{\Gamma_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})}^{-1}e^{\rm glob}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}}) = ((\mathrm{loc}_{\boldsymbol{\alpha}}\widetilde{\Gamma_{v,\widetilde{\rho}}(\tau^{\widetilde{\rho}})})^{-1}\beta(\tau^{\widetilde{\rho}})^{-1}e^{\rm loc}_{\boldsymbol{\alpha}}(\tau^{\widetilde{\rho}}))_{u(\tau\rho)} = ({}^{(\tau^{\widetilde{\rho}})^{-1}}\beta(\tau^{\widetilde{\rho}}))^{-1}_{u(\tau\rho)}\overline{g}_{\boldsymbol{\alpha}^+,v,\widetilde{\rho}}(\tau).$$

Thus $\overline{g}_{\eta_{D/E,*}\boldsymbol{\alpha}^+,v,\widetilde{\rho}}(\tau)$ equals the product of $N_{D/E}\overline{g}_{\boldsymbol{\alpha}^+,v,\widetilde{\rho}}(\tau)$ with $(\tau^{\widetilde{\rho}})^{-1}$ applied to

$$\begin{aligned}
& \left({}^{\tau^{\widetilde{\rho}}}\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}\right)N_{D/E}(\beta(\tau^{\widetilde{\rho}})^{-1}_{u(\widetilde{\rho})})\left(\prod_{\eta\in{\rm Gal}\,(D/E)}(\alpha^{\rm glob}(\tau^{\widetilde{\rho}},\eta)/\alpha^{\rm glob}(\eta,\tau^{\widetilde{\rho}}))\right) \\
& \prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\tau^{\widetilde{\rho}})_{\eta u(\widetilde{\rho})}\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)^{-1}_{\eta u(\widetilde{\rho})} \\
= \; & \prod_{\eta\in{\rm Gal}\,(D/E)}(\alpha^{\rm loc}(\tau^{\widetilde{\rho}},\eta)\beta(\tau^{\widetilde{\rho}}\eta)/\beta(\tau^{\widetilde{\rho}})^{\tau^\rho}\beta(\eta))_{\tau^{\widetilde{\rho}}\eta(\tau^{\widetilde{\rho}})^{-1}u(\widetilde{\rho})}(\beta(\eta)^\eta\beta(\tau^\rho)/\alpha^{\rm loc}(\eta,\tau^{\widetilde{\rho}})\beta(\eta\tau^{\widetilde{\rho}}))_{\eta u(\widetilde{\rho})} \\
& \prod_{\eta\in{\rm Gal}\,(D/E)}{}^{\tau^{\widetilde{\rho}}}\beta(\eta)_{\eta u(\widetilde{\rho})}({}^\eta\beta(\tau^{\widetilde{\rho}}))^{-1}_{\eta u(\widetilde{\rho})}\beta(\tau^{\widetilde{\rho}})_{\tau^{\widetilde{\rho}}\eta(\tau^{\widetilde{\rho}})^{-1}u(\widetilde{\rho})}\beta(\eta)^{-1}_{\eta u(\widetilde{\rho})} \\
\equiv \; & {}^{\tau^{\widetilde{\rho}}}\prod_{\eta\in{\rm Gal}\,(D/E)}\beta(\eta)_{\eta u(\widetilde{\rho})}/\beta(\eta)_{\eta u(\tau\widetilde{\rho})},
\end{aligned}$$

and the fourth part of the lemma follows. □

Finally we obtain:

Lemma 6.9. *Suppose that $D \supset E \supset F$ are finite extensions of number fields with E and D Galois over F, and that v is a place of F. Suppose also that $\mathfrak{a}_E^+ \in \mathcal{H}(E/F)^+$ and $\mathfrak{a}_D^+ \in \mathcal{H}(D/F)^+$. Then we can find a $t \in T_{2,E}(\mathbb{A}_D)$ such that ${}^t\inf_{\mathrm{Gal}\,(E/F)}^{\mathrm{Gal}\,(D/F)} \mathfrak{a}_E^+ = \eta_{D/E,*}\mathfrak{a}_D^+ \in \mathcal{H}(E/F)_D^+$.*

Suppose moreover that T/F is a torus that splits over E, that $\mu \in X_*(T)(\overline{F_v})$ and that $\tau \in \mathrm{Aut}\,(\overline{F_v}/F)$. Then

$$\overline{b}_{\mathfrak{a}_E^+,v,\mu,\tau} = \overline{b}_{\mathfrak{a}_D^+,v,\mu,\tau} \prod_\rho ({}^{\rho^{-1}}\mu)(t_{w(\tau\rho)}/t_{w(\rho)}) \in T(\mathbb{A}_E)/\overline{T(F)T(F_\infty)^0 T(F_v)}T(E)T(E_v),$$

where ρ runs over F-linear embeddings $E \hookrightarrow \overline{F_v}$, and $w(\rho)$ is the corresponding place of E.

Proof: Choose $\boldsymbol{\alpha}_E \in \mathfrak{a}_E$ and $\boldsymbol{\alpha}_D \in \mathfrak{a}_D$. Then we can find $t \in T_{2,E}(\mathbb{A}_D)$ and $\boldsymbol{\gamma} = (\gamma^{\mathrm{glob}}, \gamma^{\mathrm{loc}})$, where $\gamma^{\mathrm{glob}} : \mathrm{Gal}\,(D/F) \to \mathcal{E}^{\mathrm{glob}}(E/F)_D^0$ and $\gamma^{\mathrm{loc}} : \mathrm{Gal}\,(D/F) \to \prod_{u \in V_D} D_u^\times$ such that

$${}^t \inf_{D/E} \boldsymbol{\alpha}_E = {}^{\boldsymbol{\gamma}}\eta_{D/E,*}\boldsymbol{\alpha}_D.$$

If $\eta \in \mathrm{Gal}\,(E/F)$ we will select a lift $\widetilde{\eta}$ of η to $\mathrm{Gal}\,(D/F)$. Then we have

$$\begin{array}{rl} & \overline{b}_{\mathfrak{a}_D^+,v,\mu,\tau} \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \prod_{\zeta\in \mathrm{Gal}\,(D/E)} \widetilde{\eta}^{-1}\zeta^{-1}({}^{\rho^{-1}}\mu)((\beta_D(\zeta\widetilde{\eta})_{u(\tau\rho)}/\beta_D(\zeta\widetilde{\eta})_{u(\rho)})\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau)) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \prod_{\zeta\in \mathrm{Gal}\,(D/E)} \widetilde{\eta}^{-1}({}^{\rho^{-1}}\mu)({}^{\zeta^{-1}}(\pi_{u(\tau\rho)}/\pi_{u(\rho)})(\beta_D(\zeta)^\zeta\beta_D(\widetilde{\eta})\alpha_D^{\mathrm{glob}}(\zeta,\widetilde{\eta})/\alpha_D^{\mathrm{loc}}(\zeta,\widetilde{\eta})) \\ & ({}^{\zeta^{-1}}\overline{g}_{\mathfrak{a}^+,v,\rho}(\tau))) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \widetilde{\eta}^{-1}\Big(({}^{\rho^{-1}}\mu)(\overline{g}_{\eta_{D/E,*}\boldsymbol{\alpha}_D^+,v,\rho}(\tau)) \\ & \prod_{\zeta\in \mathrm{Gal}\,(D/E)}({}^{\rho^{-1}}\mu)((\pi_{\zeta^{-1}u(\tau\rho)}/\pi_{\zeta^{-1}u(\rho)})({}^{\zeta^{-1}}\beta_D(\zeta)\beta_D(\widetilde{\eta}))(\beta(\zeta^{-1})_{\zeta^{-1}u(\tau\rho)}/\beta(\zeta^{-1})_{\zeta^{-1}u(\rho)}))\Big) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \widetilde{\eta}^{-1}\Big(({}^{\rho^{-1}}\mu)(\overline{g}_{\boldsymbol{\gamma}\eta_{D/E,*}\boldsymbol{\alpha}_D^+,v,\rho}(\tau)(\eta_{D/E}\beta_D(\widetilde{\eta}))_{w(\tau\rho)}/(\eta_{D/E}\beta_D(\widetilde{\eta}))_{w(\rho)}) \\ & \prod_{\zeta\in \mathrm{Gal}\,(D/E)}({}^{\rho^{-1}}\mu)((\pi_{\zeta^{-1}u(\tau\rho)}/\pi_{\zeta^{-1}u(\rho)})({}^{\zeta^{-1}}\beta_D(\zeta)\beta_D(\zeta^{-1})))\Big) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \widetilde{\eta}^{-1}\Big(({}^{\rho^{-1}}\mu)(\overline{g}_{{}^t\inf_{D/E}\boldsymbol{\alpha}_E^+,v,\rho}(\tau)(\pi_{w(\tau\rho)}/\pi_{w(\rho)})(({}^t\beta_E)(\widetilde{\eta})\gamma^{\mathrm{glob}}(\eta)^{-1}\gamma^{\mathrm{loc}}(\widetilde{\eta}))) \\ & \prod_{\zeta\in \mathrm{Gal}\,(D/E)}({}^{\rho^{-1}}\mu)((\pi_{\zeta^{-1}u(\tau\rho)}/\pi_{\zeta^{-1}u(\rho)})(\beta_D(1)\alpha_D^{\mathrm{loc}}(\zeta^{-1},\zeta)/\alpha_D^{\mathrm{glob}}(\zeta^{-1},\zeta)))\Big) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \widetilde{\eta}^{-1}\Big(({}^{\rho^{-1}}\mu)(\overline{g}_{\inf_{D/E}\boldsymbol{\alpha}_E^+,v,\rho}(\tau)(t_{w(\rho)}/t_{w(\tau\rho)})(\pi_{w(\tau\rho)}/\pi_{w(\rho)})(\beta_E(\widetilde{\eta})(t/{}^{\widetilde{\eta}}t))) \\ & \prod_{\zeta\in \mathrm{Gal}\,(D/E)}({}^{\rho^{-1}}\mu)((\pi_{\zeta^{-1}u(\tau\rho)}/\pi_{\zeta^{-1}u(\rho)})(\beta_D(1)))\Big) \\ = & \prod_{\eta\in \mathrm{Gal}\,(E/F)} \widetilde{\eta}^{-1}\Big(({}^{\rho^{-1}}\mu)(\overline{g}_{\boldsymbol{\alpha}_E^+,v,\rho}(\tau)(\pi_{w(\tau\rho)}/\pi_{w(\rho)})(\beta_E(\widetilde{\eta})^{\widetilde{\eta}}t^{-1})) \\ & \prod_{\zeta\in \mathrm{Gal}\,(D/E)}({}^{\rho^{-1}}\mu)((\pi_{\zeta^{-1}u(\tau\rho)}/\pi_{\zeta^{-1}u(\rho)})(\alpha_D^{\mathrm{loc}}(1,1)/\alpha_D^{\mathrm{glob}}(1,1)))\Big) \\ = & \overline{b}_{\mathfrak{a}_E^+,v,\mu,\tau}\prod_{\eta\in \mathrm{Gal}\,(E/F)}\widetilde{\eta}^{-1}({}^{\rho^{-1}}\mu)(\widetilde{\eta}(\pi_{\eta^{-1}w(\tau\rho)}/\pi_{\eta^{-1}w(\rho)})(t^{-1})) \\ = & \overline{b}_{\mathfrak{a}_E^+,v,\mu,\tau}\prod_{\eta\in \mathrm{Gal}\,(E/F)}({}^{\eta^{-1}\rho^{-1}}\mu)((\pi_{w(\rho\eta)}/\pi_{w(\tau\rho\eta)})(t)), \end{array}$$

and the lemma follows. □

7. Taniyama groups

The results of this section are not necessary for stating our main results, but will be needed in the proofs in order to compare our cohomological constructions with those of Langlands in [L].

7.1. **The Serre torus.** We will write R_E for the restriction of scalars from E to $\mathbb{Q}$ of $\mathbb{G}_m$. Thus $X^*(R_E) = \mathrm{Map}\,(\mathrm{Gal}\,(E/\mathbb{Q}), \mathbb{Z})$ with $\mathrm{Gal}\,(E/\mathbb{Q})$ action given by $({}^\sigma\varphi)(\tau) = \varphi(\sigma^{-1}\tau)$.

If $\tau \in \mathrm{Gal}\,(E/\mathbb{Q})$ we define an automorphism $[\tau]$ of $R_E/\mathbb{Q}$ by $X^*([\tau])(\varphi)(\tau') = \varphi(\tau'\tau^{-1})$. We have $[\tau_1\tau_2] = [\tau_1][\tau_2]$, i.e. this gives a left action of $\mathrm{Gal}\,(E/\mathbb{Q})$ on $R_E/\mathbb{Q}$.

There is also a cocharacter $\mu^{\mathrm{can}} = \mu_E^{\mathrm{can}} : \mathbb{G}_m \to R_E$ over E characterized by $X^*(\mu^{\mathrm{can}})(\varphi) = \varphi(1)$. We have ${}^\sigma\mu^{\mathrm{can}} = [\sigma^{-1}] \circ \mu^{\mathrm{can}}$. The map $\prod_{\eta \in \mathrm{Gal}\,(E/\mathbb{Q})} \eta \circ \mu^{\mathrm{can}}$ gives isomorphisms $E^\times \xrightarrow{\sim} R_E(\mathbb{Q})$ and $E_v^\times \xrightarrow{\sim} R_E(\mathbb{Q}_v)$ and $\mathbb{A}_E^\times \xrightarrow{\sim} R_E(\mathbb{A})$.

If $T/\mathbb{Q}$ is a torus and $\mu \in X_*(T)(E)$ then there is a unique map of tori $\widetilde{\mu} : R_E \to T$ over $\mathbb{Q}$ such that $\mu = \widetilde{\mu} \circ \mu^{\mathrm{can}}$. $(X^*(\widetilde{\mu})(\chi)(\tau) = \chi \circ {}^\tau\mu.)$ If $\sigma \in \mathrm{Gal}\,(E/\mathbb{Q})$ then $\widetilde{{}^\sigma\mu} = \widetilde{\mu} \circ [\sigma^{-1}]$.

If D is a finite Galois extension of $\mathbb{Q}$ containing E then $N_{D/E} = \widetilde{\mu_E^{\mathrm{can}}} : R_D \to R_E$ is a homomorphism also characterized by $X^*(N_{D/E})(\varphi)(\tau) = \varphi(\tau|_E)$. If $\mu \in X^*(T)(E)$ then we get $\widetilde{\mu}_E : R_E \to T$ and $\widetilde{\mu}_D : R_D \to T$. They satisfy:

$$\widetilde{\mu}_E \circ N_{D/E} = \widetilde{\mu}_D.$$

There is a slightly different way of thinking about this torus that will be slightly more convenient: we want the canonical cocharacter to be defined over $\mathbb{C}$ rather than over E. If $\rho : E \hookrightarrow \mathbb{C}$ we will set $R_{E,\rho} = R_E$, a torus over $\mathbb{Q}$, and $\mu_\rho^{\mathrm{can}} = {}^\rho\mu^{\mathrm{can}} \in X_*(R_E)(\mathbb{C})$ and, if $\tau \in \mathrm{Aut}\,(\mathbb{C})$ we set $[\tau]_\rho = [\tau^\rho] \in \mathrm{Aut}\,(R_E/\mathbb{Q})$. If $\sigma \in \mathrm{Gal}\,(E/\mathbb{Q})$, then we identify $R_{E,\rho}$ and $R_{E,\rho\circ\sigma}$ via $[\sigma^{-1}] : R_{E,\rho} \xrightarrow{\sim} R_{E,\rho\circ\sigma}$. This identification carries μ_ρ^{can} to $\mu_{\rho\circ\sigma}^{\mathrm{can}}$, and $[\tau]_\rho$ to $[\tau]_{\rho\circ\sigma}$. Thus we get a well defined torus $R_{E,\mathbb{C}}/\mathbb{Q}$ with an action of $\mathrm{Aut}\,(\mathbb{C})$, which we denote $\tau \mapsto [\tau]_\mathbb{C} \in \mathrm{Aut}\,(R_{E,\mathbb{C}}/\mathbb{Q})$, and a canonical cocharacter $\mu_\mathbb{C}^{\mathrm{can}} \in X^*(R_{E,\mathbb{C}})(\mathbb{C})$. We have ${}^\tau\mu_\mathbb{C}^{\mathrm{can}} = [\tau^{-1}]_\mathbb{C} \circ \mu_\mathbb{C}^{\mathrm{can}}$ (for $\tau \in \mathrm{Aut}\,(\mathbb{C})$). Moreover if $T/\mathbb{Q}$ is a torus and $\mu \in X_*(T)(\mathbb{C})$ is actually defined over the image of E in $\mathbb{C}$, then there is a unique map of tori $\widetilde{\mu} : R_{E,\mathbb{C}} \to T$ over $\mathbb{Q}$ such that $\mu = \widetilde{\mu} \circ \mu_\mathbb{C}^{\mathrm{can}}$. If D is another finite Galois extension of $\mathbb{Q}$ containing E, we also get a map $N_{D/E} : R_{D,\mathbb{C}} \to R_{E,\mathbb{C}}$ over $\mathbb{Q}$. If $\mu \in X^*(T)(\mathbb{C})$ then we get $\widetilde{\mu}_E : R_E \to T$ and $\widetilde{\mu}_D : R_D \to T$. They satisfy:

$$\widetilde{\mu}_E \circ N_{D/E} = \widetilde{\mu}_D.$$

We will also write S_E for the torus over $\mathbb{Q}$ characterized by

$$X^*(S_E) = \{(\varphi, w) \in \mathrm{Map}\,(\mathrm{Gal}\,(E/\mathbb{Q}), \mathbb{Z}) \times \mathbb{Z} : \ \varphi(c_v\tau) + \varphi(\tau) = w \ \forall \tau \in \mathrm{Gal}\,(E/\mathbb{Q}) \text{ and } \forall c_v \in [c]\},$$

with a left action of $\mathrm{Gal}\,(\overline{E}/\mathbb{Q})$ given by

$$\sigma(\varphi, w) = (\tau \mapsto \varphi(\sigma^{-1}\tau), w).$$

It is called the *Serre torus*. There is an obvious injection $X^*(S_E) \hookrightarrow X^*(R_E)$ (sending (φ, w) to φ) with torsion free cokernel; and hence an epimorphism $R_E \twoheadrightarrow S_E$ with connected kernel R_E^1. the exact sequence

$$(0) \longrightarrow R_E^1 \longrightarrow R_E \longrightarrow S_E \longrightarrow (0)$$

splits over E. The action of $\mathrm{Gal}\,(E/\mathbb{Q})$ on R_E (via $\tau \mapsto [\tau]$) over $\mathbb{Q}$ descends to an action on S_E, which we will denote in the same way. It is also characterized by

$$X^*([\tau])(\varphi, w) = (\tau' \mapsto \varphi(\tau'\tau^{-1}), w).$$

We will again denote the composite of μ^{can} with the map $R_E \twoheadrightarrow S_E$ by $\mu^{\mathrm{can}} \in X_*(S_E)(E)$. It is also characterized by

$$X^*(\mu^{\mathrm{can}})(\varphi, w) = \varphi(1) \in \mathbb{Z} \cong X^*(\mathbb{G}_m).$$

Again we have ${}^\sigma\mu^{\mathrm{can}} = [\sigma^{-1}] \circ \mu^{\mathrm{can}}$. Also note that the $\{[\tau] \circ \mu^{\mathrm{can}}\}_{\tau \in \mathrm{Gal}\,(E/\mathbb{Q})}$ spans $X_*(S_E)$. There is a second cocharacter $\mathrm{wt} : \mathbb{G}_m \to S_E$ over $\mathbb{Q}$ characterized by

$$X^*(\mathrm{wt})(\varphi, w) = w \in \mathbb{Z} \cong X^*(\mathbb{G}_m).$$

Note that $\mathrm{wt} = {}^{c_v+1}\mu^{\mathrm{can}} = ([c_v\tau]\mu^{\mathrm{can}})([\tau]\mu^{\mathrm{can}})$ for any $c_v \in [c] \subset \mathrm{Gal}\,(E/\mathbb{Q})$ and any $\tau \in \mathrm{Gal}\,(E/\mathbb{Q})$. If $D \supset E$ is another finite Galois extension of $\mathbb{Q}$ then $N_{D/E} : R_D \to R_E$ descends to a map $N_{D/E} : S_D \to S_E$, also characterized by

$$X^*(N_{D/E})(\varphi, w) = (\tau \mapsto \varphi(\tau|_E), w).$$

If E_0 denotes the maximal CM subfield of E then

$$N_{E/E_0} : S_E \xrightarrow{\sim} S_{E_0}.$$

We recall (for instance from [MS1]) that $S_E(\mathbb{Q})$ is a discrete subgroup of $S_E(\mathbb{A}^\infty)$ and that $\ker^1(\mathbb{Q}, S_E) = (0)$.

If $T/\mathbb{Q}$ is any torus and $\mu \in X_*(T)(E)$ satisfies

- ${}^{c_v}\mu$ is independent of $c_v \in [c]$,
- and $(c_v + 1)\mu \in X_*(T)(\mathbb{Q})$ for one, and hence any, $c_v \in [c]$;

then there is a unique morphism $\widetilde{\mu} : S_E \to T$ over $\mathbb{Q}$ such that $\mu = \widetilde{\mu} \circ \mu^{\mathrm{can}}$. We have

$$X^*(\widetilde{\mu})(\chi) = (\tau \mapsto \chi \circ {}^\tau\mu, \chi \circ ({}^c\mu\mu)).$$

Note that

$$\widetilde{{}^\sigma\mu} = \widetilde{\mu} \circ [\sigma^{-1}].$$

Lemma 7.1. *Suppose that $\chi \in \mathbb{Z}[V_{E,\infty}]_0 \otimes X_*(S_E) \subset X^*(T_{3,E}) \otimes X_*(S_E)$. Then*

$$\prod_{\eta \in \mathrm{Gal}\,(E/\mathbb{Q})} {}^\eta\chi = 1.$$

Proof: By linearity we only need to consider the case $\chi = [\tau]\mu^{\mathrm{can}}(\pi_{v_1}/\pi_{v_2})$. As $[\tau]$ is rationally defined we are further reduced to the case $\chi = \mu^{\mathrm{can}}(\pi_{v_1}/\pi_{v_2})$. Thought of as an element of $\mathrm{Hom}\,(T_{2,E}, S_E)$ we have

$$\begin{array}{rcl} \prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})} {}^{\eta}(\mu^{\mathrm{can}}\circ\pi_v) & = & \prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})}({}^{\eta}\mu^{\mathrm{can}})\circ\pi_{\eta v} \\ & = & \prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})/\mathrm{Gal}\,(E_v/\mathbb{R})}({}^{\eta}\mu^{\mathrm{can}\,\eta c_v}\mu^{\mathrm{can}})\circ\pi_{\eta v} \\ & = & \prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})/\mathrm{Gal}\,(E_v/\mathbb{R})}({}^{\eta}\mathrm{wt})\circ\pi_{\eta v} \\ & = & \mathrm{wt}\circ\prod_{w|\infty}\pi_w. \end{array}$$

As this does not depend on v, the lemma follows. □

Again there is a second way to think about this: there is a torus $S_{E,\mathbb{C}}/\mathbb{Q}$ (abstractly isomorphic to S_E) together with a cocharacter $\mu_{\mathbb{C}}^{\mathrm{can}} \in X_*(S_{E,\mathbb{C}})(\mathbb{C})$ and an action of $\mathrm{Aut}\,(\mathbb{C})$ denoted $\tau \mapsto [\tau]_{\mathbb{C}}$. If $\tau \in \mathrm{Aut}\,(\mathbb{C})$ then ${}^{\tau}\mu_{\mathbb{C}}^{\mathrm{can}} = [\tau^{-1}]_{\mathbb{C}}\circ\mu_{\mathbb{C}}^{\mathrm{can}}$. There is a $\mathrm{Aut}\,(\mathbb{C})$-equivariant surjection $R_{E,\mathbb{C}} \twoheadrightarrow S_{E,\mathbb{C}}$ taking $\mu_{\mathbb{C}}^{\mathrm{can}}$ to $\mu_{\mathbb{C}}^{\mathrm{can}}$. The kernel $R^1_{E,\mathbb{C}}$ is again a torus. There is a cocharacter $\mathrm{wt} = \mu_{\mathbb{C}}^{\mathrm{can}\,c}\mu_{\mathbb{C}}^{\mathrm{can}} \in X_*(S_{E,\mathbb{C}})(\mathbb{Q})$, which commutes with the action of $\mathrm{Aut}\,(\mathbb{C})$. If $D/\mathbb{Q}$ is another finite Galois extension containing E, then $N_{D/E} : S_{D,\mathbb{C}} \to S_{E,\mathbb{C}}$.

