Brunet-Derrida velocity shift for branching-selection particle systems

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[1] Joint work with Jean-Baptiste Gouéré

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$$X_n = (X_n^1, \dots, X_n^N)$$

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- QCD (!) : (Munier and Peschanski 2003)

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Examples :

- $\mu = p\delta_1 + (1-p)\delta_0$, $0 < p < 1/2$
- $\mu = \mathcal{N}(0, 1)$
- $\mu = Unif(0, 1)$

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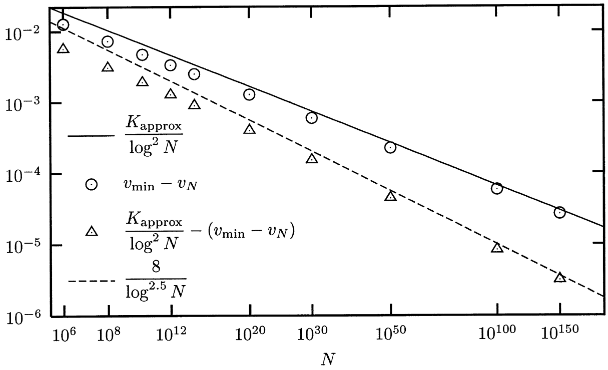
The previous examples satisfy the assumptions of the Theorem.
No t^* for $\mu = p\delta_1 + (1-p)\delta_0$ when $p \geq 1/2$. The conclusion of the Theorem does not hold either.

Brunet-Derrida velocity shift

Numerical simulations (Brunet and Derrida 2001)



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- Given n, as $N \to +\infty$,

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{X_n^i} \xrightarrow{P} c_n \text{ probability measure on } \mathbb{R}$$

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- Setting $g_n(x) := c_n(]x, +\infty[)$, we obtain the equation :

$$g_{n+1} = \Psi(g_n)$$

$$\Psi(g) := \min(2g \star \mu, 1).$$

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- The equation rewrites :

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- Comparison with F-KPP :

$$\frac{\partial g}{\partial t} = \Delta g + \Phi(g)$$

$$\Phi(g) = g(1 - g)$$

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(Brunet and Derrida 1997, 1999, Benguria and al. 2007, Dumortier and al. 2007)

One obtains the $v_\infty - v_N \sim_{N \to +\infty} C(\log N)^{-2}$ behavior with these truncated equations.

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- Other model : F-KPP equation with stochastic noise

$$\frac{\partial u}{\partial t} = \Delta u + u(1-u) + \sqrt{\frac{u(1-u)}{N}}\dot{W}$$

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The population size may vary and extinction is possible

Properties of the model

Theorem (Kingman 1975, Biggins 1977)

Without killing ($\alpha = -\infty$),

$$\lim_{n \to +\infty} \frac{\max Z_n}{n} = v_\infty, \quad v_\infty := \Lambda'(t^*)$$

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Theorem (Gantert, Hu and Shi 2008)

Let $\rho(\infty, \epsilon)$ be the survival probability with $\alpha := v_\infty - \epsilon$.

$$\rho(\infty, \epsilon) = \exp\left(- \left[\frac{C + o(1)}{\epsilon} \right]^{1/2} \right).$$

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- If $\gamma_N \sim_{N \to +\infty} C_1(\log N)^{-2}$ with $C_1 < C$, $\rho(\infty, \gamma_N) << 1/N$

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- "The growth speed of $\max X_n$ cannot exceed $v_\infty - \gamma_N$"
- If $\delta_N \sim_{N \to +\infty} C_2(\log N)^{-2}$ with $C_2 > C$, $\rho(\infty, \delta_N) >> 1/N$

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- "The growth speed of $\max X_n$ cannot exceed $v_\infty - \gamma_N$"
- If $\delta_N \sim_{N \to +\infty} C_2(\log N)^{-2}$ with $C_2 > C$, $\rho(\infty, \delta_N) >> 1/N$
- "The growth speed of $\min X_n$ cannot be less than $v_\infty - \delta_N$"

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- with $\gamma_N \sim C_1(\log N)^{-2}$, $C_1 < C$, and $m_N := A(\log N)^3$, $A >> 1$,

$$\rho(m_N, \gamma_N) \approx \rho(\infty, \gamma_N)$$

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- with $D_1 < C_1$ and $n >> m_N$,

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implies with high probability that at least one event of probability $\rho(m_N, \gamma_N)$ occurs

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- first moment bound

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- with $\delta_N \sim C_2 (\log N)^{-2}$, $C_2 > C$, and $\alpha := v_\infty - \delta_N$, the probability that none of the N BRW survives is $<< 1$
- with $D_2 > C_2$, $m_N := A(\log N)^3$,

$$\min X_n \le n(v_\infty - D_2(\log N)^{-2}) \text{ for all } 0 \le n \le m_N$$

implies with high probability that none of the N BRW survives

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- One really looks at :

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- "whence" $\log \rho(m_N, \epsilon_N) \propto -\epsilon_n^{-1/2}$

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- when ϵ is small, do the perturbative study of the corresponding linearized equation

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- comparison yields the asymptotics of $\rho(x, \infty, \epsilon)$ for small ϵ

Perspectives

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- Genealogies (convergence to the Bolthausen-Sznitman coalescent ?) (J. Berestycki, N. Berestycki, and J. Schweinsberg 2010)