If $T/\mathbb{Q}$ is any torus and $\mu \in X_*(T)(\mathbb{C})$ satisfies

- μ is defined over the image of E in $\mathbb{C}$,
- ${}^{\tau c\tau^{-1}}\mu$ is independent of $\tau \in \mathrm{Aut}\,(\mathbb{C})$,
- and ${}^{(c+1)}\mu \in X_*(T)(\mathbb{Q})$;

then there is a unique morphism $\widetilde{\mu} : S_{E,\mathbb{C}} \to T$ over $\mathbb{Q}$ such that $\mu = \widetilde{\mu}\circ\mu^{\mathrm{can}}$. We have

$$\widetilde{{}^{\tau}\mu} = \widetilde{\mu}\circ[\tau^{-1}].$$

Corollary 7.2. *If* $\chi \in \mathbb{Z}[V_{E,\infty}]_0 \otimes X_*(S_{E,\mathbb{C}})(E) \subset X^*(T_{3,E}) \otimes X_*(S_{E,\mathbb{C})}(E)$. *Then*

$$\prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})} {}^{\eta}\chi = 1.$$

7.2. **The Taniyama group.** Langlands considers extensions

$$1 \longrightarrow S_E \longrightarrow \widetilde{S} \longrightarrow \mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q}) \longrightarrow 1$$

(as a pro-algebraic group over $\mathbb{Q}$) such that the induced action of $\mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q})$ on S_E is given by [ ], together with a continuous group theoretic section $\mathrm{sp} : \mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q}) \to \widetilde{S}(\mathbb{A}^\infty)$. (We will follow [MS1], which in turn followed [L]. However the two articles use different conventions so it is hard to directly compare the details in the two sources.) Note that such a pair $(\widetilde{S}, \mathrm{sp})$ has no automorphisms (where we consider $\widetilde{S}$ with its structure of an extension of $\mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q})$ by S_E). Also note that $\widetilde{S}(E) \twoheadrightarrow \mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q})$. (See [MS1] p235.)

Langlands showed that giving such a pair is equivalent to giving an element

$$\bar{b} \in Z^1(\mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q}), (S_E(\mathbb{A}_E^\infty)/S_E(E))^{\mathrm{Gal}\,(E/\mathbb{Q})}),$$

where $\mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q})$ acts via $[\ ]$ and $\mathrm{Gal}(E/\mathbb{Q})$ acts by its Galois action on $\mathbb{A}_E^\infty$, such that $\overline{b}$ lifts to a continuous map

$$b : \mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q}) \longrightarrow S_E(\mathbb{A}_E^\infty)$$

such that

$$\begin{array}{rcl} \mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q})^2 & \longrightarrow & S_E(E) \\ (\tau_1, \tau_2) & \longmapsto & b(\tau_1)[\tau_1](b(\tau_2))b(\tau_1\tau_2)^{-1} \end{array}$$

is locally constant. (See proposition 2.7 of [MS1].) We will will write

$$Z^1_{\mathrm{cts}}(\mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q}), (S_E(\mathbb{A}_E^\infty)/S_E(E))^{\mathrm{Gal}(E/\mathbb{Q})})$$

for the set of elements $\overline{b} \in Z^1(\mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q}), (S_E(\mathbb{A}_E^\infty)/S_E(E))^{\mathrm{Gal}(E/\mathbb{Q})})$ with such a lift.

If $\tau \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q})$ then its preimage $S_{E,\bar{b},\tau}$ in $\widetilde{S}$ is a *right* S_E-torsor. If $\alpha \in S_{E,\bar{b},\tau}$, then

$$c_{\bar{b},\tau,\alpha} : \sigma \longmapsto \alpha^{-1\sigma}\alpha$$

lies in $Z^1(\mathrm{Gal}(E/\mathbb{Q}), S_E(E))$. The class $[c_{\bar{b},\tau,\alpha}] \in H^1(\mathrm{Gal}(E/\mathbb{Q}), S_E(E))$ does not depend on α so we will denote it simply $[c_{\bar{b},\tau}]$. It characterizes the right torsor $S_{E,\bar{b},\tau}$. We set

$$b_{\bar{b},\tau,\alpha} = \alpha^{-1}\mathrm{sp}(\tau) \in S_E(\mathbb{A}_E^\infty),$$

so that

$$c_{\bar{b},\tau,\alpha}(\sigma) = b_{\bar{b},\tau,\alpha}{}^{\sigma}b_{\bar{b},\tau,\alpha}^{-1}.$$

Note that $b_{\bar{b},\tau,\alpha}$ lifts $[\tau^{-1}](\overline{b}(\tau)^{-1})$. (This normalization seems unfortunate, but we keep it to aid reference to the papers of Langlands [L] and Milne-Shih [MS1].) If $\alpha \in S_E(\mathbb{Q})$ lifts $\tau \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q})$ then $\overline{b}(\tau) \in S_E(\mathbb{A}^\infty)/S_E(\mathbb{Q})$ and $b_{\bar{b},\tau,\alpha} = \alpha^{-1}\mathrm{sp}(\tau) \in S_E(\mathbb{A}^\infty)$ lifts $[\tau^{-1}](\overline{b}(\tau)^{-1})$. Also note that if $\gamma \in S_E(E)$ then

$$c_{\bar{b},\tau,\gamma\alpha} = {}^{\gamma^{-1}}c_{\bar{b},\tau,\alpha}$$

and

$$b_{\bar{b},\tau,\gamma\alpha} = \gamma^{-1}b_{\bar{b},\tau,\alpha}.$$

Moreover

$$c_{\bar{b},\tau_1\tau_2,\alpha_1\alpha_2}(\sigma) = [\tau_2]^{-1}(c_{\bar{b},\tau_1,\alpha_1})c_{\bar{b},\tau_2,\alpha_2}(\sigma)$$

and

$$b_{\bar{b},\tau_1\tau_2,\alpha_1\alpha_2} = [\tau_2^{-1}](b_{\bar{b},\tau_1,\alpha_1})b_{\bar{b},\tau_2,\alpha_2}.$$

If v is an infinite place of E^{ab} Langlands defines a particular element

$$\overline{b}^{\mathrm{Tan}}_{E,v} \in Z^1_{\mathrm{cts}}(\mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q}), (S_E(\mathbb{A}_E^\infty)/S_E(E))^{\mathrm{Gal}(E/\mathbb{Q})})$$

as follows. There is an exact sequence

$$1 \longrightarrow \mathbb{A}_E^\times/E^\times \longrightarrow W_{E^{\mathrm{ab}}/\mathbb{Q}} \longrightarrow \mathrm{Gal}(E/\mathbb{Q}) \longrightarrow 1.$$

Choose preimages $w_\eta \in W_{E^{\mathrm{ab}}/\mathbb{Q}}$ of each $\eta \in \mathrm{Gal}(E/\mathbb{Q})$ such that the following conditions hold:

- $w_1 = 1$
- and there is a set of representatives $1 \in H \subset \mathrm{Gal}\,(E/\mathbb{Q})$ for $\mathrm{Gal}\,(E/\mathbb{Q})/\mathrm{Gal}\,(E_v/\mathbb{R})$ such that $w_{\eta c_v} = w_\eta \theta_v(j)$ for all $\eta \in H$.

If $w \in W_{E^{\mathrm{ab}}/\mathbb{Q}}$ then
$$w_\eta w = \overline{u}_{v,\eta,w} w_{\eta\overline{w}}$$
where $\overline{u}_{v,\eta,w} \in \mathbb{A}_E^\times/E^\times$ and where $\overline{w}$ denotes the image of w in $\mathrm{Gal}\,(E/\mathbb{Q})$. Then Langlands takes
$$\overline{b}_{E,v}^{\mathrm{Tan}}(w) = \prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})} ({}^\eta\mu^{\mathrm{can}})(\overline{u}_{v,\eta,w}) \in S_E(\mathbb{A}_E^\infty)/S_E(E).$$

He verifies that it lies in $(S_E(\mathbb{A}_E^\infty)/S_E(E))^{\mathrm{Gal}\,(E/\mathbb{Q})}$; that it doesn't depend on the choices of preimages w_η (as long as they satisfy the above conditions); that it only depends on the image τ of w in $\mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q})$ (so we will often write $\overline{b}_{E,v}^{\mathrm{Tan}}(\tau)$); and that $\overline{b}_{E,v}^{\mathrm{Tan}} \in Z^1_{\mathrm{cts}}(\mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q}), (S_E(\mathbb{A}_E^\infty)/S_E(E))^{\mathrm{Gal}\,(E/\mathbb{Q})})$. It also doesn't depend on the choice of θ_v associated to v. (If θ_v is replaced by $a\theta_v a^{-1}$ with $a \in \overline{E^\times(E_\infty^\times)^0}$, then w_η gets replaced by $w_\eta{}^{c_v}a/a$ if $\eta \notin H$ and is unchanged if $\eta \in H$. Thus $\overline{u}_{v,\eta,w}$ gets multiplied by
$$\left\{ \begin{array}{ll} {}^{\eta c_v}a/{}^\eta a & \text{if } \eta \notin H \\ 1 & \text{if } \eta \in H \end{array} \right\} \left\{ \begin{array}{ll} {}^{\eta\overline{w}}a/{}^{\eta\overline{w}c_v}a & \text{if } \eta\overline{w} \notin H \\ 1 & \text{if } \eta\overline{w} \in H \end{array} \right\}.$$

Thus $\overline{b}_{K,v}^{\mathrm{Tan}}(w)$ changes by
$$\begin{array}{cl} & \prod_{\eta\notin H} \eta\left(\mu^{\mathrm{can}}({}^{c_v}a/a)({}^{\overline{w}^{-1}}\mu^{\mathrm{can}})(a/{}^{c_v}a)\right) \\ = & \left(\prod_{\eta\in H}\eta\mathrm{wt}(a)/\prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})}\eta\mu(a)\right)\left(\prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})}\eta^{\overline{w}^{-1}}\mu(a)/\prod_{\eta\in H}\eta\mathrm{wt}(a)\right) \\ = & \prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})}\eta({}^{\overline{w}^{-1}}\mu/\mu)(a) \\ \in & \overline{S_E(\mathbb{Q})S_E(\mathbb{R})} = S_E(\mathbb{Q})S_E(\mathbb{R}) \subset S_E(E)S_E(E_\infty), \end{array}$$
i.e. it is unchanged.)

We will write $(\widetilde{S}_{E,v}, \mathrm{sp}_{E,v})$ for the corresponding extension with a finite adelic section. It is usually referred to as the 'Taniyama group' (although as it depends on the choice of v it would be better called 'the Taniyama group with respect to v'). We also write $\widetilde{S}_{E,v,\tau}$ for the pre-image in $\widetilde{S}_{E,v}$ of $\tau \in \mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q})$, a right S_E-torsor, i.e. $\widetilde{S}_{E,v,\tau} = S_{E,\overline{b}_{E,v}^{\mathrm{Tan}},\tau}$. Note that if $\alpha \in \widetilde{S}_{E,v}(E)$ lies above $\tau \in \mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q})$, then
$$\begin{array}{rcl} b_{\overline{b}_{E,v}^{\mathrm{Tan}},\tau,\alpha} & \equiv & [\tau^{-1}]\prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})}({}^\eta\mu^{\mathrm{can}})(w_\eta\widetilde{\tau}w_{\eta\tau}^{-1})^{-1} \\ & = & \prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})}({}^{\eta\tau}\mu^{\mathrm{can}})(w_{\eta\tau}\widetilde{\tau}^{-1}w_\eta^{-1}) \\ & = & \prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})}({}^\eta\mu^{\mathrm{can}})(w_\eta\widetilde{\tau}^{-1}w_{\eta\tau^{-1}}^{-1}) \\ & = & \prod_{\eta\in\mathrm{Gal}\,(E/\mathbb{Q})}\eta\mu^{\mathrm{can}}(\widetilde{\tau}^{-1}w_{\eta\tau^{-1}}^{-1}w_\eta) \bmod S_E(E), \end{array}$$
where $\widetilde{\tau} \in W_{E^{\mathrm{ab}}/\mathbb{Q}}$ lifts τ.

Note that if $w \in \mathbb{A}_E^\times/E^\times$ then $\overline{b}_{E,v}^{\mathrm{Tan}}(\mathrm{Art}_E w) = \prod_{\eta \in \mathrm{Gal}(E/\mathbb{Q})} \eta\mu^{\mathrm{can}}(w) \in S_E(\mathbb{A}^\infty)/S_E(\mathbb{Q})$. If we choose a lift $b_{E,v}^{\mathrm{Tan}}(\mathrm{Art}_E w) \in S_E(\mathbb{A}^\infty)$ then $a(\mathrm{Art}_E w) = b_{E,v}^{\mathrm{Tan}}(\mathrm{Art}_E w)\mathrm{sp}_{E,v}(\mathrm{Art}_E w) \in \widetilde{S}_{E,v}(\mathbb{Q})$ and maps to $\mathrm{Art}_E w \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$. Thus we have shown that every element of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ has a lift in $\widetilde{S}_{E,v}(\mathbb{Q})$ (and not simply in $\widetilde{S}_{E,v}(E)$). If $\alpha \in \widetilde{S}_{E,v}(\mathbb{Q})$ lifts $\tau \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$, then

$$b_{\overline{b}_{E,v}^{\mathrm{Tan}},\tau,\alpha} \equiv \prod_{\eta \in \mathrm{Gal}(E/\mathbb{Q})} \eta\mu^{\mathrm{can}}(\mathrm{Art}_E^{-1}\tau)^{-1} \bmod S_E(\mathbb{Q}).$$

Lemma 7.3. *If $\sigma \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q})$ then there is a unique isomorphism*

$$\{\sigma\} : \widetilde{S}_{E,v} \xrightarrow{\sim} \widetilde{S}_{E,\sigma v}$$

such that

$$\begin{array}{ccccccccc} (0) & \longrightarrow & S_E & \longrightarrow & \widetilde{S}_{E,v} & \longrightarrow & \mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q}) & \longrightarrow & (0) \\ & & [\sigma]\downarrow & & \{\sigma\}\downarrow & & \mathrm{conj}_\sigma \downarrow & & \\ (0) & \longrightarrow & S_E & \longrightarrow & \widetilde{S}_{E,\sigma v} & \longrightarrow & \mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q}) & \longrightarrow & (0) \end{array}$$

commutes, and

$$\{\sigma\} \circ \mathrm{sp}_{E,v} = \mathrm{sp}_{E,\sigma v} \circ \mathrm{conj}_\sigma.$$

Proof: We may construct $\overline{b}_{E,\sigma v}^{\mathrm{Tan}}$ from $w'_\eta = \mathrm{conj}_\sigma(w_{\sigma^{-1}\eta\sigma})$ and $H' = \sigma H \sigma^{-1}$, which gives

$$\overline{u}_{\sigma v,\eta,w} = \sigma \overline{u}_{v,\sigma^{-1}\eta\sigma,\mathrm{conj}_{\sigma^{-1}}(w)}$$

and

$$\begin{array}{rcl} \overline{b}_{E,\sigma v}^{\mathrm{Tan}}(w) & = & \prod_{\eta \in \mathrm{Gal}(E/\mathbb{Q})} \eta\mu^{\mathrm{can}}(\eta^{-1}\sigma\overline{u}_{v,\sigma^{-1}\eta\sigma,\mathrm{conj}_{\sigma^{-1}}(w)}) \\ & = & \prod_{\eta \in \mathrm{Gal}(E/\mathbb{Q})} \sigma\eta\sigma^{-1}\mu^{\mathrm{can}}(\sigma\eta^{-1}\overline{u}_{v,\eta,\mathrm{conj}_{\sigma^{-1}}(w)}) \\ & = & \prod_{\eta \in \mathrm{Gal}(E/\mathbb{Q})} \sigma^\eta([\sigma]\mu^{\mathrm{can}})(\overline{u}_{v,\eta,\mathrm{conj}_{\sigma^{-1}}(w)}) \\ & = & [\sigma]\sigma\overline{b}_{E,v}^{\mathrm{Tan}}(\mathrm{conj}_{\sigma^{-1}}w) \\ & = & [\sigma]\overline{b}_{E,v}^{\mathrm{Tan}}(\mathrm{conj}_\sigma^{-1}w) \end{array}$$

The lemma follows. □

Corollary 7.4. *If $\sigma \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q})$ then*

$$b_{\overline{b}_{E,\sigma v}^{\mathrm{Tan}},\sigma\tau\sigma^{-1},\{\sigma\}\alpha} = [\sigma]b_{\overline{b}_{E,v}^{\mathrm{Tan}},\tau,\alpha}.$$

Lemma 7.5. $\mathrm{sp}_{E,v}(c_v) \in \widetilde{S}_{E,v}(\mathbb{Q})$ *and $\{c_v\} : \widetilde{S}_{E,v} \to \widetilde{S}_{E,v}$ is simply conjugation by* $\mathrm{sp}_{E,v}(c_v)$.

Proof: We have

$$\overline{u}_{v,\eta,w_{c_v}} = \begin{cases} 1 & \text{if } \eta \in H \\ -1_{\eta v} & \text{if } \eta \notin H. \end{cases}$$

Thus $\overline{b}^{\mathrm{Tan}}_{E,v}(c_v) = 1$ and $\mathrm{sp}_{E,v}(c_v) \in \widetilde{S}_{E,v}(E) \cap \widetilde{S}_{E,v}(\mathbb{A}^\infty) = \widetilde{S}_{E,v}(\mathbb{Q})$. The second part of the lemma follows immediately. □

If $\rho : E^{\mathrm{ab}} \hookrightarrow \mathbb{C}$ we can pull the extension $\widetilde{S}_{E,v(\rho)}$ back along the map

$$\begin{array}{ccc} \mathrm{Aut}\,(\mathbb{C}) & \longrightarrow & \mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q}) \\ \tau & \longmapsto & \tau^\rho \end{array}$$

to obtain an extension

$$(0) \longrightarrow S_E \longrightarrow \widetilde{S}_{E,\rho} \longrightarrow \mathrm{Aut}\,(\mathbb{C}) \longrightarrow (0),$$

where $\mathrm{Aut}\,(\mathbb{C})$ acts on S_E via $\tau \mapsto [\tau^\rho]_{\mathbb{C}}$, and a splitting

$$\mathrm{sp}_\rho : \mathrm{Aut}\,(\mathbb{C}) \longrightarrow \widetilde{S}_{E,\rho}(\mathbb{A}^\infty).$$

It follows from lemma 7.3 that if $\sigma \in \mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q})$ we have a commutative diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & S_E & \longrightarrow & \widetilde{S}_{E,\rho} & \longrightarrow & \mathrm{Aut}\,(\mathbb{C}) & \longrightarrow & (0) \\ & & [\sigma^{-1}]\downarrow & & \{\sigma^{-1}\}\downarrow & & || & & \\ (0) & \longrightarrow & S_E & \longrightarrow & \widetilde{S}_{E,\rho\sigma} & \longrightarrow & \mathrm{Aut}\,(\mathbb{C}) & \longrightarrow & (0) \end{array}$$

and that $\mathrm{sp}_{\rho\sigma} = \{\sigma^{-1}\} \circ \mathrm{sp}_\rho$. Thus we have a canonical extension

$$(0) \longrightarrow S_{E,\mathbb{C}} \longrightarrow \widetilde{S}_{E,\mathbb{C}} \longrightarrow \mathrm{Aut}\,(\mathbb{C}) \longrightarrow (0),$$

where $\tau \in \mathrm{Aut}\,(\mathbb{C})$ acts on $S_{E,\mathbb{C}}$ via $[\tau]_{\mathbb{C}}$, together with a section $\mathrm{sp} : \mathrm{Aut}\,(\mathbb{C}) \to \widetilde{S}_{E,\mathbb{C}}(\mathbb{A}^\infty)$.

If $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ has image $\overline{\alpha} \in \mathrm{Aut}\,(\mathbb{C})$ we will write

$$b_{E,\mathrm{Tan},\alpha} = \alpha^{-1}\mathrm{sp}(\overline{\alpha}) \in S_{E,\mathbb{C}}(\mathbb{A}^\infty_E).$$

We have the following properties:

(1) If $\gamma \in S_{E,\mathbb{C}}(E)$, then $b_{E,\mathrm{Tan},\gamma\alpha} = \gamma^{-1} b_{E,\mathrm{Tan},\alpha}$.

(2)
$$b_{E,\mathrm{Tan},\alpha} \equiv \prod_{\eta \in \mathrm{Gal}\,(E/\mathbb{Q})} \eta({}^{\rho^{-1}}\mu^{\mathrm{can}}_{\mathbb{C}})(\widetilde{\tau}^{-1} w^{-1}_{\eta\tau^{-1}} w_\eta) \bmod S_{E,\mathbb{C}}(E),$$
where $\rho : E^{\mathrm{ab}} \hookrightarrow \mathbb{C}$ and $\widetilde{\tau} \in W_{E^{\mathrm{ab}}/\mathbb{Q}}$ lifts $\overline{\alpha}^\rho \in \mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q})$ and where the liftings $w_\eta \in W_{E^{\mathrm{ab}}/\mathbb{Q}}$ are chosen to satisfy the conditions listed earlier in this section with respect to $v(\rho)$.

(3) In particular, if $\alpha \in \widetilde{S}_E(\mathbb{Q})$ and $\overline{\alpha}$ fixes the image of E in $\mathbb{C}$, then $b_{E,\mathrm{Tan},\alpha} \in S_{E,\mathbb{C}}(\mathbb{A}^\infty)$ and
$$b_{E,\mathrm{Tan},\alpha} \equiv \prod_{\rho : E \hookrightarrow \mathbb{C}} ({}^{\rho^{-1}}\mu^{\mathrm{can}}_{\mathbb{C}})(\mathrm{Art}\,_E^{-1}\overline{\alpha}^\rho)^{-1} \bmod S_{E,\mathbb{C}}(\mathbb{Q}).$$

We will call $\alpha \in \widetilde{S}_{E,\mathbb{C}}(\mathbb{Q})$ *well placed* if $\overline{\alpha}$ fixes the image of E in $\mathbb{C}$ and

$$b_{E,\mathrm{Tan},\alpha} = \prod_{\rho: E \hookrightarrow \mathbb{C}} ({}^{\rho^{-1}}\mu_{\mathbb{C}}^{\mathrm{can}})(\mathrm{Art}\,_E^{-1}\overline{\alpha}^{\rho})^{-1}.$$

If $\alpha \in \widetilde{S}_{E,\mathbb{C}}(\mathbb{Q})$ and $\overline{\alpha}$ fixes the image of E in $\mathbb{C}$, then $S_{E,\mathbb{C}}(\mathbb{Q})\alpha$ contains a unique well placed element. The element $1 \in \widetilde{S}_{E,\mathbb{C}}(\mathbb{Q})$ is well placed.

(4)

$$b_{E,\mathrm{Tan},\alpha_1\alpha_2} \equiv [\overline{\alpha}_2^{-1}]_{\mathbb{C}}(b_{E,\mathrm{Tan},\alpha_1})b_{E,\mathrm{Tan},,\alpha_2}.$$

(Indeed $(\alpha_1\alpha_2)^{-1}\mathrm{sp}(\overline{\alpha}_1\overline{\alpha}_2) = \alpha_2^{-1}(\alpha_1^{-1}\mathrm{sp}(\overline{\alpha}_1))\alpha_2(\alpha_2^{-1}\mathrm{sp}(\overline{\alpha}_2))$.)

(5) If $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ and $\sigma \in \mathrm{Gal}\,(E/\mathbb{Q})$, then

$$b_{E,\mathrm{Tan},\alpha}{}^{\sigma}(b_{E,\mathrm{Tan},\alpha})^{-1} \in S_{E,\mathbb{C}}(E).$$

(Indeed ${}^{\sigma}(\alpha^{-1}\mathrm{sp}(\overline{\alpha})) = (({}^{\sigma}\alpha)^{-1}\alpha)(\alpha^{-1}\mathrm{sp}(\overline{\alpha}))$.)

7.3. **Relationship between Taniyama groups and the elements $\overline{b}_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\tau}$.**

Lemma 7.6. *If $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ has image $\overline{\alpha} \in \mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q})$, then*

$$\overline{b}_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\overline{\alpha}} \in S_{E,\mathbb{C}}(\mathbb{A}_E^{\infty})/S_{E,\mathbb{C}}(E) = R_{E,\mathbb{C}}(\mathbb{A}_E^{\infty})/\overline{R_{E,\mathbb{C}}(\mathbb{Q})}R_{E,\mathbb{C}}(E)R_{E,\mathbb{C}}^1(\mathbb{A}_E^{\infty})$$

equals the image of $b_{E,\mathrm{Tan},\alpha}$.

Proof: Replacing $\mathfrak{a}^+$ by ${}^t\mathfrak{a}^+$ leaves $\overline{b}_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\overline{\alpha}}$ unchanged. (Use corollary 7.2, lemma 6.4 and note that $\overline{S_{E,\mathbb{C}}(\mathbb{Q})S_{E,\mathbb{C}}(\mathbb{R})} = S_{E,\mathbb{C}}(\mathbb{Q})S_{E,\mathbb{C}}(\mathbb{R})$.) Thus it suffices to prove the assertion with $\mathfrak{a}^+ = [\boldsymbol{\alpha}_0^+]$, the class defined at the end of section 6.1. In this case we may take $w_\eta = \varphi(e^{\mathrm{glob}}_{\boldsymbol{\alpha}_0}(\eta))$, and the result then follows from comparing the formula at the end of section 6.2 with the formula two paragraphs before lemma 7.3. □

Lemma 7.7. *Suppose that $s : S_{E,\mathbb{C}}(E) \to R_{E,\mathbb{C}}(E)$ is a group theoretic section. For $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$, we may choose elements $b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} \in R_{E,\mathbb{C}}(\mathbb{A}_E^{\infty})$ with the following properties. Write $\overline{\alpha}$ for the image of α in $\mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q})$.*

(1) $b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$ lifts $\overline{b}_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\overline{\alpha}} \in R_{E,\mathbb{C}}(\mathbb{A}_E^{\infty})/\overline{R_{E,\mathbb{C}}(\mathbb{Q})}R_{E,\mathbb{C}}(E)$.

(2) $b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$ lifts $b_{E,\mathrm{Tan},\alpha} \in S_{E,\mathbb{C}}(\mathbb{A}_E^{\infty})$.

(3) If $\alpha \in \widetilde{S}_{E,\mathbb{C}}(\mathbb{Q})$ is well placed, then $b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} = \prod_{\rho:E\hookrightarrow\mathbb{C}}({}^{\rho^{-1}}\mu_{\mathbb{C}}^{\mathrm{can}})(\mathrm{Art}\,_E^{-1}\overline{\alpha}^{\rho})^{-1}$.

(4) If $\gamma \in S_{E,\mathbb{C}}(E)$ then $b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\gamma\alpha} = \widetilde{\gamma}^{-1}b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$, for some lift $\widetilde{\gamma} \in R_{E,\mathbb{C}}(E)$ of γ.

The first two of these properties determine $b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$ up to multiplication by an element of $\overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}R^1_{E,\mathbb{C}}(E)$.

Then we also have that:

(5)

$$b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha_1\alpha_2} \equiv [\overline{\alpha}_2^{-1}]_{\mathbb{C}}(b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha_1})b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha_2} \bmod \overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}R^1_{E,\mathbb{C}}(E).$$

(6) If $t \in T_{2,E}(\mathbb{A}_E)$ *then*

$$b_{t\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} \equiv b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} \prod_{\rho:E\hookrightarrow\mathbb{C}} (({}^{\rho^{-1}}(\mu/{}^{\overline{\alpha}}\mu)) \circ \pi_{w(\rho)})(t) \bmod \overline{R^1_{E,\mathbb{C}}(\mathbb{Q})} R^1_{E,\mathbb{C}}(E).$$

(7) If $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ *and* $\sigma \in \mathrm{Gal}\,(E/\mathbb{Q})$*, then*

$$b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}{}^{\sigma}(b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha})^{-1} \in R^1_{E,\mathbb{C}}(\mathbb{A}_E^\infty)R_{E,\mathbb{C}}(E) \subset R_{E,\mathbb{C}}(\mathbb{A}_E^\infty).$$

Proof: Let A denote a set of representatives for $\widetilde{S}_{E,\mathbb{C}}(E)/S_{E,\mathbb{C}}(E)$, which we can and will assume to be the unique well-placed representative whenever the appropriate coset contains some well-placed element. If $\alpha \in A$ is well placed, then we set

$$b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} = \prod_{\rho:E\hookrightarrow\mathbb{C}} ({}^{\rho^{-1}}\mu_{\mathbb{C}}^{\mathrm{can}})(\mathrm{Art}\,_E^{-1}\overline{\alpha}^{\rho})^{-1}.$$

This also satisfies the first two conditions. If $\alpha \in A$ is not well-placed, then define $b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$ to be any common lift of $\overline{b}_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\overline{\alpha}}$ and $b_{E,\mathrm{Tan},\alpha}$. This exists tautologically. We then use (4) to extend the definition to all $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$. It is easy to check that this definition satisfies properties (1)-(4).

The uniqueness assertion follows from the equality $R^1_{E,\mathbb{C}}(\mathbb{A}_E^\infty) \cap \overline{R_{E,\mathbb{C}}(\mathbb{Q})}R_{E,\mathbb{C}}(E) = \overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}R^1_{E,\mathbb{C}}(E)$. (If $\beta \in R_E(E)$ and $\gamma_i \in R_E(\mathbb{Q})$ and $\gamma_i \to g$ in $R_E(\mathbb{A}^\infty)$ and $\beta g \in R^1_E(\mathbb{A}_E^\infty)$, then we may suppose that all the γ_i have the same image in $S_E(\mathbb{Q})$, which is discrete in $S_E(\mathbb{A}^\infty)$, and that this image is the inverse of the image of β. Then $\beta g = \lim_{\leftarrow}(\beta\gamma_0)(\gamma_0^{-1}\gamma_i) \in R^1_E(E)\overline{R^1_E(\mathbb{Q})}$.)

The final three assertions follow from this uniqueness and the corresponding relations for $\overline{b}_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\overline{\alpha}}$ (see lemma 6.4) and $b_{E,\mathrm{Tan},\alpha}$. $\square$

The particular element $b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$ was constructed by Langlands in [L] using explicit formulae. He verified some of the above properties including property (7).

Note the following simple remark:

Lemma 7.8. *Suppose* $c_1, c_2 \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, T(E))$ *for some torus* $T/\mathbb{Q}$*, and that* $[c_1] = [c_2]$*. Suppose also that* ${}_v\mathrm{loc}^\infty(c_i) = {}^{b_i}1$ *and that* $b_1^{-1}b_2 \in T(\mathbb{A}^\infty)$*. Then* $c_1 = c_2$*.*

(The point being that $c_2 = {}^{\gamma}c_1$ with $\gamma \in T(E)$, so that $\gamma b_1 b_2^{-1} \in T(\mathbb{A}^\infty)$. Thus $\gamma \in T(\mathbb{A}^\infty) \cap T(E) = T(\mathbb{Q})$.)

Lemma 7.9. *(1) If* $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$*, there exists a unique class*

$$c_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} \in \boldsymbol{\phi}_{R_{E,\mathbb{C}},\{\mu_{\mathbb{C}}^{\mathrm{can}}\},\overline{\alpha}} = [\mathrm{cor}\,({}^{\rho^{-1}}\mu_{\mathbb{C}}^{\mathrm{can}} \circ \pi_{w(\rho)}/\pi_{w(\overline{\alpha}\rho)})] \subset Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, R_{E,\mathbb{C}})$$

with $\mathrm{loc}^\infty_{\mathfrak{a}} c_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} = {}^{b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}}1$*. (For any* $\rho : E^{\mathrm{ab}} \hookrightarrow \mathbb{C}$*.) The class* $c_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$ *is independent of the suppressed choices (made in lemma 7.7) up to the action of* $R^1_{E,\mathbb{C}}(E)$*.*

(2) *If* $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ *and* $\gamma \in S_{E,\mathbb{C}}(E)$ *then we can find* $\widetilde{\gamma} \in R_{E,\mathbb{C}}(E)$ *lifting* γ *such that* $c_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\gamma\alpha} = {}^{\widetilde{\gamma}^{-1}}c_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$ *and* $b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\gamma\alpha} = \widetilde{\gamma}^{-1}b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$.

(3) *If* $\alpha \in \widetilde{S}_{E,\mathbb{C}}(\mathbb{Q})$ *is well-placed, then* $c_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} = 1$ *and*

$$b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} = \prod_{\rho:E\hookrightarrow\mathbb{C}} ({}^{\rho^{-1}}\mu_{\mathbb{C}}^{\mathrm{can}})(\mathrm{Art}_E^{-1}\overline{\alpha}^{\rho})^{-1} \in R_{E,\mathbb{C}}(\mathbb{A}^\infty).$$

(4) *Given* $\alpha_i \in \widetilde{S}_{E,\mathbb{C}}(E)$ *for* $i=1,2$, *there exists* $\beta \in R^1_{E,\mathbb{C}}(E)$ *such that*

$$\beta b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha_1\alpha_2} \equiv [\overline{\alpha}_2^{-1}]_{\mathbb{C}}(b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha_1}) b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} \bmod \overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}$$

and

$${}^{\beta}c_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha_1\alpha_2} = [\overline{\alpha}_2^{-1}]_{\mathbb{C}}(c_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha_1}) c_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha_2}.$$

(5) *If* $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ *and* $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$, *then*

$$b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}{}^{\sigma}(b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha})^{-1} \in R^1_{E,\mathbb{C}}(\mathbb{A}_E^\infty)R_{E,\mathbb{C}}(E) \subset R_{E,\mathbb{C}}(\mathbb{A}_E^\infty).$$

(6) *If* $t \in T_{2,E}(\mathbb{A}_E)$ *then there exists* $\beta \in R^1_{E,\mathbb{C}}(E)$ *with*

$$\beta b_{{}^t\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} \equiv b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} \prod_{\rho:E\hookrightarrow\mathbb{C}} (({}^{\rho^{-1}}(\mu/{}^{\overline{\alpha}}\mu)) \circ \pi_{v(\rho)})(t) \bmod \overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}$$

and

$${}^{\beta}c_{{}^t\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha} = z_t(c_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}).$$

Proof: For the first part we have that

$$\mathrm{loc}^\infty_{\boldsymbol{\alpha}} \mathrm{cor}_{\alpha^{\mathrm{glob}}}({}^{\rho^{-1}}\mu) \circ (\pi_{w(\rho)}/\pi_{w(\overline{\alpha}\rho)}) = {}^{b_{\boldsymbol{\alpha}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\overline{\alpha},\rho,\widetilde{g}}}1.$$

By lemma 7.6 we can find $\gamma \in R_{E,\mathbb{C}}(E)$ so that $\gamma b_{\boldsymbol{\alpha}^+_{0,\varphi},\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\overline{\alpha},\rho,\widetilde{g}_{\infty,\rho,0}}$ maps to $b_{E,\mathrm{Tan},\alpha}$ in $S_{E,\mathbb{C}}(\mathbb{A}_E^\infty)$. Thus $\gamma b_{\boldsymbol{\alpha}^+_{0,\varphi},\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\overline{\alpha},\rho,\widetilde{g}_{\infty,\rho,0}}$ and $b_{\mathfrak{a}^+_{0,\varphi},\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$ differ by an element of $\overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}R^1_{E,\mathbb{C}}(E)$; or varying γ simply by an element of $\overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}$. Hence

$${}^{b_{\mathfrak{a}^+_{0,\varphi},\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}}1 = \mathrm{loc}^\infty_{\boldsymbol{\alpha}_0}{}^{\gamma}\mathrm{cor}_{\alpha_0^{\mathrm{glob}}}({}^{\rho^{-1}}\mu) \circ (\pi_{w(\rho)}/\pi_{w(\overline{\alpha}\rho)}) \in \boldsymbol{\phi}_{R_{E,\mathbb{C}},\{\mu_{\mathbb{C}}^{\mathrm{can}}\},\overline{\alpha}}.$$

Existence follows. Uniqueness results from lemma 7.8 and the uniqueness assertion in lemma 7.7.

The remaining parts follow from lemmas 7.7 and 7.8. □

8. Deligne's Shimura varieties

8.1. **Deligne's Shimura data.** In Deligne's formalism, Shimura varieties are attached to 'Shimura data'. In this section we will recall Deligne's definition and mention some variants.

Write $\mathbb{S}$ for the restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$ of $\mathbb{G}_m$ and identify

$$\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_m \times \mathbb{G}_m$$

so that $z \otimes w \in \mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times$ corresponds to $(zw, \overline{z}w)$. There is a natural inclusion $\mathbb{G}_m \hookrightarrow \mathbb{S}$ and a norm map $\mathbb{S} \to \mathbb{G}_m$, both defined over $\mathbb{Q}$. If we write $\mathbb{S}^1$ for the kernel of the norm map, then there are exact sequences

$$(0) \longrightarrow \mathbb{S}^1 \longrightarrow \mathbb{S} \longrightarrow \mathbb{G}_m \longrightarrow (0)$$

and

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbb{S} & \longrightarrow & \mathbb{S}^1 & \longrightarrow & (0) \\ & & & & z & \longmapsto & z/\overline{z}. & & \end{array}$$

Define a *Shimura datum* (or *h-Shimura datum* when we need to distinguish it from an equivalent formulation) to be a pair (G, X), where $G/\mathbb{Q}$ is a reductive algebraic group and X is a $G(\mathbb{R})$-conjugacy class of morphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ of algebraic groups over $\mathbb{R}$ satisfying the following axioms:

(1) if $h \in X$ then the adjoint action of $\mathbb{S} \times_{\mathbb{R}} \mathbb{C} \cong \mathbb{G}_m \times \mathbb{G}_m$ on $(\operatorname{Lie} G)_{\mathbb{C}}$ has all its characters in the set $\{(1, -1), (0, 0), (-1, 1)\}$;

(2) if $h \in X$ then $\operatorname{ad} h(i)$ is a Cartan involution for the adjoint group G^{ad}.

If $h \in X$ then $h_{\mathbb{C}} : \mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_m^2 \to G_{\mathbb{C}}$ has the form $(\mu_h, {}^c\mu_h)$ for a unique cocharacter $\mu_h : \mathbb{G}_m \to G_{\mathbb{C}}$. Also note that if $h \in X$, then $\operatorname{ad} h|_{\mathbb{G}_m} = 1$. The space X is uniquely a complex manifold in such that a way that $z \in (\mathbb{C}^\times)^{N_{\mathbb{C}/\mathbb{R}}=1}$ acts on $T_h X = \operatorname{Lie} G(\mathbb{R})/\operatorname{Lie} \operatorname{Stab}_{G(\mathbb{R})}(h)$ as $\operatorname{ad} \mu_h(z)$. (Note that $\operatorname{ad} h$ factors through $u_h : \mathbb{S}^1 \to G^{\operatorname{ad}}$ over $\mathbb{R}$ and u_h on $\mathbb{S}^1(\mathbb{R})$ it equals the restriction of μ_h to $(\mathbb{C}^\times)^{N_{\mathbb{C}/\mathbb{R}}=1}$. According to proposition 5.9 and theorem 2.14 of [Mi3], the complex structure on $T_h X = \operatorname{Lie} G(\mathbb{R})/\operatorname{Lie} \operatorname{Stab}_{G(\mathbb{R})}(h)$ is such that $(\mathbb{C}^\times)^{N_{\mathbb{C}/\mathbb{R}}=1}$ acts by the adjoint action of u_h, i.e. by $\operatorname{ad} \mu_h$.) A morphism $\phi : (G_1, X_1) \to (G_2, X_2)$ is a morphism $\phi : G_1 \to G_2$ of algebraic groups over $\mathbb{Q}$ which takes X_1 to X_2. For instance if $\gamma \in \mathbb{G}^{\operatorname{ad}}(\mathbb{Q})_{\mathbb{R}}$ then

$$\operatorname{conj}_\gamma : (G, X) \to (G, X).$$

The G-conjugacy class $[\mu_h]$ can be thought of as a variety over $\mathbb{C}$. If we set

$$E(G, X) = \mathbb{C}^{\operatorname{Stab}_{\operatorname{Aut}(\mathbb{C})}([\mu_h])} \subset \mathbb{C}$$

then $E(G, X)$ is a number field (called the *reflex field* of (G, X)). It comes with a preferred embedding

$$\imath_{(G,X)} : E(G, X) \hookrightarrow \mathbb{C}$$

and hence with a preferred infinite place $v(G, X)$. The variety $[\mu_h]$ can be defined over $E(G, X)$, and if it can be defined over any subfield $E \subset \mathbb{C}$ then $E(G, X) \subset E$.

In [D2], Deligne imposes a further condition

(3) $G^{\rm ad}$ has no $\mathbb{Q}$ factors on which the projection of h is trivial.

Most subsequent authors have continued to impose this assumption. If (G, X) satisfies this further condition we will call it a *NCF-Shimura datum.* (Here 'NCF' stands for 'no compact factor'.)

We want to explain a straightforward reformulation of this concept. By a *μ-Shimura datum* we will mean a pair (G, Y) where $G/\mathbb{Q}$ is a reductive group and Y is a $G(\mathbb{R})$-conjugacy class of miniscule compactifying cocharacters $\mu : \mathbb{G}_m \to G_{/\mathbb{C}}$. Note that as $\mathrm{ad}\, \mu^c \mu$ is trivial, we have $\mathrm{ad}\, \mu(-1) \in G^{\rm ad}(\mathbb{R})$. A morphism $\phi : (G_1, Y_1) \to (G_2, Y_2)$ is a morphism $\phi : G_1 \to G_2$ of algebraic groups over $\mathbb{Q}$ which takes Y_1 to Y_2. We define the reflex field $E(G, Y)$ to be the field of definition of the G-conjugacy class of μ. We will call (G, Y) an *NCF-μ-Shimura datum* if G also satisfies condition (3) above.

There is a 1-1 correspondence between h-Shimura data and μ-Shimura data which preserves morphisms, reflex fields and the NCF condition. This correspondence sends an h-Shimura datum (G, X) to (G, X^μ), where

$$X^\mu = \{\mu_h : h \in X\}.$$

Note that $\mu_h{}^c \mu_h = h|_{\mathbb{G}_m}$ so that $\mathrm{ad}\, \mu_h{}^c \mu_h = 1$; and that $\mathrm{ad}\, h(i) = \mathrm{ad}\, \mu_h(i)^c \mu_h(-i) = \mathrm{ad}\, \mu_h(-1) \mathrm{ad}\, (\mu_h{}^c \mu_h)(-i) = \mathrm{ad}\, \mu_h(-1)$. In the other direction if (G, Y) is a μ-Shimura datum and $\mu \in Y$, then μ and μ^c commute (as μ^c is a central character times μ^{-1}) and so

$$(\mu, {}^c\mu) : \mathbb{G}_m^2 \longrightarrow G_{/\mathbb{C}}$$

descends to a homomorphism

$$h_\mu : \mathbb{S} \longrightarrow G_{/\mathbb{R}}.$$

Note that $\mathrm{ad}\, h_\mu(i) = \mathrm{ad}\, (\mu(i)({}^c\mu)(-i)) = \mathrm{ad}\, \mu(-1) \mathrm{ad}\, (\mu^c \mu)(-i) = \mathrm{ad}\, \mu(-1)$. We send (G, Y) to (G, Y^h), where $Y_h = \{h_\mu : \mu \in Y\}$.

It follows from lemma **??** that a Shimura datum (G, Y) is completely determined by the triple $(G, [Y]_G, \widehat{\boldsymbol{\lambda}}_G(Y))$, where $[Y]_G$ denotes the G-conjugacy class of cocharacters containing Y.

An element $\mu \in Y$ is called *special* if it factors through a sub-torus $T \subset G$ which is defined over $\mathbb{Q}$. We will call it *E-special* if we may choose T such that in addition T is split by E.

Lemma 8.1. *(1) If $\mu \in Y$ is special it factors through a maximal torus defined over $\mathbb{Q}$.*

(2) If $\mu \in Y$ is special and factors through a torus T defined over $\mathbb{Q}$, then $T^{\rm ad}(\mathbb{R})$ is compact, i.e. c acts on $X_*(T^{\rm ad})$ by -1.

(3) If $T \subset G$ is a maximal torus defined over $\mathbb{Q}$ and if $T^{\rm ad}(\mathbb{R})$ is compact then there is a $\mu \in Y$ which factors through T.

(4) If G contains a maximal torus T defined over $\mathbb{Q}$ and split by E with $T^{\rm ad}(\mathbb{R})$ compact, then the E-special points in Y are dense. In any case the special points in Y are dense.

(5) If $\mu \in Y$ is special and $E/\mathbb{Q}$ is Galois, then μ is E-special if and only if μ is defined over E.

(6) If $E/\mathbb{Q}$ is finite Galois and if $\mu \in Y$ is E-special factoring through a torus $T \subset G$ defined over $\mathbb{Q}$ and split by E, then there is a commutative diagram

$$\begin{array}{ccccc} R^1_{E,\mathbb{C}} & \longrightarrow & Z(G)\cap T & \subset & Z(G) \\ \downarrow & & \downarrow & & \downarrow \\ R_{E,\mathbb{C}} & \xrightarrow{\widetilde{\mu}} & T & \subset & G \\ \downarrow & & \downarrow & & \downarrow \\ S_{E,\mathbb{C}} & \longrightarrow & T^{\mathrm{ad}} & \subset & G^{\mathrm{ad}}. \end{array}$$

Moreover the the restriction $\widetilde{\mu}|_{R^1_{E,\mathbb{C}}}$ depends only on E, but not on μ or T. We will denote it $\widetilde{\mu}_{Y,E}$. If $D \supset E$ is another finite Galois extension of $\mathbb{Q}$ then $\widetilde{\mu}_{Y,D} = \widetilde{\mu}_{Y,E} \circ N_{D/E}$.

Proof: For the first part suppose that μ factors through a torus $T \subset G$ defined over $\mathbb{Q}$. Then one can replace T by a maximal torus of $Z_G(T)$ defined over $\mathbb{Q}$.

For the second part note that T^{ad} embeds over $\mathbb{R}$ into the inner form of G^{ad} determined by the cocycle $c \mapsto \mathrm{ad}\,\mu(-1)$, whose real points are compact.

For the third part choose any $\mu_1 \in Y$ and chose a maximal torus $T_1 \subset G$ defined over $\mathbb{R}$ through which μ_1 factors. Then $T_1^{\mathrm{ad}}(\mathbb{R})$ is compact (as in part 2)). Thus T and T_1 are fundamental tori in $G_{/\mathbb{R}}$ and hence $T = gT_1g^{-1}$ for some $g \in G(\mathbb{R})$. Then $\mu = g\mu_1 g^{-1}$ will do.

For the first assertion of the fourth part, because $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$, it suffices to see that there is some E-special point. This follows from the previous part. For the second assertion choose a maximal torus $T_1 \subset G$ defined over $\mathbb{R}$ with $T_1^{\mathrm{ad}}(\mathbb{R})$ compact. Then T_1 is $G(\mathbb{R})$-conjugate to some maximal torus $T \subset G$ defined over $\mathbb{Q}$, and we see that $T^{\mathrm{ad}}(\mathbb{R})$ is also compact. Choosing a finite Galois extension $E/\mathbb{Q}$ which splits T, and the second assertion follows from the first.

For the fifth part note that if μ factors through a torus $T \subset G$ defined over $\mathbb{Q}$ and split over E, then μ, like any cocharacter of T, is defined over E. Conversely, if μ is defined over E and factors through a torus $T_1 \subset G$ defined over $\mathbb{Q}$, then let T be the minimal subtorus of T_1 defined over $\mathbb{Q}$ through which μ factors. Because $X_*(T_1)^{\mathrm{Gal}\,(\overline{E}/E)}$ is $\mathrm{Gal}\,(\overline{E}/\mathbb{Q})$-invariant, we see that T splits over E.

For the sixth part note that one, and hence every, complex conjugation acts on $X_*(T^{\mathrm{ad}})$ by -1. If μ_1 and $\mu_2 \in Y$ are two E-special points, then the composites $\mu_i : \mathbb{G}_m \to G \to C(G)$ are equal and hence so are the composites $\widetilde{\mu}_i : R_{E,\mathbb{C}} \to G \to C(G)$. Because $Z(G) \to C(G)$ is an isogeny we see that $\widetilde{\mu}_1|_{R^1_{E,\mathbb{C}}} = \widetilde{\mu}_2|_{R^1_{E,\mathbb{C}}}$ □

8.2. **Conjugation of Shimura data: general theory.** Suppose that (G, Y) is an μ-Shimura datum. If $\tau \in \mathrm{Aut}\,(\mathbb{C})$ and $\phi \in \boldsymbol{\phi}_{G,Y,\tau}$, then $({}^{\phi}G, {}^{\tau,\phi}Y)$ is another Shimura datum. If $G = T$ is a torus then $({}^{\phi}T, {}^{\tau,\phi}\{\mu\}) = (T, \{{}^{\tau}\mu\})$.

We will say that a finite Galois extension $E/\mathbb{Q}$ is *acceptable* for G if

- G contains a maximal torus T defined over $\mathbb{Q}$ and split by E and with $T^{\rm ad}(\mathbb{R})$ compact;
- E is totally imaginary;
- $B(\mathbb{Q},G)_{\{\infty\},{\rm basic}}$ is contained in the image of $H^1_{\rm alg}(\mathcal{E}_3(E/\mathbb{Q}),G(E))_{\rm basic}$.

The existence of some such E follows from the last paragraph of section 2.3 and lemma 5.8.

If (G,Y) is a Shimura datum, E is a number field acceptable for G and $\mathfrak{a}\in\mathcal{H}(E/\mathbb{Q})$, then we will write ${\rm Conj}_{E,\mathfrak{a},}(G,Y)$ for the set of triples (τ,ϕ,b), where

- $\tau\in{\rm Aut}(\mathbb{C})$;
- $\phi\in\boldsymbol{\phi}_{G,Y,\tau}$;
- $b\in G(\mathbb{A}_E^\infty)$ satisfies ${\rm res}^\infty{\rm loc}_{\mathfrak{a}}\phi={}^b1$.

We will sometimes write ${}^{(\tau,\phi,b)}(G,Y)=({}^\phi G,{}^{\tau,\phi}Y)$. We have

$$\mathrm{conj}_b:G\times\mathbb{A}^\infty\stackrel{\sim}{\longrightarrow}{}^\phi G\times\mathbb{A}^\infty.$$

Note that if $t\in T_{2,E}(\mathbb{A}_E)$ then there is a natural bijection

$$\begin{array}{rccl} c_t:{\rm Conj}_{E,\mathfrak{a}}(G,Y) & \stackrel{\sim}{\longrightarrow} & {\rm Conj}_{E,{}^t\mathfrak{a}}(G,Y) \\ (\tau,\phi,b) & \longmapsto & (\tau,z_t(\phi),({\rm loc}_{\mathfrak{a}}\phi)(t)b). \end{array}$$

More generally if $D\supset E$ is another finite Galois extension of $\mathbb{Q}$, if $\mathfrak{a}_E\in\mathcal{H}(E/\mathbb{Q})$ and $\mathfrak{a}_D\in\mathcal{H}(D/\mathbb{Q})$ and if $t\in T_{2,E}(\mathbb{A}_D)$ with ${}^t\inf_{{\rm Gal}(E/\mathbb{Q})}^{{\rm Gal}(D/\mathbb{Q})}\mathfrak{a}_E=\eta_{D/E,*}\mathfrak{a}_D$, and if $(\tau,\phi,b)\in{\rm Conj}_{E,\mathfrak{a}_E}(G,Y)$, then

$$\inf_{D/E,t}(\tau,\phi,b)=(\tau,\inf_{3,D/E,t}\phi,({\rm loc}_{\mathfrak{a}_E}\phi)(t)b)\in{\rm Conj}_{D,\mathfrak{a}_D}(G,Y).$$

If $(\tau_1,\phi_1,b_1)\in{\rm Conj}_{E,\mathfrak{a}}(G,Y)$ and $(\tau_2,\phi_2,b_2)\in{\rm Conj}_{E,\mathfrak{a}^{\tau_1,\phi_1,b_1}}(G,Y)$, then $(\tau_2\tau_1,\phi_2\phi_1,b_2b_1)\in{\rm Conj}_{E,\mathfrak{a}}(G,Y)$, and we have

$${}^{\tau_2,\phi_2,b_2}({}^{\tau_1,\phi_1,b_1}(G,Y))={}^{\tau_2\tau_1,\phi_2\phi_1,b_2b_1}(G,Y).$$

If $(\tau,\phi,b)\in{\rm Conj}_{E,\mathfrak{a}}(G_1,Y_1)$ and $f:(G_1,Y_1)\to(G_2,Y_2)$ then $(\tau,f\circ\phi,f(b))\in{\rm Conj}_{E,\mathfrak{a}}(G_2,Y_2)$ and f induces a map

$${}^{\tau,\phi,b}f:{}^{\tau,\phi,b}(G_1,Y_1)\longrightarrow{}^{\tau,f\circ\phi,f(b)}(G_2,Y_2).$$

Moreover

$${\rm conj}_{f(b)}\circ f={}^{\tau,\phi,b}f\circ{\rm conj}_b.$$

If we fix $\tau\in{\rm Aut}(\mathbb{C})$ we will write ${\rm Conj}_{E,\mathfrak{a}}(G,Y)_\tau$ for the subset of ${\rm Conj}_{E,\mathfrak{a}}(G,Y)$ consisting of those triples with first element τ. The group $G(E)\times G(\mathbb{A}^\infty)$ acts transitively on ${\rm Conj}_{E,\mathfrak{a}}(G,Y)_\tau$ via

$$(\gamma,h)(\tau,\phi,b)=(\tau,{}^\gamma\phi,\gamma bh^{-1}).$$

The stabilizer of (τ,ϕ,b) is identified with ${}^\phi G(\mathbb{Q})$ via $\delta\mapsto(\delta,b^{-1}\delta b)$. We have

$$\inf_{D/E,t}((\gamma,h)(\tau,\phi,b))=(\gamma,h)\inf_{D/E,t}(\tau,\phi,b).$$

8.3. **Conjugation of Shimura data: Langlands' theory.** Now suppose that $\mathfrak{a}^+ \in \mathcal{H}(E/\mathbb{Q})^+$ lies above $\mathfrak{a}$ and that $\alpha \in \widetilde{S}_{E,\mathbb{C},\tau}(E)$. Suppose also that $\mu \in Y$ is E-special, and choose a torus $T \subset G$ defined over $\mathbb{Q}$ and split by E, through which μ factors. Then $\widetilde{\mu} : R_{E,\mathbb{C}} \to T$ over $\mathbb{Q}$.

We define

$$\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu} = \widetilde{\mu}(c_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}) \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))$$

and

$$b_{E,\mathfrak{a}^+,\tau,\alpha,\mu} = \widetilde{\mu}(b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}) \in T(\mathbb{A}_E^\infty) \subset G(\mathbb{A}_E^\infty).$$

There was a somewhat arbitrary choice of lifting made in the definition of $b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$. Varying this choice will only vary $b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ by an element of $Z(G)(E)\overline{Z(G)(\mathbb{Q})}$, which is independent of μ, and $\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ by the corresponding element of $Z(G)(E)$. We have the following observations:

(1) $\mathrm{res}^\infty \mathrm{loc}_{\mathfrak{a}} \phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu} = {}^{b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}}1$.
(2) $\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}|_{T_{3,E}(E)} = \prod_{\rho:E\hookrightarrow\mathbb{C}}({}^{\rho^{-1}}(\mu/{}^\tau\mu)) \circ \pi_{v(\rho)}$, which by lemma 7.1 is valued in $Z(G)(E)$.
(3) $[\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}] = \boldsymbol{\phi}_{G,Y,\tau} \in H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}}$.
(4) $(\tau, \phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}, b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}) \in \mathrm{Conj}\,_{E,\mathfrak{a}}(G, Y)$.
(5) Given $\tau_1, \tau_2 \in \mathrm{Aut}\,(\mathbb{C})$ and $\alpha_i \in \widetilde{S}_{E,\mathbb{C},\tau_i}(E)$, there exists $\beta \in Z(G)(E)$, independent of $\mu \in Y$ E-special, such that

$$\beta b_{E,\mathfrak{a}^+,\tau_1\tau_2,\alpha_1\alpha_2,\mu} \equiv b_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu} b_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu} \bmod \overline{Z(G)(\mathbb{Q})}$$

and

$${}^\beta\phi_{E,\mathfrak{a}^+,\tau_1\tau_2,\alpha_1\alpha_2,\mu} = \phi_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu}\phi_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu}.$$

(6) If $\gamma \in S_{E,\mathbb{C}}(E)$ then γ has a lift $\widetilde{\gamma} \in R_{E,\mathbb{C}}(E)$ such that

$$\phi_{E,\mathfrak{a}^+,\tau,\alpha\gamma^{-1},\mu} = {}^{\widetilde{\mu}(\widetilde{\gamma})}\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$$

and

$$b_{E,\mathfrak{a}^+,\tau,\alpha\gamma^{-1},\mu} = \widetilde{\mu}(\widetilde{\gamma}) b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}.$$

(7) We may take

$$\phi_{E,{}^t\mathfrak{a}^+,\tau,\alpha,\mu} = z_t(\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu})$$

and

$$b_{E,{}^t\mathfrak{a}^+,\tau,\alpha,\mu} = b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}(\mathrm{res}^\infty \mathrm{loc}_{\mathfrak{a}} \phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu})(t).$$

We will write

$$\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu} = \mathrm{ad}\,\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu} \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), G^{\mathrm{ad}}\,(E)),$$

and

$$b^{\mathrm{ad}}_{E,\tau,\alpha,\mu} = \mathrm{ad}\, b_{E,\mathfrak{a}^+,\tau,\alpha,\mu} \in G^{\mathrm{ad}}\,(\mathbb{A}_E^\infty).$$

As the notation suggests these do not depend on the choice of $\mathfrak{a}^+$ (nor on the somewhat arbitrary choice of lifting made in the definition of $b_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$). If $\gamma \in S_{E,\mathbb{C}}(E)$ then γ has a lift $\widetilde{\gamma} \in R_{E,\mathbb{C}}(E)$ such that

$$\phi^{\mathrm{ad}}_{E,\tau,\alpha\gamma^{-1},\mu} = {}^{\mathrm{ad}\,\widetilde{\mu}(\widetilde{\gamma})}\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu}$$

and

$$b^{\mathrm{ad}}_{E,\tau,\alpha\gamma^{-1},\mu} = \mathrm{ad}\,\widetilde{\mu}(\widetilde{\gamma}) b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}.$$

(Note that $\mathrm{ad}\,\widetilde{\mu}(\widetilde{\gamma})$ depends only on γ, i.e. is independent of the particular lift $\widetilde{\gamma}$.) The cocycle $\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu}$ equals the cocycle $\sigma \mapsto c_\sigma(\tau,\mu_{\mathrm{ad}})^{-1}$ of section 6 of [L]. Moreover the element $b^{\mathrm{ad}}_{E,\tau,\alpha,\mu} \in G^{\mathrm{ad}}(\mathbb{A}_E^\infty)$ equals the element denoted $\mathrm{ad}\,\widetilde{b}(\tau,\mu)^{-1}$ in section 6 of [L]. Note that Langlands does not mention the chosen lift α in his notation. This is presumably because by point (6) above there is a canonical relationship between these quantities for different choices of α. Nonetheless we find it less confusing to keep track of the α.

Following Langlands we will set

$${}^{\tau,\mu,\alpha}(G,Y) = ({}^{\tau,\mu,\alpha}G, {}^{\tau,\mu,\alpha}Y) = {}^{(\tau,\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}, b_{E,\mathfrak{a}^+,\tau,\alpha,\mu})}(G,Y),$$

so that

$$\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}} : G \times \mathbb{A}^\infty \xrightarrow{\sim} {}^{\tau,\mu,(\rho,\alpha)}G \times \mathbb{A}^\infty$$

and ${}^\tau\mu \in {}^{\tau,\mu,\alpha}Y$. Note that

$${}^{\tau,\mu,\alpha}(G,Y) = ({}^{\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}G, {}^{\tau,\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}Y),$$

and so ${}^{\tau,\mu,\alpha}(G,Y)$ does not depend on the choice of $\mathfrak{a}^+$. This notation is consistent with Langlands notation in [L], except again Langlands suppresses the choice of α in his notation. If $\gamma \in S_{E,\mathbb{C}}(E)$, then there is a canonical identification

$$\mathrm{conj}_{\widetilde{\mu}(\gamma)} : {}^{\tau,\mu,\alpha}(G,Y) \xrightarrow{\sim} {}^{\tau,\mu,\alpha\gamma^{-1}}(G,Y)$$

and

$$b^{\mathrm{ad}}_{E,\tau,\alpha\gamma^{-1},\mu} = (\mathrm{ad}\,\widetilde{\mu})(\gamma) b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}.$$

This may be seen as explaining Langlands choice to suppress the α in his notation, but again we feel it is clearer to make it explicit.

As

$$\phi^{\mathrm{ad}}_{E,\tau_1\tau_2,\alpha_1\alpha_2,\mu} = \phi^{\mathrm{ad}}_{E,\tau_1,\alpha_1,{}^{\tau_2}\mu}\phi^{\mathrm{ad}}_{E,\tau_2,\alpha_2,\mu},$$

we see that

$${}^{\tau_1\tau_2,\mu,\alpha_1\alpha_2}(G,Y) = {}^{\tau_1,{}^{\tau_2}\mu,\alpha_1}({}^{\tau_2,\mu,\alpha_2}(G,Y)).$$

Similarly

$$b^{\mathrm{ad}}_{E,\tau_1\tau_2,\alpha_1\alpha_2,\mu} = b^{\mathrm{ad}}_{E,\tau_1,\alpha_1,{}^{\tau_2}\mu} b^{\mathrm{ad}}_{E,\tau_2,\alpha_2,\mu}.$$

If $f : (G_1,Y_1) \to (G_2,Y_2)$ is a morphism of Shimura data and $\mu_1 \in Y_1$ is special, then we get a morphism

$${}^{\tau,\mu_1,\alpha}f : {}^{\tau,\mu_1,\alpha}(G_1,Y_1) \longrightarrow {}^{\tau,f(\mu_1),\alpha}(G_2,Y_2).$$

Moreover

$$\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,f(\mu_1)}} \circ f = {}^{\tau,\mu_1,\alpha} f \circ \mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_1}}.$$

If $\mu_1, \mu_2 \in Y$ are both E-special then we set

$$\phi_{E,\tau,\alpha,\mu_1,\mu_2} = \phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu_2} \phi^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu_1} \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\tau,\mu_1,\alpha}G^{\mathrm{ad}}\,(E))$$

and

$$b_{E,\tau,\alpha,\mu_1,\mu_2} = b_{E,\mathfrak{a}^+,\tau,\alpha,\mu_2} b^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu_1} \in G(\mathbb{A}^\infty_E).$$

As the notation suggests, these do not depend on the somewhat arbitrary choices made in the definition of $b_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}}_{\mathbb{C}},\alpha}$ nor on the choice of $\mathfrak{a}^+$.

(1)

$$\mathrm{ad}\, b_{E,\tau,\alpha,\mu_1,\mu_2} = b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2} (b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_1})^{-1} \in G^{\mathrm{ad}}\,(\mathbb{A}^\infty_E).$$

If $\widetilde{\gamma} \in R_{E,\mathbb{C}}(E)$ maps to $\gamma \in S_{E,\mathbb{C}}(E)$ then $b_{E,\tau,\gamma\alpha,\mu_1,\mu_2} = \widetilde{\mu}_2(\widetilde{\gamma})^{-1} b_{E,\tau,\alpha,\mu_1,\mu_2} \widetilde{\mu}_1(\widetilde{\gamma})$.

(2)

$$\phi_{E,\tau,\alpha,\mu_1,\mu_2} \longmapsto \phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2} (\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu_1})^{-1} \in Z^1(\mathrm{Gal}\,(K/\mathbb{Q}), {}^{\tau,\mu_1,\alpha}G^{\mathrm{ad}}\,(E)).$$

We also have

$$\phi_{E,\tau,\alpha,\mu_1,\mu_2}(\sigma) = b_{E,\tau,\alpha,\mu_1,\mu_2} \mathrm{conj}_{\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu_1}(\sigma)} ({}^\sigma b_{E,\tau,\alpha,\mu_1,\mu_2})^{-1}.$$

If $\widetilde{\gamma} \in R_{E,\mathbb{C}}(E)$ maps to $\gamma \in S_{E,\mathbb{C}}(E)$ then

$$\phi_{E,\tau,\gamma\alpha,\mu_1,\mu_2}(\sigma) = \widetilde{\mu}_2(\widetilde{\gamma})^{-1} \mathrm{conj}_{\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2}(\sigma)} ({}^\sigma \widetilde{\mu}_2(\widetilde{\gamma})^{-1}) \phi_{E,\tau,\alpha,\mu_1,\mu_2}(\sigma) \mathrm{conj}_{\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu_1}(\sigma)} ({}^\sigma \widetilde{\mu}_1(\widetilde{\gamma})) \widetilde{\mu}_1(\widetilde{\gamma})^{-1}.$$

(3) $[\phi_{E,\tau,\alpha,\mu_1,\mu_2}] \in H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\tau,\mu_1,\alpha}G)$ is trivial, so that

$$\phi_{E,\tau,\alpha,\mu_1,\mu_2}(\sigma) = \gamma_{E,\tau,\alpha,\mu_1,\mu_2} \mathrm{conj}_{\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu_1}(\sigma)} ({}^\sigma \gamma^{-1}_{E,\tau,\alpha,\mu_1,\mu_2})$$

for some $\gamma_{E,\tau,\alpha,\mu_1,\mu_2} \in G(E)$ well defined up to right multiplication by an element of ${}^{\tau,\mu_1,\alpha}G(\mathbb{Q})$. We see that

$$\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu_2} = {}^{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}} \phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu_2}$$

and

$$\mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}} : {}^{\tau,\mu_1,\alpha}G \xrightarrow{\sim} {}^{\tau,\mu_2,\alpha}G,$$

and

$$b_{E,\tau,\alpha,\mu_1,\mu_2} \gamma^{-1}_{E,\tau,\alpha,\mu_1,\mu_2} \in {}^{\tau,\mu_2,\alpha}G(\mathbb{A}^\infty).$$

Moreover

$$\mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}}({}^{\tau,\mu_1,\alpha}Y) = {}^{\tau,\mu_2,\alpha}Y.$$

The cocycle $\phi_{E,\tau,\alpha,\mu_1,\mu_2} \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\tau,\mu_1,\alpha}G)$ equals the cocycle denoted $\sigma \mapsto \gamma_\sigma$ in 'the first lemma of comparison' in section 6 of [L]. Moreover $b_{E,\tau,\alpha,\mu_1,\mu_2} \in G(\mathbb{A}^\infty_E)$ is the element denoted $B(\tau) = B(\tau, \mu_1, \mu_2)$ in section 6 of [L]. Finally the element $\gamma_{E,\tau,\alpha,\mu_1,\mu_2} \in G(E)$ is denoted u in the 'second lemma of comparison' in section 6 of [L].

8.4. **Deligne's Shimura varieties.** Given a Shimura datum (G, X) and a neat compact open subgroup $U \subset G(\mathbb{A}^\infty)$ the complex analytic manifold

$$\mathrm{Sh}(G, X)_U(\mathbb{C}) = G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)/U \times X)$$

arises from a unique quasi-projective variety $\mathrm{Sh}(G, X)_U$ over $\mathbb{C}$. Moreover to each morphism $f : (G_1, X_1) \to (G_2, X_2)$ of Shimura data, each neat open compact subgroup $U_i \subset G_i(\mathbb{A}^\infty)$ and each $g \in G_2(\mathbb{A}^\infty)$ such that $gf(U_1)g^{-1} \subset U_2$ the map

$$\begin{array}{rcl} G_1(\mathbb{Q})\backslash(G_1(\mathbb{A}^\infty)/U_1 \times X_1) & \longrightarrow & G_2(\mathbb{Q})\backslash(G_2(\mathbb{A}^\infty)/U_2 \times X_2) \\ G_1(\mathbb{Q})(hU_1, x) & \longmapsto & G_2(\mathbb{Q})(f(h)g^{-1}U_2, f(x), \end{array}$$

is holomorphic and arises from an algebraic map

$$\mathrm{Sh}(g, f) : \mathrm{Sh}(G_1, X_1)_{U_1} \longrightarrow \mathrm{Sh}(G_2, X_2)_{U_2}.$$

If $G = T$ is a torus then we have an isomorphism

$$\begin{array}{rcl} \Pi_{T,\{\mu\}} : T(\mathbb{Q})\backslash T(\mathbb{A}^\infty)/U & \xrightarrow{\sim} & \mathrm{Sh}(T, \{\mu\})_U(\mathbb{C}) \\ T(\mathbb{Q})tU & \longmapsto & [(t, \mu)] \end{array}$$

Note that

(1) If $f_1 : (G_1, X_1) \to (G_2, X_2)$ and $f_2 : (G_2, X_2) \to (G_3, X_3)$ and if $U_i \subset G_i(\mathbb{A}^\infty)$ is a neat open compact subgroup and if $g_i \in G_i(\mathbb{A}^\infty)$ (for $i = 2, 3$) satisfy $g_2 f_1(U_1) g_2^{-1} \subset U_2$ and $g_3 f_2(U_2) g_3^{-1} \subset U_3$, then

$$\mathrm{Sh}(g_3, f_2) \circ \mathrm{Sh}(g_2, f_1) = \mathrm{Sh}(g_3 f_2(g_2), f_2 \circ f_1).$$

In particular as U varies over neat open compact subgroups of $G(\mathbb{A}^\infty)$ the filtered inverse system $\{\mathrm{Sh}(G, X)_U\}$ (with transition maps $\mathrm{Sh}(1, 1)$) has a right action of $G(\mathbb{A}^\infty)$, where g acts by $\mathrm{Sh}(g^{-1}, 1)$.

(2) If $\gamma \in G(\mathbb{Q})$ and $u \in U$ then $\mathrm{Sh}(u\gamma^{-1}, \mathrm{conj}_\gamma)$ is the identity on $\mathrm{Sh}(G, X)_U$. In particular if $z \in \overline{Z(G)(\mathbb{Q})}$ then $\mathrm{Sh}(z, 1) = 1$.

(3) If $x \in \lim_{\leftarrow V} \mathrm{Sh}(G, X)_V(\mathbb{C})$ then the image of $G(\mathbb{A}^\infty)x$ is dense in $\mathrm{Sh}(G, X)_U(\mathbb{C})$, for any U.

This implies the following: If $T \subset G$ is a maximal torus defined over $\mathbb{Q}$ with $T^{\mathrm{ad}}(\mathbb{R})$ is compact, if $i : t \hookrightarrow G$ denotes this embedding, and if $\mu \in X^\mu$ factors through T (such a μ always exists); then

$$\bigcup_{g \in G(\mathbb{A}^\infty)} \mathrm{Sh}(g, i)(\mathrm{Sh}(T, \{\mu\})_{g^{-1}Ug \cap T(\mathbb{A}^\infty)}(\mathbb{C}))$$

is dense in $\mathrm{Sh}(G, \mathbb{C})_U(\mathbb{C})$.

(4) The group of automorphisms of the variety $\mathrm{Sh}(G, X)_U$ is finite.

(For most of this see sections 1.8 and 1.14 of [D1]. For the uniqueness of the quasi-projective algebraic structure on $\mathrm{Sh}(G, X)_U$ see [B]. Item (3) above follows from the density of $G(\mathbb{Q})x$ in X for any $x \in X$, or even from the density of $G(\mathbb{Q})$ in $G(\mathbb{R})$. Item (4) follows from lemma 2.6.3 of [Ma]. (See also lemma 2.2 of [Mi2].))

As best we understand the main theorem of [Mi1] (proving a conjecture of Langlands from [L]), it asserts the following:

Theorem 8.2 (Milne)**.** *Suppose that (G, Y) is an NCF μ-Shimura datum, that $E/\mathbb{Q}$ is a finite Galois extension, and that $\mu \in Y$ is an E-special point. Suppose also that $\tau \in \mathrm{Aut}\,(\mathbb{C})$ and choose $\alpha \in \widetilde{S}_{E,\mathbb{C},\tau}$. Then there is a unique morphism*

$$\Phi(\tau, \mu, \alpha) : \tau\mathrm{Sh}(G, Y)_U \xrightarrow{\sim} \mathrm{Sh}({}^{\tau,\mu,\alpha}G, {}^{\tau,\mu,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(U)}$$

such that

$$\Phi(\tau, \mu, \alpha)(1, \mu) = (1, {}^{\tau}\mu)$$

and

$$\Phi(\tau, \mu, \alpha) \circ {}^{\tau}\mathrm{Sh}(g, 1) = \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g), 1) \circ \Phi(\tau, \mu, \alpha)$$

for all $g \in G(\mathbb{A}^\infty)$.

If μ_1 and $\mu_2 \in Y$ are two E-special points, then

$$\Phi(\tau, \mu_2, \alpha) = \mathrm{Sh}(b_{E,\tau,\alpha,\mu_1,\mu_2}\gamma^{-1}_{E,\tau,\alpha,\mu_1,\mu_2}, \mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}}) \circ \Phi(\tau, \mu_1, \alpha).$$

(Note that the right hand side is unchanged if $\gamma_{E,\tau,\alpha,\mu_1,\mu_2}$ is replaced by $\gamma_{E,\tau,\alpha,\mu_1,\mu_2}\beta$ with $\beta \in {}^{\tau,\mu_2,\alpha}G(\mathbb{Q})$, and so the ambiguity in $\gamma_{E,\tau,\alpha,\mu_1,\mu_2}$ is unimportant.)

From these assertions the following additional formulae are easily deduced:

1. If $\gamma \in S_{E,\mathbb{C}}(E)$ then $\Phi(\tau, \mu, \alpha\gamma)) = \mathrm{Sh}(1, \mathrm{conj}_{\widetilde{\mu}(\gamma)^{-1}})\Phi(\tau, \mu, \alpha)$.
2. If $f : (G_1, Y_1) \to (G_2, Y_2)$ and $g \in G_2(\mathbb{A}^\infty)$ and $\mu_1 \in Y_1$ is an E-special point, then $\Phi(\tau, f \circ \mu_1, \alpha) \circ {}^{\tau}\mathrm{Sh}(g, f) = \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,f\circ\mu}}(g), {}^{\tau,\mu_1,\alpha}f) \circ \Phi(\tau, \mu_1, \alpha)$.
3. $\Phi(\tau_1\tau_2, \mu, \alpha_1\alpha_2) = \Phi(\tau_1, {}^{\tau_2}\mu, \alpha_1) \circ {}^{\tau_1}\Phi(\tau_2, \mu, \alpha_2)$.
4. If $G = T$ is a torus then $\Phi(\tau, \mu, \alpha) \circ \tau \circ \Pi_{T,\{\mu\}} = \Pi_{T,\{{}^{\tau}\mu\}}$.

8.5. **Removing the NCF-condition.** We start with the following lemma.

Lemma 8.3. *Suppose that (G, Y) is a Shimura datum. Suppose also that $H \subset G$ is a normal connected reductive subgroup such that $(G/H)(\mathbb{R})$ is compact and the image of one, and hence every, $\mu \in Y$ in $(G/H)(\mathbb{R})$ is trivial. We will write i for the inclusion $H \hookrightarrow G$. Also suppose that U is a neat open compact subgroup of $G(\mathbb{A}^\infty)$.*

(1) Then Y is a single $H(\mathbb{R})$-conjugacy class so that (H, Y) is also a Shimura datum.

(2) $G(\mathbb{Q})H(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U$ has finite cardinality.

(3) $(G/H)(\mathbb{Q}) \cap \mathrm{Im}\,(U \to (G/H)(\mathbb{A}^\infty)) = \{1\}$.

(4) $G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)/U \times Y) = \coprod_{h\in G(\mathbb{Q})H(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} H(\mathbb{Q})\backslash(H(\mathbb{A}^\infty)/(hUh^{-1} \cap H(\mathbb{A}^\infty)) \times Y)h$.

(5) $\mathrm{Sh}(G, Y)_U = \coprod_{h\in G(\mathbb{Q})H(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} \mathrm{Sh}(H, Y)_{hUh^{-1}\cap H(\mathbb{A}^\infty)}$, where we map

$$\mathrm{Sh}(H, Y)_{hUh^{-1}\cap H(\mathbb{A}^\infty)} \hookrightarrow \mathrm{Sh}(G, Y)_U$$

via $\mathrm{Sh}(h^{-1}, i)$.

Proof: The exact sequence

$$(0) \longrightarrow H^{\rm ad} \longrightarrow G^{\rm ad} \longrightarrow (G/H)^{\rm ad} \longrightarrow (0)$$

has a unique splitting in which $(G/H)^{\rm ad}$ lifts to a normal subgroup of $G^{\rm ad}$. Write H' for the pre-image in G of $(G/H)^{\rm ad} \subset H^{\rm ad} \times (G/H)^{\rm ad} = G^{\rm ad}$, so that $(H')^{\rm ad} \stackrel{\sim}{\to} G/H$. Note that $H'(\mathbb{R}) \twoheadrightarrow (G/H)(\mathbb{R})$ (as $(G/H)(\mathbb{R})$ is compact) and acts trivially on Y. If $\mu, \mu' \in Y$ then $\mu' = {\rm conj}_g \circ \mu$ for some $g \in G(\mathbb{R})$. Let $h \in H'(\mathbb{R})$ have the same image as g in $(G/H)(\mathbb{R})$. Thus $gh^{-1} \in H(\mathbb{R})$ and ${\rm conj}_{gh^{-1}} \circ \mu = \mu'$. The first part of the lemma follows.

The set

$$G(\mathbb{Q})H(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U = G(\mathbb{Q})H(\mathbb{A}^\infty)\backslash G(\mathbb{A})/UG(\mathbb{R})$$

is finite by theorem 5.1 of [PR].

For the third part we see that $(G/H)(\mathbb{Q}) \cap {\rm Im}\,(U \to (G/H)(\mathbb{A}^\infty))$ is finite (because $(G/H)(\mathbb{R})$ is compact) and hence $\{1\}$ because U is neat.

For the fourth part, first note that

$$G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)/U \times Y) = \coprod_{h \in G(\mathbb{Q})H(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} G(\mathbb{Q})\backslash(G(\mathbb{Q})H(\mathbb{A}^\infty)hU/U \times Y).$$

Next suppose that for $g_1, g_2 \in H(\mathbb{A}^\infty)$ and $\mu_1, \mu_2 \in Y$ we have

$$\gamma(g_1hu, \mu_1) = (g_2h, \mu_2),$$

for some $\gamma \in G(\mathbb{Q})$ and $u \in U$. Then we see that the image of γ in $(G/H)(\mathbb{Q})$ lies in $(G/H)(\mathbb{Q}) \cap {\rm Im}\,(hUh^{-1} \to (G/H)(\mathbb{A}^\infty)) = \{1\}$. Thus $\gamma \in H(\mathbb{Q})$ and $huh^{-1} \in H(\mathbb{A}^\infty)$. We conclude that

$$H(\mathbb{Q})\backslash(H(\mathbb{A}^\infty)/(hUh^{-1} \cap H(\mathbb{A}^\infty)) \times Y) \stackrel{h}{\longrightarrow} G(\mathbb{Q})\backslash(G(\mathbb{Q})H(\mathbb{A}^\infty)hU/U \times Y)$$

is an isomorphism, and the third part of the lemma follows. The fifth part follows from the fourth and the uniqueness assertion in section 8.4. $\Box$

Suppose that (G, Y) is a μ-Shimura datum. We have $G^{\rm ad} = G^{{\rm ad},nc} \times G^{{\rm ad},c}$, where $G^{{\rm ad},c}(\mathbb{R})$ is compact, but if H is any simple factor of $G^{{\rm ad},nc}/\mathbb{Q}$, then $H(\mathbb{R})$ is not compact. We will write G^{nc} (resp. G^c) for the connected component of the identity of $\ker(G \twoheadrightarrow G^{{\rm ad},c})$ (resp. $\ker(G \twoheadrightarrow G^{{\rm ad},nc})$) and $\overline{G}^{nc}$ (resp. $\overline{G}^{c}$) for G/G^c (resp. G/G^{nc}). Thus

$$G^c \twoheadrightarrow \overline{G}^c \twoheadrightarrow G^{{\rm ad},c} \stackrel{\sim}{\longrightarrow} G^{c,{\rm ad}}$$

and

$$G^{nc} \twoheadrightarrow \overline{G}^{nc} \twoheadrightarrow G^{{\rm ad},nc} \stackrel{\sim}{\longrightarrow} G^{nc,{\rm ad}},$$

where the central maps have finite central kernels. We also have $Z(G^c) = Z(G) \cap G^c$ and $Z(G^{nc}) = Z(G) \cap G^{nc}$. Moreover G^c and G^{nc} centralize each other. (Indeed if we let G^c act on G^{nc} by conjugation, we see that, given $h \in G^{nc}$, there is a character $\chi_h : G^c \to Z(G) \cap G^{nc}$ such that ${\rm conj}_g(h) = \chi_h(g)h$. The character χ_h must factor through $C(G^c)$, but is trivial on $Z(G)\cap G^c \twoheadrightarrow C(G^c)$. Thus $\chi_h = 1$ and G^c centralizes h as desired.) We have an exact sequence

$$(0) \longrightarrow Z(G^c)\cap Z(G^{nc}) \longrightarrow G^{nc}\times G^c \longrightarrow G \longrightarrow (0).$$

Note $(G/G^{nc})(\mathbb{R})$ is compact and hence connected. Thus $G^c(\mathbb{R}) \twoheadrightarrow (G/G^{nc})(\mathbb{R})$.

If $\mu \in Y$ then the composition of μ with $G \to G^{\mathrm{ad},c}$ takes -1 to 1 and hence factors through the squaring map $\mathbb{G}_m \to \mathbb{G}_m$. As this composition is miniscule we see that it must actually be trivial, i.e. $\mu \in X_*(G^{nc})$ and G^c centralizes μ. By lemma 8.3 (G^{nc}, Y) is a NCF-Shimura datum. Write i for the map $G^{nc} \hookrightarrow G$, so that $i : (G^{nc}, Y) \to (G, Y)$. Further by lemma 8.3 we have

$$\mathrm{Sh}(G,Y)_U = \coprod_{h\in G(\mathbb{Q})G^{nc}(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} \mathrm{Sh}(G^{nc},Y)_{hUh^{-1}\cap G^{nc}(\mathbb{A}^\infty)},$$

where $G(\mathbb{Q})G^{nc}(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U$ is finite, and where

$$\mathrm{Sh}(h^{-1},i) : \mathrm{Sh}(G^{nc},Y)_{hUh^{-1}\cap G^{nc}(\mathbb{A}^\infty)} \hookrightarrow \mathrm{Sh}(G,Y)_U.$$

If $h' = g\gamma hu$ with $g \in G^{nc}(\mathbb{A}^\infty)$, $\gamma \in G(\mathbb{Q})$ and $u \in U$, then

$$\begin{array}{ccc} \mathrm{Sh}(G^{nc},Y)_{hUh^{-1}\cap G^{nc}(\mathbb{A}^\infty)} & & \\ & \searrow \mathrm{Sh}(h^{-1},i) & \\ \mathrm{Sh}(g,\mathrm{conj}_\gamma) \downarrow \wr & & \mathrm{Sh}(G,Y)_U \\ & \nearrow \mathrm{Sh}((h')^{-1},i) & \\ \mathrm{Sh}(G^{nc},Y)_{h'U(h')^{-1}\cap G^{nc}(\mathbb{A}^\infty)} & & \end{array}$$

commutes.

Now suppose that $f : (G_1, Y_1) \to (G_2, Y_2)$ is a morphism of Shimura data, that $U_i \subset G_i(\mathbb{A}^\infty)$ are neat open compact subgroups and that $g \in G_2(\mathbb{A}^\infty)$ such that $gf(U_1)g^{-1} \subset U_2$. Note that $f : G_1^{nc} \to G_2^{nc}$. If $h \in G(\mathbb{A}^\infty)$ then

$$\begin{array}{ccc} \mathrm{Sh}(G_1^{nc},Y_1)_{hU_1h^{-1}\cap G_1^{nc}(\mathbb{A}^\infty)} & \stackrel{\mathrm{Sh}(h^{-1},i_1)}{\hookrightarrow} & \mathrm{Sh}(G_1,Y_1)_{U_1} \\ \mathrm{Sh}(1,f)\downarrow & & \downarrow \mathrm{Sh}(g,f) \\ \mathrm{Sh}(G_2^{nc},Y_2)_{f(h)g^{-1}U_2(f(h)g^{-1})^{-1}\cap G_2^{nc}(\mathbb{A}^\infty)} & \stackrel{\mathrm{Sh}(gf(h^{-1}),i_2)}{\hookrightarrow} & \mathrm{Sh}(G_2,Y_2)_{U_2} \end{array}$$

commutes.

Our next aim is to extend theorem 8.2 to this setting. So suppose that (G, Y) is a Shimura datum and that $\mu \in Y$ is an E-special point. Suppose also that $\tau \in \mathrm{Aut}\,(\mathbb{C})$ and $\alpha \in \widetilde{S}_{E,\mathbb{C},\tau}$.

Note that $\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ and $b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ as defined for G equal those defined for G^{nc}. Thus we will denote them with the same symbol. Hence $\overline{G}^c = \overline{{}^{\tau,\mu,\alpha}G}^c$ and ${}^{\tau,\mu,\alpha}(G^{nc}) = ({}^{\tau,\mu,\alpha}G)^{nc}$. We claim that the images of $G(\mathbb{Q})$ and ${}^{\tau,\mu,\alpha}G(\mathbb{Q})$ in $\overline{G}^c(\mathbb{Q})$ are equal, from which it follows that

$$\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(G(\mathbb{Q})G^{nc}(\mathbb{A}^\infty)) = {}^{\tau,\mu,\alpha}G(\mathbb{Q})^{\tau,\mu,\alpha}G^{nc}(\mathbb{A}^\infty),$$

and hence that $\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}$ gives a bijection

$$G(\mathbb{Q})G^{nc}(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U \xrightarrow{\sim} {}^{\tau,\mu,\alpha}G(\mathbb{Q})^{\tau,\mu,\alpha}G^{nc}(\mathbb{A}^\infty)\backslash {}^{\tau,\mu,\alpha}G(\mathbb{A}^\infty)/\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(U).$$

To prove the claim suppose that $\gamma \in G(\mathbb{Q})$. Then we have $\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\circ\mu} \in G^{nc}(E)$ satisfying

$$\begin{array}{rl} & \phi_{E,\mathfrak{a}^+,\tau,\alpha,\mathrm{conj}_{\gamma^{-1}}\circ\mu}(\sigma)\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}(\sigma)^{-1} \\ = & \gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\circ\mu}\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}(\sigma)^\sigma\gamma^{-1}_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\circ\mu}\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}(\sigma)^{-1}\end{array}$$

i.e.

$$\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu} = \mathrm{conj}_{\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu}(\sigma)}({}^\sigma(\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu})).$$

Hence

$$\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu} \in {}^{\tau,\mu,\alpha}G(\mathbb{Q})$$

and has the same image in $\overline{G}^c(E)$ as γ. Thus the image of $G(\mathbb{Q})$ in $\overline{G}^c(\mathbb{Q})$ is contained in the image of ${}^{\tau,\mu,\alpha}G(\mathbb{Q})$ in $\overline{G}^c(\mathbb{Q})$. Using the identification ${}^{\tau^{-1},{}^\tau\mu,\alpha^{-1}}({}^{\tau,\mu,\alpha}G) = G$ we get the reverse inclusion.

Now define

$$\Phi(\tau,\mu,\alpha) : \tau\mathrm{Sh}(G,Y)_U \xrightarrow{\sim} \mathrm{Sh}({}^{\tau,\mu,\alpha}G, {}^{\tau,\mu,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(U)}$$

to be the disjoint union over $h \in G(\mathbb{Q})G^{nc}(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U$ of the maps $\Phi(\tau,\mu,\alpha)$:

$$\tau\mathrm{Sh}(G^{nc},Y)_{hUh^{-1}\cap G^{nc}(\mathbb{A}^\infty)} \xrightarrow{\sim} \mathrm{Sh}({}^{\tau,\mu,\alpha}G^{nc}, {}^{\tau,\mu,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(h)\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(U)\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(h)^{-1}\cap {}^{\tau,\mu,\alpha}G^{nc}(\mathbb{A}^\infty)}.$$

From the claim above we see that $\Phi(\tau,\mu,\alpha)$ is an isomorphism. We must check it is independent of the choice of coset representatives h. For this suppose that $h' = g\gamma hu$ with $g \in G^{nc}(\mathbb{A}^\infty)$ and $\gamma \in G(\mathbb{Q})$ and $u \in U$. Then

$$\begin{array}{rcl}\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(h') & = & (\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g\gamma)\gamma^{-1}_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}\gamma^{-1})(\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}) \\ & & \mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(h)\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(u),\end{array}$$

with $\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu} \in {}^{\tau,\mu,\alpha}G(\mathbb{Q})$ and

$$\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g\gamma)\gamma^{-1}_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}\gamma^{-1} \in {}^{\tau,\mu,\alpha}G^{nc}(\mathbb{A}^\infty).$$

Thus what we must show is that

$$\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(g,\mathrm{conj}_\gamma) = \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g\gamma)\gamma^{-1}_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}\gamma^{-1}, \mathrm{conj}_{\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}})\circ\Phi(\tau,\mu,\alpha).$$

However

$$\begin{array}{rl} & \Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(g,\mathrm{conj}_\gamma) \\ = & \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g),\mathrm{conj}_\gamma)\circ\Phi(\tau,\mathrm{conj}_{\gamma^{-1}}\mu,\alpha) \\ = & \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g),\mathrm{conj}_\gamma)\circ\mathrm{Sh}(b_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}\gamma^{-1}_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}, \mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}})\circ\Phi(\tau,\mu,\alpha).\end{array}$$

Thus we are reduced to checking that

$$\begin{array}{rl} & \mathrm{Sh}(\gamma b_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}^{-1}\gamma^{-1},\mathrm{conj}_{\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}}) \\ = & \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(\gamma)\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}^{-1}\gamma^{-1},\mathrm{conj}_{\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}}). \end{array}$$

This is clear because

$$\begin{array}{rl} & \gamma b_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu} \\ = & \gamma\mathrm{conj}_{\gamma^{-1}}(b_{E,\mathfrak{a}^+,\tau,\alpha,\mu})b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}^{-1} \\ = & b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}\gamma b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}^{-1} \\ = & \mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(\gamma). \end{array}$$

We certainly have

$$\Phi(\tau,\mu,\alpha)(\mu,1)=({}^\tau\mu,1).$$

If $gU_1g^{-1}\subset U_2$, we claim that

$$\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(g,1)=\mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g),1)\circ\Phi(\tau,\mu,\alpha),$$

as maps

$${}^\tau\mathrm{Sh}(G,Y)_{U_1}\to\mathrm{Sh}({}^{\tau,\mu,\alpha}G,{}^{\tau,\mu,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(U_2)}.$$

However both sides when restricted to $\mathrm{Sh}(G,Y)_{hU_1h^{-1}\cap G^{nc}(\mathbb{A}^\infty)}$ are just $\Phi(\tau,\mu,\alpha)$ taking

$${}^\tau\mathrm{Sh}(G^{nc},Y)_{hU_1h^{-1}\cap G^{nc}(\mathbb{A}^\infty)}\longrightarrow\mathrm{Sh}({}^{\tau,\mu,\alpha}G^{nc},{}^{\tau,\mu,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}hg^{-1}(U_2)\cap{}^{\tau,\mu,\alpha}G^{nc}(\mathbb{A}^\infty)}.$$

Now suppose that μ_1 and μ_2 are special in Y and defined over E. We claim that

$$\Phi(\tau,\mu_2,\alpha)=\mathrm{Sh}(b_{E,\tau,\alpha,\mu_1,\mu_2}\gamma_{E,\tau,\alpha,\mu_1,\mu_2}^{-1},\mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}})\circ\Phi(\tau,\mu_1,\alpha)$$

as maps

$${}^\tau\mathrm{Sh}(G,Y)_U\longrightarrow\mathrm{Sh}({}^{\tau,\mu_2,\alpha}G,{}^{\tau,\mu_2,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2}}U}.$$

To verify this, we must show that if $h\in G(\mathbb{A}^\infty)$, then

$$\Phi(\tau,\mu_2,\alpha)=\mathrm{Sh}(b_{E,\tau,\alpha,b_1,b_2}\gamma_{E,\tau,\alpha,\mu_1,\mu_2}^{-1},1)\circ\mathrm{Sh}(1,\mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}})\circ\Phi(\tau,\mu_1,\alpha)$$

as maps from ${}^\tau\mathrm{Sh}(G^{nc},Y)_{hUh^{-1}\cap G^{nc}(\mathbb{A}^\infty)}$ to

$$\mathrm{Sh}({}^{\tau,\mu_2,\alpha}G^{nc},{}^{\tau,\mu_2,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2}}(h)\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2}}(U)\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2}}(h)^{-1}\cap{}^{\tau,\mu_2,\alpha}G^{nc}(\mathbb{A}^\infty)}.$$

However this equality is part of theorem 8.2.

Thus Milne's theorem 8.2 remains true without the NCF hypothesis. As noted immediately after the statement of that theorem, this allows us to conclude:

Theorem 8.4. *Suppose that $E/\mathbb{Q}$ is a finite Galois extension, that (G,Y) is a μ-Shimura datum, and that $\mu\in Y$ is an E-special point. Suppose also that $\tau\in\mathrm{Aut}(\mathbb{C})$ and choose $\alpha\in\widetilde{S}_{E,\mathbb{C},\tau}$. Then there is a unique morphism*

$$\Phi(\tau,\mu,\alpha):\tau\mathrm{Sh}(G,Y)_U\xrightarrow{\sim}\mathrm{Sh}({}^{\tau,\mu,\alpha}G,{}^{\tau,\mu,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(U)}$$

such that

$$\Phi(\tau,\mu,\alpha)(\mu,1)=({}^\tau\mu,1)$$

and

$$\Phi(\tau,\mu,\alpha)\circ \mathrm{Sh}(g,1)=\mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g),1)\circ\Phi(\tau,\mu,\alpha)$$

for all $g\in G(\mathbb{A}^\infty)$*. Moreover:*

(1) *If* $\gamma\in S_{E,\mathbb{C}}(E)$ *then* $\Phi(\tau,\mu,\alpha\gamma)=\mathrm{Sh}(1,\mathrm{conj}_{\widetilde{\mu}(\gamma)^{-1}})\Phi(\tau,\mu,\alpha)$.

(2) *If* $f:(G_1,Y_1)\to(G_2,Y_2)$ *and* $g\in G_2(\mathbb{A}^\infty)$ *and* $\mu_1\in Y_1$ *is a special point defined over the image of* E *in* $\mathbb{C}$*, then*

$$\Phi(\tau,f\circ\mu_1,\alpha)\circ{}^\tau\mathrm{Sh}(g,f)=\mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,f\circ\mu}}(g),{}^{\tau,\mu_1,\alpha}f)\circ\Phi(\tau,\mu_1,\alpha).$$

(3) $\Phi(\tau_1\tau_2,\mu,\alpha_1\alpha_2)=\Phi(\tau_1,{}^{\tau_2}\mu,\alpha_1)\circ{}^{\tau_1}\Phi(\tau_2,\mu,\alpha_2)$.

(4) *If* $G=T$ *is a torus then* $\Phi(\tau,\mu,\alpha)\circ\tau\circ\Pi_{T,\{\mu\}}=\Pi_{T,\{{}^\tau\mu\}}$.

(5) *If* μ_1 *and* μ_2 *are two such special points defined over* E*, then*

$$\Phi(\tau,\mu_2,\alpha)=\mathrm{Sh}(b_{E,\tau,\alpha,\mu_1,\mu_2}\gamma^{-1}_{E,\tau,\alpha,\mu_1,\mu_2},\mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}})\circ\Phi(\tau,\mu_1,\alpha).$$

8.6. **Reformulation of Milne's theorem.** We now state and prove our first main theorem, which is a reformulation of Milne's theorem.

Theorem 8.5. *Suppose that* $E/\mathbb{Q}$ *is a finite Galois extension and* $\mathfrak{a}^+\in\mathcal{H}(E/\mathbb{Q})^+$*. If* (G,Y) *is a Shimura datum with* E *acceptable for* G*, if* $(\tau,\phi,b)\in\mathrm{Conj}_{E,\mathfrak{a}}(G,Y)$ *and if* U *is a neat open compact subgroup of* $G(\mathbb{A}^\infty)$*, then there is an isomorphism*

$$\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b):{}^\tau\mathrm{Sh}(G,Y)_U\xrightarrow{\sim}\mathrm{Sh}^{(\tau,\phi,b)}(G,Y)_{bUb^{-1}}$$

with the following properties.

(1) $\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)\circ{}^\tau\mathrm{Sh}(g,1)=\mathrm{Sh}(bgb^{-1},1)\circ\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)$.

(2) $\mathrm{Sh}(1,f)\circ\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)=\Phi_{E,\mathfrak{a}^+}(\tau,f\circ\phi,f(b))\circ{}^\tau\mathrm{Sh}(1,f)$.

(3) *If* $\delta\in G(E)$ *and* $h\in G(\mathbb{A}^\infty)$*, then* $\Phi_{E,\mathfrak{a}^+}(\tau,{}^\delta\phi,\delta bh)=\mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)\circ{}^\tau\mathrm{Sh}(h,1)$.

(4) *If* $(\tau_1,\phi_1,b_1)\in\mathrm{Conj}_{E,\mathfrak{a}^+}{}^{(\tau_2,c_2,b_2)}(G,Y)$ *and* $(\tau_2,\phi_2,b_2)\in\mathrm{Conj}_{E,\mathfrak{a}^+}(G,Y)$*, then*

$$\Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2,\phi_1\phi_2,b_1b_2)=\Phi_{E,\mathfrak{a}^+}(\tau_1,\phi_1,b_1)\circ{}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2,\phi_2,b_2).$$

(5) *Suppose that* $G=T$ *is a torus, that* $\mu\in X_*(T)(\mathbb{C})$ *and that* $(\tau,\phi,b)\in\mathrm{Conj}_{E,\mathfrak{a}}(T,\{\mu\})$*. Then*

$$b^{-1}\overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau}\in T(\mathbb{A}^\infty)/\overline{T(\mathbb{Q})}\subset T(\mathbb{A}^\infty_K)/\overline{T(\mathbb{Q})}T(K).$$

Moreover

$$\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)\circ\tau\circ\Pi_{T,\{\mu\}}=\mathrm{Sh}(b\overline{b}^{-1}_{\mathfrak{a}^+,\infty,\mu,\tau},1)\circ\Pi_{T,\{{}^\tau\mu\}}.$$

In the special case that τ *fixes the image of* E *in* $\mathbb{C}$*, then* $\Pi^{-1}_{T,\{{}^\tau\mu\}}\circ\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)\circ\tau\circ\Pi_{T,\{\mu\}}$ *equals multiplication by*

$$b^{-1}\prod_{\rho:E\hookrightarrow\mathbb{C}}({}^{\rho^{-1}}\mu)(\mathrm{Art}_E^{-1}\tau^{\widetilde{\rho}})^{-1},$$

where $\widetilde{\rho}$ is any extension of ρ to E^{ab}.

(6) *If $(\tau,\phi,b)\in \mathrm{Conj}_{E,\mathfrak{a}}(G,Y)$, then $(\tau, z_t(\phi), (\mathrm{loc}_{\mathfrak{a}}\phi)(t)b)\in \mathrm{Conj}_{K,{}^t\mathfrak{a}}(G,Y)$ and $\Phi_{E,{}^t\mathfrak{a}^+}(\tau, z_t(\phi), (\mathrm{loc}_{\mathfrak{a}}\phi)(t)b)=\Phi_{G,\mathfrak{a}^+}(\tau,\phi,b)$.*

(7) *Suppose that $D\supset E$ is another finite Galois extension of $\mathbb{Q}$, that $\mathfrak{a}_D^+\in \mathcal{H}(D/\mathbb{Q})^+$ and that $t\in T_{2,E}(\mathbb{A}_D)$ with ${}^t\inf_{\mathrm{Gal}\,(E/\mathbb{Q})}^{\mathrm{Gal}\,(D/\mathbb{Q})}\mathfrak{a}^+=\eta_{D/E,*}\mathfrak{a}_D^+$. Then $\Phi_{D,\mathfrak{a}_D^+}(\inf_{D/E,t}(\tau,\phi,b))=\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)$.*

(8) *If $\mu\in Y$ is an E-special point and if $\alpha\in\widetilde{S}_{E,\mathbb{C},\tau}(E)$ then*

$$\Phi_{E,\mathfrak{a}^+}(\tau,\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}, b_{E,\mathfrak{a}^+,\tau,\alpha,\mu})=\Phi(\tau,\mu,\alpha).$$

Proof: Suppose that $T\subset G$ is a maximal torus defined over $\mathbb{Q}$ such that $T^{\mathrm{ad}}(\mathbb{R})$ is compact and T is split by E. Then we may choose $\mu\in Y$ that factors through T. It will be E-special. Choosing α as in part (8) of the theorem, we may find $\delta\in G(E)$ and $h\in G(\mathbb{A}^\infty)$ such that

$$(\tau,\phi,b)=(\tau,{}^{\delta}\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu},\delta b_{K,\mathfrak{a}^+,\tau,\alpha,\mu}h).$$

Then we are forced to set

$$\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)=\mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(h,1).$$

We must check that this is a good definition.

If $\gamma\in{}^{\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}}G(\mathbb{Q})$ then

$$\begin{aligned}&\mathrm{Sh}(1,\mathrm{conj}_{\delta\gamma})\circ\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(b^{-1}\gamma^{-1}bh,1)\\ =\ &\mathrm{Sh}(1,\mathrm{conj}_{\delta})\circ\mathrm{Sh}(\gamma,1)\circ\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(b^{-1}\gamma^{-1}bh,1)\\ =\ &\mathrm{Sh}(1,\mathrm{conj}_{\delta})\circ\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(h,1),\end{aligned}$$

and so the definition is independent of the choice of δ and h.

If we replace α by $\alpha\gamma^{-1}$ with $\gamma\in S_{E,\mathbb{C}}(E)$, then there is a lift $\widetilde{\gamma}\in R_{E,\mathbb{C}}(E)$ of γ such that $\phi_{E,\mathfrak{a}^+,\tau,\alpha\gamma^{-1},\mu}={}^{\widetilde{\mu}(\widetilde{\gamma})}\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ and $b_{E,\mathfrak{a}^+,\tau,\alpha\gamma^{-1},\mu}=\widetilde{\mu}(\widetilde{\gamma})b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ and so

$$(\tau,\phi,b)=(\tau,{}^{\delta\widetilde{\mu}(\widetilde{\gamma})^{-1}}\phi_{E,\mathfrak{a}^+,\tau,\alpha\gamma^{-1},\mu},\delta\widetilde{\mu}(\widetilde{\gamma})^{-1}b_{E,\mathfrak{a}^+,\tau,\alpha\gamma^{-1},\mu}h).$$

Then, because $\Phi(\tau,\mu,\alpha\gamma^{-1})=\mathrm{Sh}(1,\mathrm{conj}_{\widetilde{\gamma}})\circ\Phi(\tau,\mu,\alpha)$, we see that

$$\mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(h,1)=\mathrm{Sh}(1,\mathrm{conj}_{\delta\widetilde{\mu}(\widetilde{\gamma})^{-1}})\circ\Phi(\tau,\mu,\alpha\gamma^{-1})\circ{}^\tau\mathrm{Sh}(h,1),$$

and our definition is independent of the choice of α.

Finally if we replace μ by μ', then

$$(\tau,\phi,b)=(\tau,{}^{\delta\gamma^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}}\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu'},(\delta\gamma^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'})b_{E,\mathfrak{a}^+,\tau,\alpha,\mu'}(b^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu'}\gamma_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}h)).$$

We must check that

$$\begin{aligned}&\mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(h,1)\\ =\ &\mathrm{Sh}(1,\mathrm{conj}_{\delta\gamma^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}})\circ\Phi(\tau,\mu',\alpha)\circ{}^\tau\mathrm{Sh}(b^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu'}\gamma_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}h,1),\end{aligned}$$

or that

$$\begin{aligned} & \Phi(\tau,\mu,\alpha)\circ {}^{\tau}\mathrm{Sh}(b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}^{-1}\gamma_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}^{-1}b_{E,\mathfrak{a}^+,\tau,\alpha,\mu'},1) \\ = & \mathrm{Sh}(1,\mathrm{conj}_{\gamma_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}^{-1}})\circ \mathrm{Sh}(b_{E,\tau,\alpha,\mu,\mu'}\gamma_{E,\tau,\alpha,\mu,\mu'}^{-1},\mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu,\mu'}})\circ\Phi(\tau,\mu,\alpha), \end{aligned}$$

or even that

$$\begin{aligned} & \mathrm{Sh}(\gamma_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}^{-1}b_{E,\mathfrak{a}^+,\tau,\alpha,\mu'}b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}^{-1},1)\circ\Phi(\tau,\mu,\alpha) \\ = & \mathrm{Sh}(\gamma_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}^{-1}b_{E,\tau,\alpha,\mu,\mu'},1)\circ\Phi(\tau,\mu,\alpha), \end{aligned}$$

which is true.

Having checked that our definition is good we must check the desired properties. Property (8) is part of the definition, while property (3) follows easily from the definition.

Properties (1) and (2) are true for

$$(\tau,\phi,b)=(\tau,\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu},b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}),$$

because $\phi_{E,\mathfrak{a}^+,\tau,\alpha,f\circ\mu}=f\circ\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ and $b_{E,\mathfrak{a}^+,\tau,\alpha,f\circ\mu}=f(b_{E,\mathfrak{a}^+,\tau,\alpha,\mu})$. To check that they remain true for all (τ,ϕ,b), it suffices to check that if they are true (τ,ϕ,b) then they are also true for $(\tau,{}^{\delta}\phi,\delta bh)$. However we have

$$\begin{aligned} & \Phi_{E,\mathfrak{a}^+}(\tau,{}^{\delta}\phi,\delta bh)\circ{}^{\tau}\mathrm{Sh}(g,1) \\ = & \mathrm{Sh}(1,\mathrm{conj}_{\delta})\circ\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)\circ{}^{\tau}\mathrm{Sh}(hg,1) \\ = & \mathrm{Sh}(1,\mathrm{conj}_{\delta})\circ\mathrm{Sh}(bhgh^{-1}b^{-1},1)\circ\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)\circ{}^{\tau}\mathrm{Sh}(h,1) \\ = & \mathrm{Sh}(\mathrm{conj}_{\delta bh}(g),1)\circ\Phi_{E,\mathfrak{a}^+}(\tau,{}^{\delta}\phi,\delta bh) \end{aligned}$$

and

$$\begin{aligned} & \Phi_{E,\mathfrak{a}^+}(\tau,f\circ{}^{\delta}\phi,f(\delta bh))\circ{}^{\tau}\mathrm{Sh}(1,f) \\ = & \mathrm{Sh}(1,\mathrm{conj}_{f(\delta)})\circ\Phi_{E,\mathfrak{a}^+}(\tau,f\circ\phi,f(b)b)\circ{}^{\tau}\mathrm{Sh}(f(h),1)\circ{}^{\tau}\mathrm{Sh}(1,f) \\ = & \mathrm{Sh}(1,\mathrm{conj}_{f(\delta)})\circ\Phi_{E,\mathfrak{a}^+}(\tau,f\circ\phi,f(b)b)\circ{}^{\tau}\mathrm{Sh}(1,f)\circ{}^{\tau}\mathrm{Sh}(h,1) \\ = & \mathrm{Sh}(1,\mathrm{conj}_{f(\delta)})\circ\mathrm{Sh}(1,f)\circ\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)\circ{}^{\tau}\mathrm{Sh}(h,1) \\ = & \mathrm{Sh}(1,f)\circ\mathrm{Sh}(1,\mathrm{conj}_{\delta})\circ\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)\circ{}^{\tau}\mathrm{Sh}(h,1) \\ = & \mathrm{Sh}(1,f)\circ\Phi_{E,\mathfrak{a}^+}(\tau,{}^{\delta}\phi,\delta bh). \end{aligned}$$

Similarly property (5) is true in the case

$$(\tau,\phi,b)=(\tau,\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu},b_{E,\mathfrak{a}^+,\tau,\alpha,\mu}).$$

On the other hand if the claim is true for (τ,ϕ,b), then

$$\begin{aligned} & \Phi_{E,\mathfrak{a}^+}(\tau,{}^{\delta}\phi,\delta bh)\circ\tau\circ\Pi_{T,\{\mu\}} \\ = & \mathrm{Sh}(1,\mathrm{conj}_{\delta})\circ\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)\circ{}^{\tau}\mathrm{Sh}(h,1)\circ\tau\circ\Pi_{T,\{\mu\}} \\ = & \mathrm{Sh}(h,1)\circ\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)\circ\tau\circ\Pi_{T,\{\mu\}} \\ = & \mathrm{Sh}(h,1)\circ\mathrm{Sh}(b\overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau}^{-1},1)\circ\Pi_{T,\{{}^{\tau}\mu\}} \\ = & \mathrm{Sh}(\delta bh\overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau}^{-1},1)\circ\Pi_{T,\{{}^{\tau}\mu\}}, \end{aligned}$$

and so it is also true for $(\tau,{}^{\delta}\phi,\delta bh)$.

That property (7) is true in the case $G = T$ is a torus follows from property (5) because $\inf_{D/E,t}(\tau, \phi, b) = (\tau, \inf_{3,D/E,t} \phi, (\mathrm{loc}_{\mathfrak{a}}\phi)(t)b)$ and

$$\bar{b}_{\mathfrak{a}_D^+,\infty,\mu,\tau} = \bar{b}_{\mathfrak{a}^+,\infty,\mu,\tau} \prod_{\rho:E\hookrightarrow\mathbb{C}} ((^{\rho^{-1}}\mu) \circ (\pi_{w(\rho)}/\pi_{w(\tau\rho)}))(t) = \bar{b}_{\mathfrak{a}^+,\infty,\mu,\tau}(\mathrm{loc}_{\mathfrak{a}}\phi)(t).$$

Now consider the general case. Because $\inf_{D/E,t}(\gamma, h)(\tau, \phi, b) = (\gamma, h) \inf_{D/E,t}(\tau, \phi, b)$, the assertion will be true for (τ, ϕ, b) if and only if it is true for $(\gamma, h)(\tau, \phi, b)$. Choose a maximal torus $T \subset G$ defined over $\mathbb{Q}$ and split by E such that $T^{\mathrm{ad}}(\mathbb{R})$ is compact. Also choose $\mu \in Y$ which factors through T and let i denote the canonical embedding $i : T \hookrightarrow G$. Also choose $(\tau, \phi, b) \in \mathrm{Conj}_{E,\mathfrak{a}}(T, \{\mu\})$. It will suffice to prove that

$$\Phi_{E,\mathfrak{a}^+}(\tau, i \circ \phi, i(b)) = \Phi_{E,\mathfrak{a}_D^+}(\inf_{D/E,t}(\tau, i \circ \phi, i(b))).$$

Because

$$\bigcup_{g\in G(\mathbb{A}^\infty)} \mathrm{Sh}(g, i)(\mathrm{Sh}(T, \{\mu\})_{g^{-1}Ug\cap T(\mathbb{A}^\infty)})$$

is Zariski dense in $\mathrm{Sh}(G, Y)_U$, it even suffices to check that

$$\Phi_{E,\mathfrak{a}^+}(\tau, i \circ \phi, i(b)) \circ {}^\tau\mathrm{Sh}(g, i) = \Phi_{E,\mathfrak{a}_D^+}(\inf_{D/E,t}(\tau, i \circ \phi, i(b))) \circ {}^\tau\mathrm{Sh}(g, i)$$

for all $g \in G(\mathbb{A}^\infty)$. As $\mathrm{conj}_b(g) = \mathrm{conj}_{(\mathrm{loc}_{\mathfrak{a}}\phi)(t)b}(g)$ (because ϕ is basic) and $\inf_{3,D/E,t}(i\circ \phi) = i \circ \inf_{3,D/E,t} \phi$ and $i((\mathrm{loc}_{\mathfrak{a}}\phi)(t)) = (\mathrm{loc}_{\mathfrak{a}} i \circ \phi)(t)$; applying properties (1) and (2) we reduce to the equality

$$\Phi_{E,\mathfrak{a}^+}(\tau, \phi, b) = \Phi_{E,\mathfrak{a}_D^+}(\inf_{D/E,t}(\tau, \phi, b)),$$

which we have already verified.

Property (6) is a special case of property (7).

Finally we must check property (4). If

$$(\tau_1, \phi_1, b_1) = (\tau, \phi_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu}, b_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu})$$

and

$$(\tau_2, \phi_2, b_2) = (\tau, \phi_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu}, b_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu})$$

Then the result is true because for some $\beta \in Z(G)(E)$ we have

$$\phi_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu}\phi_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu} = {}^\beta\phi_{E,\mathfrak{a}^+,\tau_1\tau_2,\alpha_1\alpha_2,\mu}$$

and

$$b_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu}b_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu} \equiv \beta b_{E,\mathfrak{a}^+,\tau_1\tau_2,\alpha_1\alpha_2,\mu} \bmod \overline{Z(G)(\mathbb{Q})},$$

so that

$$\begin{aligned} & \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, \phi_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu}\phi_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu}, b_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu}b_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu}) \\ = \; & \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, \phi_{E,\mathfrak{a}^+,\tau_1\tau_2,\alpha_1\alpha_2,\mu}, b_{E,\mathfrak{a}^+,\tau_1\tau_2,\alpha_1\alpha_2,\mu}). \end{aligned}$$

Suppose now that property (4) holds for (τ_1, ϕ_1, b_1) and (τ_2, ϕ_2, b_2). Then it also holds for $(\tau_1, {}^\delta\phi_1, \delta b_1 h)$ and (τ_2, ϕ_2, b_2), because

$$\begin{array}{rl} & \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, ({}^\delta\phi_1)\phi_2, \delta b_1 h b_2) \\ = & \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, {}^\delta(\phi_1\phi_2), \delta b_1 b_2(b_2^{-1}hb_2)) \\ = & \mathrm{Sh}(1, \mathrm{conj}_\delta) \circ \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, \phi_1\phi_2, b_1b_2) \circ {}^{\tau_1\tau_2}\mathrm{Sh}(b_2^{-1}hb_2, 1) \\ = & \mathrm{Sh}(1, \mathrm{conj}_\delta) \circ \Phi_{E,\mathfrak{a}^+}(\tau_1, \phi_1, b_1) \circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2, \phi_2, b_2) \circ {}^{\tau_1\tau_2}\mathrm{Sh}(b_2^{-1}hb_2, 1) \\ = & \mathrm{Sh}(1, \mathrm{conj}_\delta) \circ \Phi_{E,\mathfrak{a}^+}(\tau_1, \phi_1, b_1) \circ {}^{\tau_1}\mathrm{Sh}(h, 1) \circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2, \phi_2, b_2) \\ = & \Phi_{E,\mathfrak{a}^+}(\tau_1, {}^\delta\phi_1, \delta b_1 h) \circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2, \phi_2, b_2). \end{array}$$

Similarly, if the property holds for $(\tau_1, \mathrm{conj}_{\delta^{-1}} \circ \phi_1, \mathrm{conj}_{\delta^{-1}} \circ b_1)$ and (τ_2, ϕ_2, b_2), then it also holds for (τ_1, ϕ_1, b_1) and $(\tau_2, {}^\delta\phi_2, \delta b_2 h)$, because

$$\begin{array}{rl} & \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, \phi_1({}^\delta\phi_2), b_1\delta b_2 h) \\ = & \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, {}^\delta((\mathrm{conj}_{\delta^{-1}} \circ \phi_1)\phi_2), \delta\mathrm{conj}_{\delta^{-1}}(b_1)b_2 h) \\ = & \mathrm{Sh}(1, \mathrm{conj}_\delta) \circ \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, (\mathrm{conj}_{\delta^{-1}} \circ \phi_1)\phi_2, \mathrm{conj}_{\delta^{-1}}(b_1)b_2) \circ {}^{\tau_1\tau_2}\mathrm{Sh}(h, 1) \\ = & \mathrm{Sh}(1, \mathrm{conj}_\delta) \circ \Phi_{E,\mathfrak{a}^+}(\tau_1, \mathrm{conj}_{\delta^{-1}} \circ \phi_1, \mathrm{conj}_{\delta^{-1}}(b_1)) \circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2, \phi_2, b_2) \circ {}^{\tau_1\tau_2}\mathrm{Sh}(h, 1) \\ = & \Phi_{E,\mathfrak{a}^+}(\tau_1, \phi_1, b_1) \circ {}^{\tau_1}\mathrm{Sh}(1, \mathrm{conj}_\delta) \circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2, \phi_2, b_2) \circ {}^{\tau_1\tau_2}\mathrm{Sh}(h, 1) \\ = & \Phi_{E,\mathfrak{a}^+}(\tau_1, \phi_1, b_1) \circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2, {}^\delta\phi_2, \delta b_2 h). \end{array}$$

Note that

$$(\tau_1, \mathrm{conj}_{\delta^{-1}} \circ \phi_{E,\mathfrak{a}^+,\tau_1,\alpha_1,\mu}, \mathrm{conj}_{\delta^{-1}} \circ b_{E,\mathfrak{a}^+,\tau_1,\alpha_1,\mu}) = (\tau_1, \phi_{E,\mathfrak{a}^+,\tau_1,\alpha_1,\mathrm{conj}_{\delta^{-1}}\circ\mu}, b_{E,\mathfrak{a}^+,\tau_1,\alpha_1,\mathrm{conj}_{\delta^{-1}}\circ\mu}).$$

Thus property (4) follows in full generality. □

We remark that properties (1), (2) and (5) completely characterize the $\Phi_{E,\mathfrak{a}^+}(\tau, \phi, b)$.

9. Rational Shimura varieties

9.1. **Rational Shimura data.** In this section we will discuss the data that we propose as an alternative formalism for Shimura varieties. Suppose that $E/\mathbb{Q}$ is a finite totally imaginary Galois extension and that $\mathfrak{a}^+ \in \mathcal{H}(E/\mathbb{Q})^+$.

By a $((E,\mathfrak{a})$-)*rational Shimura datum* over a field L of characteristic 0 we mean a triple (G,ψ,C) where

(1) G is a reductive group over $\mathbb{Q}$ for which E is acceptable;
(2) $\psi \in Z^1_{\text{alg}}(\mathcal{E}^{\text{loc}}(E/\mathbb{Q})_{\mathfrak{a}}, G(\mathbb{A}_E))_{\text{basic}}$ with $\text{res}_{\mathbb{C}/\mathbb{R}}\psi$ compact;
(3) C is a G-conjugacy class, defined over L, of miniscule cocharacacters of G such that C can be defined over a subfield $L_0 \subset L$ which admits an embedding $\rho : L_0 \hookrightarrow E$ and $\overline{\kappa}(\psi) = \lambda({}^{\rho}C) \in \Lambda_{G,\text{Gal}\,(E/\mathbb{Q})}$. (This is independent of the particular choice of L_0 and ρ.)

By a *morphism* $(\phi,g,f) : (G_1,\psi_1,C_1) \to (G_2,\psi_2,C_2)$ of $(E,\mathfrak{a})$-rational Shimura data over L, we will mean

- a cocycle $\phi \in Z^1_{\text{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G_2(E))_{\text{basic}}$,
- an element $g \in G_2(\mathbb{A}_E)$,
- and a morphism $f : G_1 \to {}^{\phi}G_2$ defined over $\mathbb{Q}$, such that $f\circ\psi_1 = {}^{g^{-1}}\psi_2 \text{loc}_{\mathfrak{a}}\phi^{-1}$ and $f(C_1) \subset C_2$.

Note that
$$\theta_{(\phi,g,f)} = \text{conj}_g \circ f : {}^{\psi_1}G_1(\mathbb{A}) \longrightarrow {}^{\psi_2}G_2(\mathbb{A}).$$
We define the composite of such morphisms by
$$(\phi_2,g_2,f_2)\circ(\phi_1,g_1,f_1) = (f_2(\phi_1)\phi_2, g_2f_2(g_1), f_2\circ f_1)$$
and set
$$\text{Id}_{(G,\psi,C)} = (1,1,1).$$
This makes $(E,\mathfrak{a})$-rational Shimura data over a field L into a category $\text{RSD}(E,\mathfrak{a};L)$.

If $Z(G)$ is connected, then (by proposition 10.4 of [K3]) any object of $\text{RSD}(E,\mathfrak{a};L)$ is isomorphic to one with G quasisplit. Thus in this case one might as well restrict to triples (G,ψ,C) with G quasi-split.

We will write
$$\widehat{G}_{E,(G,\psi,C)}(\mathbb{A}) = \widehat{G}_{E,\psi}(\mathbb{A}) = \{(\zeta,g,1) \in Z^1_{\text{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, Z(G)(E))\times G(\mathbb{A}_E)\times\{1_G\} :\ (\text{loc}_{\mathfrak{a}}\zeta)^g\psi = \psi\}.$$
Elements of $\widehat{G}_{E,\psi}(\mathbb{A})$ are examples of the triples considered in the previous paragraph. Under the composition law described above they become a group. Explicitly
$$(\zeta_2,g_2,1)(\zeta_1,g_1,1) = (\zeta_2\zeta_1, g_2g_1, 1).$$
We will often write (ζ,g) for $(\zeta,g,1)$. If $(\zeta,g) \in \widehat{G}_{(G,\psi,C)}(\mathbb{A})$, then we see that $\zeta \in Z^1(\text{Gal}\,(E/\mathbb{Q}), Z(G)(E))$. We have embeddings
$$\begin{array}{ccc} {}^{\psi}G(\mathbb{A}) & \hookrightarrow & \widehat{G}_{E,\psi}(\mathbb{A}) \\ g & \longmapsto & (1,g) \end{array}$$

and

$$\begin{array}{rcl} Z(G)(E) & \hookrightarrow & \widehat{G}_{E,\psi}(\mathbb{A}) \\ z & \longmapsto & ({}^{z}1, z^{-1}). \end{array}$$

We further define

$$\widetilde{G}_{E,\psi}(\mathbb{A}^\infty) = \widehat{G}_{E,\psi}(\mathbb{A})/Z(G)(E)\overline{Z(G)(\mathbb{Q})^\psi G(\mathbb{R})}.$$

There is a short exact sequence

$$\begin{array}{l} (0) \longrightarrow {}^\psi G(\mathbb{A}^\infty)/\overline{Z(G)(\mathbb{Q})} \longrightarrow \widetilde{G}_{E,\psi}(\mathbb{A}^\infty) \stackrel{\boldsymbol{\zeta}}{\longrightarrow} \\ \ker(H^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^\psi G(\mathbb{A}_E))) \longrightarrow (0), \end{array}$$

where $\boldsymbol{\zeta}$ is induced by $(\zeta, g) \mapsto [\zeta]$. If $(\phi, g, f) : (G_1, \psi_1, C_1) \longrightarrow (G_2, \psi_2, C_2)$, set

$$\widetilde{G}_{E,(G_1,\psi_1,C_1)}(\mathbb{A}^\infty)_f = \{[(\zeta, h)] \in \widetilde{G}_{E,(G_1,\psi_1,C_1)}(\mathbb{A}^\infty) :\ f \circ \zeta \text{ is valued in } Z(G_2)(E)\}.$$

Then we get a continuous homomorphism

$$\begin{array}{rcl} \widetilde{\theta}_{(\phi,g,f)} : \widetilde{G}_{(G_1,\psi_1,C_1)}(\mathbb{A}^\infty)_f & \longrightarrow & \widetilde{G}_{(G_2,\psi_2,C_2)}(\mathbb{A}^\infty) \\ {[(\zeta, h)]} & \longmapsto & [(f \circ \zeta, \mathrm{conj}_g(f(h)))], \end{array}$$

satisfying

$$\widetilde{\theta}_{(\phi,g,f)}|_{{}^{\psi_1}G_1(\mathbb{A}^\infty)/\overline{Z(G_1)(\mathbb{Q})}} = \theta_{(\phi,g,h)}$$

and

$$\widetilde{\theta}_{(\phi,g,f)}(\widetilde{g}) \circ (\phi, g, f) = (\phi, g, f) \circ \widetilde{g}$$

If (G, ψ, C) is a $(E, \mathfrak{a})$-rational Shimura datum over L and if $\tau : L \to L'$ is a field morphism then we define

$$\tau(G, \psi, C) = (G, \psi, {}^\tau C),$$

a $(E, \mathfrak{a})$-rational Shimura datum over L'. If $(\phi, g, f) : (G_1, \psi_1, C_1) \to (G_2, \psi_2, C_2)$ over L, then $(\phi, g, f) : \tau(G_1, \psi_1, C_1) \to \tau(G_2, \psi_2, C_2)$ over L' and τ induces a functor $\tau : \mathrm{RSD}(E, \mathfrak{a}; L) \to \mathrm{RSD}(E, \mathfrak{a}; L')$.

If $t \in T_{2,E}(\mathbb{A}_E)$ we define an equivalence of categories

$$z_t : \mathrm{RSD}(E, \mathfrak{a}; L) \longrightarrow \mathrm{RSD}(E, {}^t\mathfrak{a}; L)$$

with

$$z_t(G, \psi, C) = (G, z_t(\psi), C)$$

and

$$z_t(\phi, g, f) = (z_t(\phi), (\mathrm{loc}_{\mathfrak{a}}\phi)(t)^{-1}g, f).$$

(Note that z_t on objects only depends on $\mathfrak{a}$ and ${}^t\mathfrak{a}$, but on morphisms it depends on t itself.) Thus we get an isomorphism

$$\begin{array}{rcl} z_t : \widetilde{G}_{E,(G,\psi,C)}(\mathbb{A}^\infty) & \stackrel{\sim}{\longrightarrow} & \widetilde{G}_{E,z_t(G,\psi,C)}(\mathbb{A}^\infty) \\ {[(\zeta, g)]} & \longmapsto & [(\zeta, g)], \end{array}$$

which only depends on $\mathfrak{a}$ and ${}^t\mathfrak{a}$, but not on t.

More generally suppose that $D \supset E$ is another finite Galois extension of $\mathbb{Q}$, that $\mathfrak{a}_D \in \mathcal{H}(D/\mathbb{Q})$ and that $t \in T_{2,E}(\mathbb{A}_D)$ with $\eta_{D/E,*}\mathfrak{a}_D = {}^t \inf_{D/E} \mathfrak{a}$. Then there is a functor

$$\begin{array}{rcl} \inf_{D/E,t} : \mathrm{RSD}(E, \mathfrak{a}; L) & \longrightarrow & \mathrm{RSD}(D, \mathfrak{a}_D; L) \\ (G, \psi, C) & \longmapsto & (G, \inf^{\mathrm{loc}}_{D/E,t} \psi, C) \\ (\phi, g, f) & \longmapsto & (\inf_{3,D/E,t} \phi, (\mathrm{loc}_{\mathfrak{a}}\phi)(t)g, f). \end{array}$$

(Again note that $\inf_{D/E,t}$ on objects only depends on $\mathfrak{a}$ and $\mathfrak{a}_D$, but on morphisms it depends on t itself.) Thus we get a map

$$\begin{array}{rcl} \inf_{D/E,t} : \widetilde{G}_{E,(G,\psi,C)}(\mathbb{A}^\infty) & \longrightarrow & \widetilde{G}_{D,\inf_{D/E,t}(G,\psi,C)}(\mathbb{A}^\infty) \\ [(\zeta, g)] & \longmapsto & [(\inf^{\mathrm{Gal}\,(D/\mathbb{Q})}_{\mathrm{Gal}\,(E/\mathbb{Q})} \zeta, g)], \end{array}$$

which only depends on $\mathfrak{a}$ and ${}^t\mathfrak{a}$, but not on t. It restricts to the identity on ${}^\psi G(\mathbb{A}^\infty) = {}^{\inf^{\mathrm{loc}}_{D/E,t} \psi} G(\mathbb{A}^\infty)$. Moreover

$$\inf_{D'/D,t'} \circ \inf_{D/E,t} = \inf_{D'/E, t\eta_{D/E}(t')}.$$

9.2. **Labels.** If (G, ψ, C) is a $(E, \mathfrak{a})$-rational Shimura datum over $\mathbb{C}$, we define $\mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$ to be the set of pairs (ϕ, b) where

(1) $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$ with $\mathrm{loc}[\phi] = \widetilde{\boldsymbol{\lambda}}_{\mathrm{res}_{\mathbb{C}} {}^\psi G}(C)[\psi]$,
(2) and $b \in G(\mathbb{A}_E)$ with $\mathrm{res}^\infty \mathrm{loc}_{\mathfrak{a}}\phi = {}^b\mathrm{res}^\infty \psi$.

We also set

$$(G, \psi, C)_{(\phi,b)} = ({}^\phi G, Y(C)_{{}^\phi G}),$$

a Deligne Shimura datum. Note that ${}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}}$ acts on $Y(C)_{{}^\phi G}$; and that

$$\mathrm{conj}_b : {}^\psi G(\mathbb{A}^\infty) \xrightarrow{\sim} {}^\phi G(\mathbb{A}^\infty).$$

There is an embedding

$$\begin{array}{rcl} i_{(\phi,b)} : {}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}} & \hookrightarrow & \widetilde{G}_{E,\psi}(\mathbb{A}^\infty) \\ \gamma & \longmapsto & [((\sigma \mapsto \widetilde{\gamma}^{-1}\phi(\sigma)^\sigma\widetilde{\gamma}\phi(\sigma)^{-1}), (b^{-1}\widetilde{\gamma}b, \widehat{\gamma}^{-1}\widetilde{\gamma}))], \end{array}$$

where $\widetilde{\gamma} \in {}^\phi G(E)$ and $\widehat{\gamma} \in {}^\phi G(\mathbb{R})$ are lifts of γ. (This is independent of the choice of lifts $\widetilde{\gamma}$ and $\widehat{\gamma}$.)

We call two elements (ϕ, b) and $(\phi', b') \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$ equivalent if $[\phi] = [\phi'] \in H^1(\mathcal{E}_3(E/\mathbb{Q}), G(E))$. In this case we can find $\gamma \in G(E)$ and $h \in {}^\psi G(\mathbb{A}^\infty)$ with

$$(\phi', b') = {}^{(\gamma,h)}(\phi, b) = ({}^\gamma\phi, \gamma b h^{-1}).$$

Moreover the only ambiguity in the choice of (γ, h) is that we can replace γ by $\gamma\delta$ and h by $h\mathrm{conj}_{b^{-1}}(\delta)$ for some $\delta \in {}^\phi G(\mathbb{Q})$. Choosing some $(\phi, b) \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$, we have

$$\begin{array}{rcl} \#\mathrm{Label}_{\mathfrak{a}}(G, \psi, C)/\sim & = & \#\{\boldsymbol{\phi} \in H^1(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}} : \ \mathrm{loc}\boldsymbol{\phi} = \widetilde{\boldsymbol{\lambda}}_{\mathrm{res}_{\mathbb{C}} {}^\psi G}(C)[\psi]\} \\ & = & \#\ker(H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^\phi G(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^\phi G(\mathbb{A}_E))) \\ & < & \infty. \end{array}$$

If $(\phi, g, f) : (G_1, \psi_1, C_1) \to (G_2, \psi_2, C_2)$ then we get a map

$$\begin{array}{rcl} \mathrm{Label}_{\mathfrak{a}}(\phi, g, f) : \mathrm{Label}_{\mathfrak{a}}(G_1, \psi_1, C_1) & \longrightarrow & \mathrm{Label}_{\mathfrak{a}}(G_2, \psi_2, C_2) \\ (\phi_1, b_1) & \longmapsto & ((f \circ \phi_1)\phi, f(b_1)g^{-1}). \end{array}$$

Moreover f gives a map

$$f : {}^{\phi_1}G_1 \longrightarrow {}^{(f\circ\phi_1)\phi}G_2$$

over $\mathbb{Q}$ which takes $Y(C_1)_{\phi_1 G_1}$ to $Y(C_2)_{(f\circ\phi_1)\phi G_2}$, and we have

$$\mathrm{conj}_{f(b_1)g^{-1}} \circ \theta_{(\phi,g,f)} = f \circ \mathrm{conj}_{b_1} : {}^{\psi_1}G_1(\mathbb{A}^\infty) \longrightarrow {}^{(f\circ\phi_1)\phi}G_2(\mathbb{A}^\infty).$$

Moreover

$$\mathrm{Label}_{\mathfrak{a}}(\phi, g, f)({}^{(\gamma,h)}(\phi_1, b_1)) = {}^{(f(\gamma), gf(h)g^{-1})}\mathrm{Label}_{\mathfrak{a}}(\phi, g, f)(\phi_1, b_1),$$

and so we get an induced map

$$\mathrm{Label}_{\mathfrak{a}}(\phi, g, f) : \mathrm{Label}_{\mathfrak{a}}(G_1, \psi_1, C_1)/\sim \longrightarrow \mathrm{Label}_{\mathfrak{a}}(G_2, \psi_2, C_2)/\sim .$$

If $(\phi, b) \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$ and $(\tau, \phi', b') \in \mathrm{Conj}_{E,\mathfrak{a}}({}^{\phi}G, Y(C)_{\phi G})$, then

$$(\phi'\phi, b'b) \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, {}^{\tau}C).$$

(To verify this use the fact from the end of section 3.4 that

$$\widehat{\boldsymbol{\lambda}}_{\tilde{\lambda}_{G(C)}G}({}^{\tau}C - Y(C)_{\tilde{\lambda}_{G(C)}G})\widetilde{\boldsymbol{\lambda}}_G(C) = \widetilde{\boldsymbol{\lambda}}_G({}^{\tau}C).)$$

If $t \in T_{2,E}(\mathbb{A}_E)$, then there is a bijection

$$\begin{array}{rcl} z_t : \mathrm{Label}_{\mathfrak{a}}(G, \psi, C) & \xrightarrow{\sim} & \mathrm{Label}_{{}^t\mathfrak{a}}(G, z_t(\psi), C) \\ (\phi, b) & \longmapsto & (z_t(\phi), (\mathrm{loc}_{\mathfrak{a}}\phi)(t)b). \end{array}$$

More generally suppose that $D \supset E$ is another finite Galois extension of $\mathbb{Q}$, that $\mathfrak{a}_D \in \mathcal{H}(D/\mathbb{Q})$ and that $t \in T_{2,E}(\mathbb{A}_D)$ with $\eta_{D/E,*}\mathfrak{a}_D = {}^t\inf_{D/E}\mathfrak{a}$. Then there is a map

$$\begin{array}{rcl} \inf_{D/E,t} : \mathrm{Label}_{\mathfrak{a}}(G, \psi, C) & \longrightarrow & \mathrm{Label}_{\mathfrak{a}_D}(\inf_{D/E,t}(G, \psi, C)) \\ (\phi, b) & \longmapsto & (\inf_{3,D/E,t}\phi, (\mathrm{loc}_{\mathfrak{a}}\phi)(t)b). \end{array}$$

It induces a bijection

$$\mathrm{Label}_{\mathfrak{a}}(G, \psi, C)/\sim \xrightarrow{\sim} \mathrm{Label}_{\mathfrak{a}_D}(\inf_{D/E,t}(G, \psi, C))/\sim .$$

(Because, if $(\phi, b) \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$, then $\ker^1(\mathrm{Gal}(E/\mathbb{Q}), {}^{\phi}G(E)) \xrightarrow{\sim} \ker^1(\mathrm{Gal}(D/\mathbb{Q}), {}^{\phi}G(D))$, as E is acceptable for G.) We have

$$\inf_{D/E,t} \circ \mathrm{Label}_{\mathfrak{a}}(\phi, g, f) = \mathrm{Label}_{\mathfrak{a}_D}(\inf_{D/E,t}(\phi, g, f)) \circ \inf_{D/E,t}$$

and

$$\inf_{D'/D,t'} \circ \inf_{D/E,t} = \inf_{D'/E,t\eta_{D/E}(t')} .$$

9.3. **Rational Shimura varieties.** We now state and prove our second main theorem.

Theorem 9.1. *We have the following objects:*

(I) To any object (G,ψ,C) of $\mathrm{RSD}(E,\mathfrak{a};L)$ and any neat open compact subgroup $U\subset {}^{\psi}G(\mathbb{A}^\infty)$ we may associate a quasi-projective variety $\mathrm{Sh}(G,\psi,C)_U=\mathrm{Sh}_{E,\mathfrak{a}^+,L}(G,\psi,C)_U/L$, well defined up to canonical isomorphism.

(II) To any morphism $(\phi,g,f):(G_1,\psi_1,C_1)\to(G_2,\psi_2,C_2)$ in $\mathrm{RSD}(E,\mathfrak{a};L)$ and neat open compact subgroups $U_i\subset {}^{\psi_i}G_i(\mathbb{A}^\infty)$ with $\theta_{(\phi,g,f)}(U_1)\subset U_2$, we may associate a well defined morphism of varieties over L

$$\mathrm{Sh}(\phi,g,f)=\mathrm{Sh}_{E,\mathfrak{a}^+;L}(\phi,g,f):\mathrm{Sh}(G_1,\psi_1,C_1)_{U_1}\longrightarrow\mathrm{Sh}(G_2,\psi_2,C_2)_{U_2}$$

over L.

(III) To an embedding of fields $\tau:L\hookrightarrow L'$ we may associate well defined morphisms of varieties over L'

$$\Phi_{E,\mathfrak{a}^+}(\tau)=\Phi(\tau):{}^{\tau}\mathrm{Sh}(G,\psi,C)_U\xrightarrow{\sim}\mathrm{Sh}(G,\psi,{}^{\tau}C)_U.$$

(IV) To an embedding $\rho:L\hookrightarrow\mathbb{C}$ and $(\phi,\tau)\in\mathrm{Label}_{E,\mathfrak{a}}(G,\psi,{}^{\rho}C)$ we may associate an isomorphism of complex manifolds

$$\pi_{E,\mathfrak{a}^+;\rho,(\phi,b)}=\pi_{\rho,(\phi,b)}:{}^{\phi}G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}}\backslash(\widetilde{G}_\psi(\mathbb{A}^\infty)/U\times Y({}^{\rho}C)_{\phi G})\xrightarrow{\sim}{}^{\tau}\mathrm{Sh}(G,\psi,C)_U(\mathbb{C}).$$

(V) If $t\in T_{2,E}(\mathbb{A}_E)$ an isomorphism

$$\alpha_t:\mathrm{Sh}_{E,\mathfrak{a}^+}(G,\psi,C)_U\xrightarrow{\sim}\mathrm{Sh}_{E,{}^{t}\mathfrak{a}^+}(G,z_t(\psi),C)_U.$$

(VI) If $D\supset E$ is another finite Galois extension of $\mathbb{Q}$, if $\mathfrak{a}_D^+\in\mathcal{H}(D/\mathbb{Q})^+$ and if $t\in T_{2,E}(\mathbb{A}_D)$ with $\eta_{D/E,*}\mathfrak{a}_D^+={}^{t}\inf_{D/E}\mathfrak{a}^+$, an isomorphism

$$\alpha_t:\mathrm{Sh}_{E,\mathfrak{a}^+}(G,\psi,C)_U\xrightarrow{\sim}\mathrm{Sh}_{D,\mathfrak{a}_D^+}(\inf_{D/E}(G,\psi,C))_U.$$

These objects satisfy the following compatibilities.

(1) $\mathrm{Sh}(1,1,1)=\mathrm{Id}$ and $\mathrm{Sh}((\phi_2,g_2,f_2)\circ(\phi_1,g_1,f_1))=\mathrm{Sh}(\phi_2,g_2,f_2)\circ\mathrm{Sh}(\phi_1,g_1,f_1)$.

(2) If $z\in Z(G)(E)$ and $u\in U$ and $h\in{}^{\psi}G(\mathbb{R})$, then

$$\mathrm{Sh}({}^{z}1,z^{-1}uh,1):\mathrm{Sh}(G,\psi,C)_U\longrightarrow\mathrm{Sh}(G,\psi C)_U$$

is the identity. In particular $\widetilde{G}_\psi(\mathbb{A}^\infty)$ acts on the inverse system $\{\mathrm{Sh}(G,\psi,C)_U\}_U$.

(3) $\Phi(1)=\mathrm{Id}$ and $\Phi(\tau'\circ\tau)=\Phi(\tau')\circ{}^{\tau'}\Phi(\tau)$.

(4) $\Phi(\tau)\circ{}^{\tau}\mathrm{Sh}(\phi,g,f)=\mathrm{Sh}(\phi,g,f)\circ\Phi(\tau)$.

(5) If $\tau:L\hookrightarrow L'$ and $(\phi,b)\in\mathrm{Label}_{\mathfrak{a}}(G,\psi,{}^{\rho\tau}C)$, then ${}^{\rho}\Phi(\tau)\circ\pi_{\tau\rho,(\phi,b)}=\pi_{\rho,(\phi,b)}$.

(6) If $\widetilde{g}\in\widetilde{G}_\psi(\mathbb{A}^\infty)$, then $\mathrm{Sh}(\widetilde{g})\circ\pi_{\rho,(\phi_1,b_1)}$ equals the compositum of $\pi_{\rho,(\phi_1,b_1)}$ with right translation by $\widetilde{g}^{-1}$.

(7) If $\widetilde{k}\in\widetilde{G}_{1,\psi_1}(\mathbb{A}^\infty)_f$, then

$$\mathrm{Sh}(\phi,g,f)\circ\pi_{\rho,(\phi_1,b_1)}(\widetilde{k},\mu)=\pi_{\mathrm{Label}_{\mathfrak{a}}(\phi,g,f)(\phi_1,b_1)}(\widetilde{\theta}_{(\phi,g,f)}(\widetilde{k}),f\circ\mu).$$

(8) *If $G=T$ is a torus and $\tau \in \mathrm{Aut}(\mathbb{C})$, then*

$$(\tau \circ \pi_{\rho,(\phi,b)})(\widetilde{g},\mu) = \pi_{\tau\rho,(\phi_\tau \phi, b_\tau b)}(\widetilde{g}, {}^\tau\mu),$$

for any $(\tau,\phi_\tau,b_\tau) \in \mathrm{Conj}_{\mathfrak{a}}(T,\{\mu\})$ for which b_τ lifts $\overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau} \in T(\mathbb{A}_E^\infty)/\overline{T(\mathbb{Q})}T(E)$. Such a pair (ϕ_τ,b_τ) always exists.

(9) $\alpha_t \circ \mathrm{Sh}_{E,\mathfrak{a}^+}(\phi,g,f) = \mathrm{Sh}_{D,\mathfrak{a}_D^+}(\inf_{D/E,t}(\phi,g,f)) \circ \alpha_t$.

(10) $\alpha_t \circ \Phi_{E,\mathfrak{a}^+}(\tau) = \Phi_{D,\mathfrak{a}_D^+}(\tau)\circ\alpha_t$.

(11) $\alpha_t \circ \pi_{E,\mathfrak{a}^+;\rho,(\phi,b)} = \pi_{D,\mathfrak{a}_D^+;\rho,\inf_{D/E,t}(\phi,b)} \circ (\inf_{D/E,t}\times 1)$.

(12) *If $D' \supset D$ is another finite Galois extension of $\mathbb{Q}$, if $\mathfrak{a}_{D'}^+ \in \mathcal{H}(D'/\mathbb{Q})^+$ and if $t' \in T_{2,D}(\mathbb{A}_{D'})$ with $\eta_{D'/D,*}\mathfrak{a}_{D'}^+ = {}^{t'}\inf_{D'/D}\mathfrak{a}_D^+$; then $\eta_{D'/E,*}\mathfrak{a}_{D'}^+ = {}^{t\eta_{D/E}(t')}\inf_{D'/E}\mathfrak{a}^+$ and*

$$\alpha_{t\eta_{D/E}(t')} = \alpha_{t'}\circ\alpha_t.$$

Proof: A completely routine descent argument shows that we only need treat the case that L (and L') is $\mathbb{C}$. In case it helps the reader, we sketch the argument.

As a first reduction, we will show that it suffices to prove the theorem in the case that $L \subset E$. In the general case suppose that (G,ψ,C) is an object of $\mathrm{RSD}(E,\mathfrak{a}^+;L)$. Consider triples (L_0,τ,C_0) where $L_0 \subset E$ and $\tau_0 : L_0 \hookrightarrow L$ and C_0 is a conjugacy class of cocharacters of G defined over L_0 with ${}^\tau C_0 = C$. By definition some such triple exists. We make these into a category $\mathcal{C}$ by defining a morphism $\sigma : (L_0,\tau,C_0) \to (L_0',\tau',C_0')$ to be a map $\sigma: L_0 \to L_0'$ such that $\tau = \tau'\circ\sigma$. We set

$$\mathrm{Sh}(G,\psi,C)_{(L_0,\tau,C_0),U} = {}^\tau\mathrm{Sh}(G,\psi,C_0)_U.$$

If $\sigma : (L_0,\tau,C_0) \to (L_0',\tau',C_0')$ then

$${}^{\tau'}\Phi(\sigma) : \mathrm{Sh}(G,\psi,C)_{(L_0,\tau,C_0),U} \xrightarrow{\sim} \mathrm{Sh}(G,\psi,C)_{(L_0',\tau',C_0'),U}.$$

We set

$$\mathrm{Sh}(G,\psi,C)_U = \lim_{\rightarrow} \mathrm{Sh}(G,\psi,C)_{(L_0,\tau,C_0),U},$$

which is canonically defined and isomorphic to each $\mathrm{Sh}(G,\psi,C)_{(L_0,\tau,C_0)}$. It is easily verified that these varieties have the desired properties.

As a second reduction, we will show that it suffices to prove the theorem in the case $L=\mathbb{C}$. Indeed suppose that $L\subset E$ and (G,ψ,C) is an object of $\mathrm{RSD}(E,\mathfrak{a}^+;L)$. If $\rho: L\hookrightarrow \mathbb{C}$ then for any $\tau\in\mathrm{Aut}(\mathbb{C}/\rho L)$ we have

$$\Phi(\tau) : {}^\tau\mathrm{Sh}(G,\psi,{}^\rho C)_U \xrightarrow{\sim} \mathrm{Sh}(G,\psi,{}^\rho C)_U,$$

and these maps provide descent data, so we obtain a quasi-projective variety

$$\mathrm{Sh}(G,\psi,C)_{\rho,U}/{}^\rho L.$$

(See section 2.2 and recall from section 8.4 that the group of automorphisms of the variety $\mathrm{Sh}(G,\psi,{}^\rho C)$ is finite.) If $\rho' : L \hookrightarrow \mathbb{C}$ is a second embedding then $\rho' = \sigma\rho$ for some $\sigma \in \mathrm{Aut}\,(\mathbb{C})$. The element σ is not well defined, but the coset $\mathrm{Aut}\,(\mathbb{C}/\rho' L)\sigma$ is. Then

$$\Phi(\sigma) : {}^{\sigma}\mathrm{Sh}(G,\psi,{}^{\rho}C)_U \stackrel{\sim}{\to} \mathrm{Sh}(G,\psi,{}^{\rho'}C)_U$$

descends to

$$\Phi(\sigma) : {}^{\sigma}\mathrm{Sh}(G,\psi,{}^{\rho}C)_{\rho,U} \stackrel{\sim}{\to} \mathrm{Sh}(G,\psi,{}^{\rho'}C)_{\rho',U}$$

over ${}^{\rho'}L$, so that

$${}^{(\rho')^{-1}}\Phi(\sigma) : {}^{\rho^{-1}}\mathrm{Sh}(G,\psi,{}^{\rho}C)_{\rho,U} \stackrel{\sim}{\longrightarrow} {}^{(\rho')^{-1}}\mathrm{Sh}(G,\psi,{}^{\rho'}C)_{\rho',U}.$$

If we replace σ by $\tau\sigma$ with $\tau \in \mathrm{Aut}\,(\mathbb{C}/\rho' L)$, then ${}^{(\rho')^{-1}}\Phi(\sigma)$ is replaced by ${}^{(\rho')^{-1}}\Phi(\tau){}^{(\rho')^{-1}}\Phi(\sigma)$, but this equals ${}^{(\rho')^{-1}}\Phi(\sigma)$ because $\Phi(\tau)$ descends to the identity on $\mathrm{Sh}(G,\psi,{}^{\rho'}C)_{\rho',U}$. Thus we have a canonical isomorphism

$$\alpha_{\rho',\rho} : {}^{\rho^{-1}}\mathrm{Sh}(G,\psi,{}^{\rho}C)_{\rho,U} \stackrel{\sim}{\longrightarrow} {}^{(\rho')^{-1}}\mathrm{Sh}(G,\psi,{}^{\rho'}C)_{\rho',U}.$$

We define $\mathrm{Sh}(G,\psi,C)$ to be the limit over $\rho \in \mathrm{Hom}\,(L,\mathbb{C})$ of the ${}^{\rho^{-1}}\mathrm{Sh}(G,\psi,{}^{\rho}C)_{\rho,U}$ with respect to these maps. We also obtain maps $\mathrm{Sh}(\phi,g,f) : \mathrm{Sh}(G_1,\psi_1,C_1)_{U_1} \to \mathrm{Sh}(G_2,\psi_2,C_2)_{U_2}$ with the desired properties. If $\sigma : L \hookrightarrow L'$ and $\rho : L \hookrightarrow \mathbb{C}$ and $\rho' : L' \hookrightarrow \mathbb{C}$ we can find $\tau \in \mathrm{Aut}\,(\mathbb{C})$ with $\rho' \circ \sigma = \tau \circ \rho$, and τ is unique up to left multiplication by an element of $\mathrm{Aut}\,(\mathbb{C}/\rho L)$. We obtain

$$\Phi(\tau) : {}^{\tau}\mathrm{Sh}(G,\psi,{}^{\rho}C) \longrightarrow \mathrm{Sh}(G,\psi,{}^{\rho'\sigma}C)$$

which descends to

$$\Phi(\tau) : {}^{\tau\rho}\mathrm{Sh}(G,\psi,C) \longrightarrow {}^{\rho'}\mathrm{Sh}(G,\psi,{}^{\sigma}C)$$

over $\rho' L$, and hence we obtain

$${}^{(\rho')^{-1}}\Phi(\tau) : {}^{\sigma}\mathrm{Sh}(G,\psi,C) \longrightarrow \mathrm{Sh}(G,\psi,{}^{\sigma}C)$$

over L'. This can be checked to be independent of the choice of τ, and so we use it to define $\Phi(\sigma)$, which can be easily checked to have the desired properties.

So now assume that $L = \mathbb{C}$. If $(\phi,b) \in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)$ we define

$$\mathrm{Sh}(G,\psi,C)_{U,(\phi,b)} = \mathrm{Sh}({}^{\phi}G, Y(C)_{{}^{\phi}G})_{bUb^{-1}}.$$

Up to canonical isomorphism this only depends on the equivalence class of (ϕ,b). Indeed if $\gamma \in G(E)$ and $h \in {}^{\psi}G(\mathbb{A}^{\infty})$, then

$$\mathrm{Sh}(\mathrm{conj}_{\gamma b}(h)^{-1},\mathrm{conj}_{\gamma}) : \mathrm{Sh}(G,\psi,C)_{U,(\phi,b)} \stackrel{\sim}{\longrightarrow} \mathrm{Sh}(G,\psi,C)_{U,({}^{\gamma}\phi,\gamma bh^{-1})}.$$

If we replace (γ,h) by $(\gamma\delta, h\mathrm{conj}_{b^{-1}}(\delta))$ with $\delta \in {}^{\phi}G(\mathbb{Q})$ then this map is replaced by

$$\begin{aligned}\mathrm{Sh}(\mathrm{conj}_{\gamma\delta b}(hb^{-1}\delta b)^{-1},\mathrm{conj}_{\gamma\delta}) &= \mathrm{Sh}(\gamma bh^{-1}b^{-1}\delta^{-1}\gamma^{-1},\mathrm{conj}_{\gamma\delta}) \\ &= \mathrm{Sh}(\gamma bh^{-1}b^{-1}\gamma^{-1},\mathrm{conj}_{\gamma}) \circ \mathrm{Sh}(\delta^{-1},\mathrm{conj}_{\delta}) \\ &= \mathrm{Sh}(\mathrm{conj}_{\gamma b}(h)^{-1},\mathrm{conj}_{\gamma}).\end{aligned}$$

Thus we have a canonical isomorphism

$$\alpha_{(\phi,b),({}^{\gamma}\phi,\gamma bh^{-1})} : \mathrm{Sh}(G,\psi,C)_{U,(\phi,b)} \stackrel{\sim}{\longrightarrow} \mathrm{Sh}(G,\psi,C)_{U,({}^{\gamma}\phi,\gamma bh^{-1})}$$

which is independent of the choice of (γ, h), and $\mathrm{Sh}(G,\psi,C)_{U,(\phi,b)}$ only depends on $[(\phi,b)]$ up to canonical isomorphism. Thus we can define

$$\mathrm{Sh}(G,\psi,C)_U = \coprod_{[(\phi,b)]\in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)/\sim} \mathrm{Sh}(G,\psi,C)_{U,(\phi,b)}$$

and it is well defined up to canonical isomorphism. As the union is finite, $\mathrm{Sh}(G,\psi,C)_U$ is a quasi-projective variety.

Now suppose that $(\phi,g,f):(G_1,\psi_1,C_1)\to(G_2,\psi_2,C_2)$ and that $\theta_{(\phi,g,f)}(U_1)\subset U_2$. Then we define

$$\mathrm{Sh}(\phi,g,f)|_{\mathrm{Sh}(G_1,\psi_1,C_1)_{U_1,(\phi_1,b_1)}} = \mathrm{Sh}(1,f):\mathrm{Sh}(G_1,\psi_1,C_1)_{U_1,(\phi_1,b_1)} \longrightarrow \mathrm{Sh}(G_2,\psi_2,C_2)_{U_2,((f\circ\phi_1)\phi,f(b_1)g^{-1})}.$$

This is well defined independent of the choice of representatives (ϕ_1,b_1) because

$$\mathrm{Sh}(1,f)\circ\mathrm{Sh}(\mathrm{conj}_{\gamma b_1}(h)^{-1},\mathrm{conj}_\gamma) = \mathrm{Sh}(\mathrm{conj}_{f(\gamma)f(b_1)g^{-1}}(gf(h)g^{-1})^{-1},\mathrm{conj}_{f(\gamma)})\circ\mathrm{Sh}(1,f).$$

We have $\mathrm{Sh}(1,1,1)=\mathrm{Id}$ and $\mathrm{Sh}((\phi',g_2,f_2)\circ(\phi,g_1,f_1)) = \mathrm{Sh}(\phi',g_2,f_2)\circ\mathrm{Sh}(\phi,g_1,f_1)$. To verify the latter suppose that $(\phi_1,b_1)\in \mathrm{Label}_{\mathfrak{a}}(G_1,\psi_1,C_1)$. Then we have to verify that $\mathrm{Sh}(1,f_2)\circ\mathrm{Sh}(1,f_1)=\mathrm{Sh}(1,f_1f_2)$ as maps

$$\begin{array}{rcl} \mathrm{Sh}(G_1,\psi_1,C_1)_{U_1,(\phi_1,b_1)} & \longrightarrow & \mathrm{Sh}(G_3,\psi_3,C_3)_{U_3,((f_2\circ((f_1\circ\phi_1)\phi))\phi',f_2(f_1(b_1)g_1^{-1})g_2^{-1})} \\ & = & \mathrm{Sh}(G_3,\psi_3,C_3)_{U_3,((f_2\circ f_1\circ\phi_1)((f_2\circ\phi)\phi'),(f_2\circ f_1)(b_1)(g_2f(g_1))^{-1})} \end{array},$$

which is clear.

We must check that if $z\in Z(G)(E)$ and $u\in U$ and $h\in {}^\psi G(\mathbb{R})$ then $\mathrm{Sh}({}^z1,z^{-1}uh,1)=\mathrm{Id}$. Indeed its restriction to $\mathrm{Sh}(G,\psi,C)_{U,(\phi_1,b_1)}$ is

$$\mathrm{Sh}(1,1)=\mathrm{Id}:\mathrm{Sh}({}^{\phi_1}G,Y(C)_{\phi_1G})_{b_1Ub_1^{-1}} \longrightarrow \mathrm{Sh}({}^{\phi_1}G,Y(C)_{\phi_1G})_{b_1Ub_1^{-1}} = \mathrm{Sh}(G,\psi,C)_{U,(\phi_1{}^z1,b_1h^{-1}u^{-1}z)}.$$

thus it suffices to check that

$$\alpha_{(\phi_1,b_1),({}^z\phi_1,b_1h^{-1}u^{-1}z)} = \mathrm{Sh}(\mathrm{conj}_{zb_1}(h^{-1}u^{-1}),\mathrm{conj}_z) = \mathrm{Sh}(b_1h^{-1}u^{-1}b_1^{-1},1)$$

is the identity on $\mathrm{Sh}({}^{\phi_1}G,Y(C)_{\phi_1G})_{b_1Ub_1^{-1}}$, which it is.

Next suppose that $\tau\in\mathrm{Aut}(\mathbb{C})$. We simply define

$$\Phi(\tau) = \coprod_{[(\phi_1,b_1)]\in\mathrm{Label}_{\mathfrak{a}}(G,\psi,C)/\sim} \Phi_{E,\mathfrak{a}^+}(\tau,\phi,b): {}^\tau\mathrm{Sh}(G,\psi,C)_{U,(\phi_1,b_1)} \xrightarrow{\sim} \mathrm{Sh}(G,\psi,{}^\tau C)_{U,(\phi\phi_1,bb_1)}$$

for any $(\tau,\phi,b)\in\mathrm{Conj}_{E,\mathfrak{a}}({}^{\phi_1}G,Y(C)_{\phi_1G})$. This is independent of the choice of (τ,ϕ,b) because if $\gamma\in G(E)$ and $h\in{}^{\phi_1}G(\mathbb{A}^\infty)$, then

$$\mathrm{Sh}(\mathrm{conj}_{\gamma bb_1}(b_1^{-1}h^{-1}b_1),\mathrm{conj}_\gamma)\circ\Phi(\tau,\phi,b) = \Phi(\tau,{}^\gamma\phi,\gamma bh^{-1}),$$

as both sides equals $\mathrm{Sh}(1,\mathrm{conj}_\gamma)\circ\Phi(\tau,\phi,b)\circ{}^\tau\mathrm{Sh}(h^{-1},1)$. It is also independent of the choice of representatives (ϕ_1,b_1) because, if $\gamma\in G(E)$ and $h\in{}^\psi G(\mathbb{A}^\infty)$ then

$$\Phi(\tau,\mathrm{conj}_\gamma\circ\phi,\mathrm{conj}_\gamma(b))\circ{}^\tau\alpha_{(\phi_1,b_1),({}^\gamma\phi_1,\gamma b_1h^{-1})} = \alpha_{(\phi\phi_1,bb_1),((\mathrm{conj}_\gamma\circ\phi){}^\gamma\phi_1,\gamma bb_1h^{-1})}\circ\Phi(\tau,\phi,b).$$

To see this, note that $(\mathrm{conj}_\gamma \circ \phi)^\gamma \phi_1 = {}^\gamma(\phi\phi_1)$ and decode the equality to to

$$\Phi(\tau, \mathrm{conj}_\gamma \phi, \gamma b \gamma^{-1}) \circ {}^\tau\mathrm{Sh}(\mathrm{conj}_{\gamma b_1}(h^{-1}), \mathrm{conj}_\gamma) = \mathrm{Sh}(\mathrm{conj}_{\gamma b b_1}(h^{-1}), \mathrm{conj}_\gamma) \circ \Phi(\tau, \phi, b),$$

which holds.

We have $\Phi(1) = \mathrm{Id}$ and $\Phi(\tau'\tau) = \Phi(\tau') \circ {}^{\tau'}\Phi(\tau)$. The latter because if $(\tau, \phi, b) \in \mathrm{Conj}\,({}^{\phi_1}G, Y(C)_{{}^{\phi}_{1}G})$ and $(\tau', \phi', b') \in \mathrm{Conj}\,({}^{\phi\phi_1}G, Y({}^\tau C)_{{}^{\phi\phi_1}G})$, then $(\tau'\tau, \phi'\phi, b'b) \in \mathrm{Conj}\,({}^{\phi_1}G, Y(C)_{{}^{\phi_1}G})$ and

$$\Phi(\tau'\tau, \phi'\phi, b'b) = \Phi(\tau', \phi', b') \circ {}^{\tau'}\Phi(\tau, \phi, b).$$

We must also check that $\Phi(\tau) \circ {}^\tau\mathrm{Sh}(\phi, g, f)) = \mathrm{Sh}(\phi, g, f) \circ \Phi(\tau)$. Indeed we will consider the restriction of both sides to

$${}^\tau\mathrm{Sh}(G_1, \psi_1, C_1)_{U_1,(\phi_1,b_1)} = {}^\tau\mathrm{Sh}({}^{\phi_1}G_1, Y(C_1)_{{}^{\phi_1}G_1})_{b_1 U_1 b_1^{-1}}.$$

Choose $(\phi', b') \in \mathrm{Conj}\,({}^{\phi_1}G_1, Y(C_1)_{{}^{\phi_1}G_1})$, so that $(f\circ\phi', f(b')) \in \mathrm{Conj}\,({}^{(f\circ\phi_1)\phi}G_2, Y(C_2)_{{}^{(f\circ\phi_1)\phi}G_2})$. then we are required to check that

$$\Phi(\tau, f\circ\phi', f(b')) \circ {}^\tau\mathrm{Sh}(1, f) = \mathrm{Sh}(1, f) \circ \Phi(\tau, \phi', b')$$

as maps

$$\begin{array}{rcl} {}^\tau\mathrm{Sh}({}^{\phi_1}G_1, Y(C_1)_{{}^{\phi_1}G_1})_{b_1 U_1 b_1^{-1}} & \longrightarrow & \mathrm{Sh}({}^{(f\circ(\phi'\phi_1))\phi}G_2, Y(C_2)_{{}^{(f\circ(\phi'\phi_1))\phi}G_2})_{f(b'b_1)g^{-1}U_2 g f(b'b_1)^{-1}} \\ & = & \mathrm{Sh}(G_2, \psi_2, C_2)_{U_2,((f\circ(\phi'\phi_1))\phi, f(b'b_1)g^{-1})}. \end{array}$$

This is true.

If $(\phi, b) \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$, then we define a map of complex analytic spaces

$$\begin{array}{rcl} \pi_{1,(\phi,b)} : {}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}} \backslash (\widetilde{G}_\psi(\mathbb{A}^\infty)/U \times Y(C)_{{}^\phi G}) & \longrightarrow & \mathrm{Sh}(G, \psi, C)_U(\mathbb{C}) \\ (\widetilde{g}, \mu) & \longmapsto & \mathrm{Sh}(\widetilde{g}^{-1})^\phi G(\mathbb{Q})(b\widetilde{g}U\widetilde{g}^{-1}b^{-1}, \mu), \end{array}$$

where ${}^\phi G(\mathbb{Q})(b\widetilde{g}U\widetilde{g}^{-1}b^{-1}, \mu) \in \mathrm{Sh}(G, \psi, C)_{\widetilde{g}U\widetilde{g}^{-1},(\phi,b)}(\mathbb{C})$. To see it is well defined, suppose that $\gamma \in {}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}}$ has lifts $\widetilde{\gamma} \in {}^\phi G(E)$ and $\widehat{\gamma} \in {}^\phi G(\mathbb{R})$, so that $i_{(\phi,b)}(\gamma^{-1}) = ({}^{\widetilde{\gamma}}1, (b^{-1}\widetilde{\gamma}^{-1}b, \widetilde{\gamma}^{-1}\widehat{\gamma}))$. Then we must check that

$$\mathrm{Sh}({}^{\widetilde{\gamma}}1, (b^{-1}\widetilde{\gamma}^{-1}b, \widetilde{\gamma}^{-1}\widehat{\gamma}), 1)({}^\phi G(\mathbb{Q})(b(b^{-1}\widetilde{\gamma}b)\widetilde{g}U\widetilde{g}^{-1}(b^{-1}\widetilde{\gamma}b)^{-1}b^{-1}, \mathrm{conj}_{\widehat{\gamma}} \circ \mu))$$

equals $({}^\phi G(\mathbb{Q})(b\widetilde{g}U\widetilde{g}^{-1}b^{-1}, \mu))$ in $\mathrm{Sh}(G, \psi, C)_U(\mathbb{C})$. However the former of these equals

$$({}^\phi G(\mathbb{Q})(\widetilde{\gamma}b\widetilde{g}U\widetilde{g}^{-1}b^{-1}\widetilde{\gamma}^{-1}, \mathrm{conj}_{\widehat{\gamma}} \circ \mu)) \in \mathrm{Sh}(G, \psi, C)_{U,({}^{\widetilde{\gamma}}\phi, \widetilde{\gamma}b)}(\mathbb{C}),$$

so we must check that

$$\alpha_{({}^{\widetilde{\gamma}}\phi,\widetilde{\gamma}b),(\phi,b)}({}^\phi G(\mathbb{Q})(\widetilde{\gamma}b\widetilde{g}U\widetilde{g}^{-1}b^{-1}\widetilde{\gamma}^{-1}, \mathrm{conj}_{\widehat{\gamma}}\circ\mu)) = ({}^\phi G(\mathbb{Q})(b\widetilde{g}U\widetilde{g}^{-1}b^{-1}, \mu)) \in \mathrm{Sh}(G, \psi, C)_U(\mathbb{C}),$$

i.e. that

$$\mathrm{Sh}(1, \mathrm{conj}_{\widetilde{\gamma}^{-1}})({}^\phi G(\mathbb{Q})(\widetilde{\gamma}b\widetilde{g}U\widetilde{g}^{-1}b^{-1}\widetilde{\gamma}^{-1}, \mathrm{conj}_{\widehat{\gamma}}\circ\mu)) = ({}^\phi G(\mathbb{Q})(b\widetilde{g}U\widetilde{g}^{-1}b^{-1}, \mu)) \in \mathrm{Sh}(G, \psi, C)_U(\mathbb{C}),$$

which is clear.

We next prove that $\pi_{1,(\phi,b)}$ is an isomorphism.

Note that if $\widetilde{g} = [(\zeta, g)] \in \widetilde{G}_\psi(\mathbb{A}^\infty)$, then

$$\mathrm{Sh}(\widetilde{g}^{-1}) : \mathrm{Sh}(G, \psi, C)_{U,(\phi,b)} \xrightarrow{\sim} \mathrm{Sh}(G, \psi, C)_{\widetilde{g}^{-1}U\widetilde{g},(\zeta^{-1}\phi,bg)}.$$

Thus $\widetilde{G}_\psi(\mathbb{A}^\infty)$ acts on $\mathrm{Label}_{\mathfrak{a}}(G, \psi, C)/\sim$ via

$$[(\zeta, g)] : [(\phi, b)] \mapsto [(\zeta^{-1}\phi, bg)].$$

This action factors through the abelian group

$$\ker(H^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^\psi G(\mathbb{A}_E)).$$

We claim this action is transitive. Suppose that (ϕ, b) and $(\phi', b') \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$, then $\mathrm{loc}[\mathrm{ad}\,\phi] = \mathrm{loc}[\mathrm{ad}\,\phi'] \in H^1(\mathrm{Gal}\,(E/\mathbb{Q}), G^{\mathrm{ad}}(\mathbb{A}_E))$ and so $[\mathrm{ad}\,\phi] = [\mathrm{ad}\,\phi'] \in H^1(\mathrm{Gal}\,(E/\mathbb{Q}), G^{\mathrm{ad}}(E))$. Thus there is $[\zeta] \in H^1_{\mathrm{alg}}(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E))$ such that $[\zeta][\phi] = [\phi']$. Moreover $\mathrm{loc}[\zeta\phi] = \mathrm{loc}[\phi] \in H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q}), G(\mathbb{A}_E))_{\mathrm{basic}}$. Thus if $v|\infty$ then $\mathrm{res}_v\mathrm{loc}[\zeta] = o(\gamma)$ for some $\gamma \in {}^\phi G^{\mathrm{ad}}(\mathbb{R})$. By theorem 7.8 of [PR] we may take $\gamma \in {}^\phi G^{\mathrm{ad}}(\mathbb{Q})$. Then $[\phi] = [\phi]o(\gamma)^{-1}$ and $[\phi'] = [\zeta'\phi] \in H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}}$, where $\zeta' = \zeta o(\gamma)^{-1} \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E))$. Moreover $\mathrm{res}_v\mathrm{loc}[\zeta'] = 1 \in H^1(\mathrm{Gal}\,(E_v/\mathbb{R}), Z(G)(E_v))$. Thus

$$[\zeta'] \in \ker(H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), Z(G)(E)) \to H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q}), {}^\psi G(\mathbb{A}_E))),$$

and we can find $g \in G(\mathbb{A}_E)$ such that $\mathrm{loc}_{\mathfrak{a}}\zeta' = {}^g 1 \in Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}}, {}^\psi G(\mathbb{A}_E))$, i.e. $(\zeta', g) \in \widetilde{G}_\psi(\mathbb{A}^\infty)$. We can also find $\gamma \in G(E)$ with ${}^\gamma\phi' = \zeta'\phi$. We have $[(\zeta, g)]([(\phi', b')]) = [(\phi, \gamma b' g)] = [(\phi, b)]$.

As the action of $\widetilde{G}_\psi(\mathbb{A}^\infty)$ on $\mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$ is transitive and factors through an abelian group, the stabilizer of each point is equal. We will denote it $\widetilde{G}_\psi(\mathbb{A}^\infty)^1$. Note that $\widetilde{G}_\psi(\mathbb{A}^\infty)^1 \supset i_{(\phi,b)}{}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}}$ and $\widetilde{G}_\psi(\mathbb{A}^\infty)^1 \supset {}^\psi G(\mathbb{A}^\infty)/\overline{Z(G)(\mathbb{Q})}$. We have an exact sequence

$$\begin{array}{l}(0) \longrightarrow {}^\psi G(\mathbb{A}^\infty)/\overline{Z(G)(\mathbb{Q})} \longrightarrow \widetilde{G}_\psi(\mathbb{A}^\infty)^1 \longrightarrow \\ \ker(H^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^\phi G(E)) \oplus H^1(\mathrm{Gal}\,(\mathbb{C}/\mathbb{R}), {}^\phi G(\mathbb{C}))) \longrightarrow (0),\end{array}$$

and hence an exact sequence

$$(0) \longrightarrow {}^\psi G(\mathbb{A}^\infty)/\overline{Z(G)(\mathbb{Q})} \longrightarrow \widetilde{G}_\psi(\mathbb{A}^\infty)^1 \longrightarrow {}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}}/{}^\phi G(\mathbb{Q})^{\mathrm{ad}} \longrightarrow (0).$$

The composite map

$${}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}} \xrightarrow{i_{(\phi,b)}} \widetilde{G}_\psi(\mathbb{A}^\infty)^1 \twoheadrightarrow {}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}}/{}^\phi G(\mathbb{Q})^{\mathrm{ad}}$$

is the natural projection, so that

$$\widetilde{G}_\psi(\mathbb{A}^\infty)^1 = {}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}}{}^\psi G(\mathbb{A}^\infty)$$

and

$${}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}} \cap {}^\psi G(\mathbb{A}^\infty) = G(\mathbb{Q})^{\mathrm{ad}}.$$

If we write

$$\widetilde{G}_\psi(\mathbb{A}^\infty) = \coprod_i \widetilde{G}_\psi(\mathbb{A}^\infty)^1[(\zeta_i, g_i)]$$

then

$${}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}} \backslash (\widetilde{G}_\psi(\mathbb{A}^\infty)/U \times Y(C)_{{}^\phi G}) = \coprod_i {}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}} \backslash (\widetilde{G}_\psi(\mathbb{A}^\infty)^1/\mathrm{conj}_{g_i}(U) \times Y(C)_{{}^\phi G})$$

and $\pi_{1,(\phi,b)}$ is a disjoint union of maps

$$\begin{array}{rcl} {}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}} \backslash (\widetilde{G}_\psi(\mathbb{A}^\infty)^1/\mathrm{conj}_{g_i}(U) \times Y(C)_{{}^\phi G}) & \xrightarrow{\pi_{1,(\phi,b)}} & \mathrm{Sh}(G,\psi,C)_{\mathrm{conj}_{g_i}(U),(\phi,b)}(\mathbb{C}) \\ & \xrightarrow{\mathrm{Sh}([(\zeta_i^{-1}, g_i^{-1})])} & \mathrm{Sh}(G,\psi,C)_{\mathrm{conj}_{g_i}(U),(\zeta_i^{-1}\phi, bg_i)}(\mathbb{C}). \end{array}$$

The latter map is an isomorphism. Thus to show that $\pi_{1,(\phi,b)}$ is an isomorphism, it suffices to show that

$$\pi_{1,(\phi,b)} : {}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}} \backslash (\widetilde{G}_\psi(\mathbb{A}^\infty)^1/\mathrm{conj}_{g_i}(U) \times Y(C)_{{}^\phi G}) \longrightarrow \mathrm{Sh}({}^\phi G, Y(C)_{{}^\phi G})_{\mathrm{conj}_{bg_i}(U)}(\mathbb{C})$$

is an isomorphism. However

$$\begin{array}{rcl} {}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}} \backslash (\widetilde{G}_\psi(\mathbb{A}^\infty)^1/\mathrm{conj}_{g_i}(U) \times Y(C)_{{}^\phi G}) & = & {}^\phi G(\mathbb{Q})^{\mathrm{ad}} \backslash ({}^\psi G(\mathbb{A}^\infty)/\mathrm{conj}_{g_i}(U) \times Y(C)_{{}^\phi G}) \\ & \cong & {}^\phi G(\mathbb{Q})^{\mathrm{ad}} \backslash ({}^\phi G(\mathbb{A}^\infty)/\mathrm{conj}_{bg_i}(U) \times Y(C)_{{}^\phi G}) \end{array}$$

and the map

$${}^\phi G(\mathbb{Q})^{\mathrm{ad}} \backslash ({}^\phi G(\mathbb{A}^\infty)/\mathrm{conj}_{bg_i}(U) \times Y(C)_{{}^\phi G}) \longrightarrow \mathrm{Sh}({}^\phi G, Y(C)_{{}^\phi G})_{\mathrm{conj}_{bg_i}(U)}(\mathbb{C})$$

is the usual isomorphism.

In conclusion, we have defined an isomorphism of complex manifolds:

$$\pi_{1,(\phi,b)} : {}^\phi G^{\mathrm{ad}}(\mathbb{Q})_{E,\mathbb{R}} \backslash (\widetilde{G}_\psi(\mathbb{A}^\infty)/U \times Y(C)_{{}^\phi G}) \xrightarrow{\sim} \mathrm{Sh}(G,\psi,C)_U(\mathbb{C}).$$

If $\rho \in \mathrm{Aut}(\mathbb{C})$ we define

$$\pi_{\rho,(\phi,b)} = \Phi(\rho)^{-1} \circ \pi_{1,(\phi,b)},$$

which makes property (5) trivially true. Properties (6) and (7) follow immediately from the definitions.

To verify property (8) it suffices to treat the case $\widetilde{g} = 1$ and $\rho = 1$. Then $\pi_{1(\phi,b)}(1,\mu)$ is represented by $\Pi_{T,\{\mu\}}(1) \in \mathrm{Sh}(T,\{\mu\})_U = \mathrm{Sh}(T,\psi,\{\mu\})_{(\phi,b)}$ and $\tau \circ \pi_{1,(\phi,b)}(1,\mu)$ is represented by $\Phi(\tau,\phi_\tau,b_\tau)^{-1}\Pi_{T,\{{}^\tau\mu\}}(1) \in {}^\tau\mathrm{Sh}(T,\{\mu\})_U = {}^\tau\mathrm{Sh}(T,\psi,\{\mu\})_{(\phi,b)}$. On the other hand $\pi_{\tau,(\phi_\tau\phi,b_\tau b)}(1,{}^\tau\mu) = \Phi(\tau)^{-1}\pi_{1,(\phi_\tau\phi,b_\tau b)}(1,{}^\tau\mu)$ is represented by $\Phi(\tau)^{-1}\Pi_{T,\{{}^\tau\mu\}}(1)$, where $\Pi_{T,\{{}^\tau\mu\}}(1) \in \mathrm{Sh}(T,\{{}^\tau\mu\})_U = \mathrm{Sh}(T,\psi,\{{}^\tau\mu\})_{U,(\phi_\tau\phi,b_\tau b)}(\mathbb{C})$. Thus $\pi_{\tau,(\phi_\tau\phi,b_\tau b)}(1,{}^\tau\mu)$ is represented by $\Phi_{\mathfrak{a}^+}(\tau,\phi_\tau,b_\tau)^{-1}\Pi_{T,\{{}^\tau\mu\}}(1)$ in ${}^\tau\mathrm{Sh}(T,\{\mu\})_U = {}^\tau\mathrm{Sh}(T,\psi,\{\mu\})_{U,(\phi,b)}$, and part (8) follows.

In the setting of part VI we define

$$\alpha_t : \mathrm{Sh}_{E,\mathfrak{a}^+}(G,\psi,C)_U \xrightarrow{\sim} \mathrm{Sh}_{D,\mathfrak{a}_D^+}(G, \operatorname*{inf}^{\mathrm{loc}}_{D/E,t}(\psi), C)_U$$

to be the disjoint union of the maps

$$\begin{array}{ccc} \mathrm{Sh}_{E,\mathfrak{a}^+}(G,\psi,C)_{U,(\phi,b)} & \stackrel{\sim}{\longrightarrow} & \mathrm{Sh}_{D,\mathfrak{a}_D^+}(G,\inf_{D/E,t}^{\mathrm{loc}}(\psi),C)_{U,(\inf_{3,D/E,t}(\phi),(\mathrm{loc}_{\mathfrak{a}}\phi)(t)b)} \\ || & & || \\ \mathrm{Sh}({}^{\phi}G,Y(C)_{{}^{\phi}G})_{bUb^{-1}} & = & \mathrm{Sh}({}^{\inf_{3,D/E,t}(\phi)}G,Y(C)_{{}^{\inf_{3,D/E,t}(\phi)}G})_{(\mathrm{loc}_{\mathfrak{a}}\phi)(t)bUb^{-1}(\mathrm{loc}_{\mathfrak{a}}\phi)(t)^{-1})}. \end{array}$$

(Recall that $(\mathrm{loc}_{\mathfrak{a}}\phi)(t) \in Z({}^{\psi}G)(\mathbb{A}_E)$.) Properties (9) and (12) are immediate, while property (10) follows from part (7) of theorem 8.5. By part (12) it suffices to check property (11) in the case $\rho = 1$. In this case it follows easily from property (9) and the fact that the restriction of $\inf_{D/E,t} : \widetilde{G}_{E,(G,\psi,C)}(\mathbb{A}^{\infty}) \to \widetilde{G}_{D,\inf_{D/E,t}(G,\psi,C)}(\mathbb{A}^{\infty})$ to ${}^{\psi}G(\mathbb{A}^{\infty})$ equals the identity.

□

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Email address: j.sempliner@imperial.ac.uk

Imperial College, London

Email address: rltaylor@stanford.edu

Stanford University