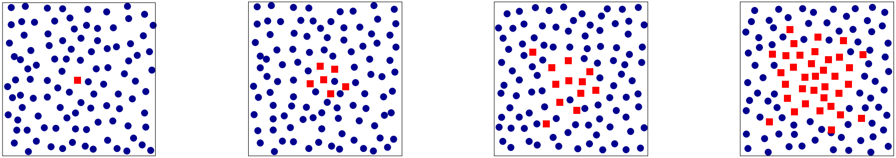
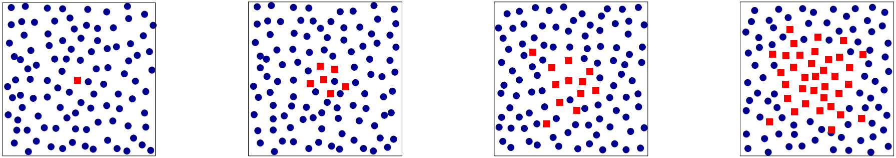


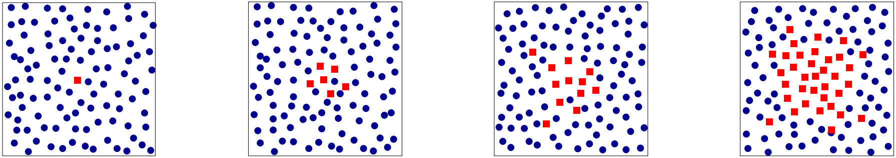
The Fisher-KPP Equation and other Pulled Fronts

Éric Brunet

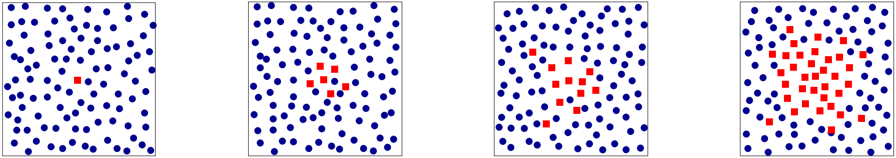
Laboratoire de Physique Statistique, É.N.S., UPMC, Paris

Banff 2010





A and B diffuse, $\qquad A + B \to 2A$



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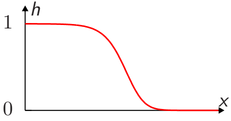
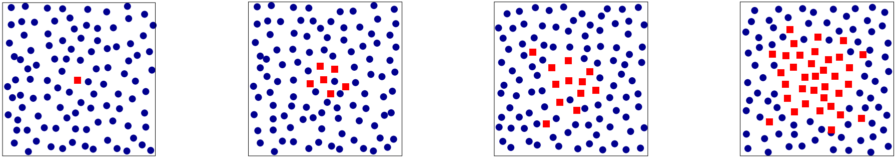
Let $h(x,t) =$ proportion of A around x at time t





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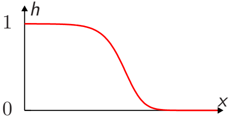
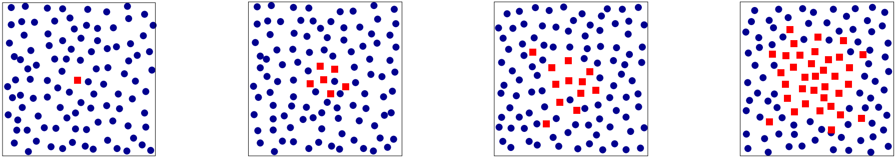


In the limit of infinite concentration:

$$\partial_t h = \partial_x^2 h + h(1-h) \qquad \text{Fisher-KPP equation}$$



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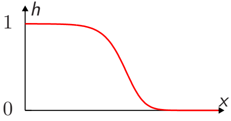

In the limit of infinite concentration:

$$\partial_t h = \partial_x^2 h + h(1-h) \qquad \text{Fisher-KPP equation}$$

For large but finite concentration:

$$\partial_t h = \partial_x^2 h + h(1-h) + (\text{small noise term}) \qquad \text{Stochastic Fisher-KPP equation}$$

Before starting...

I am a physicist

There won't be any rigorous proof

but only. . .

Heuristics Arguments Ideas Hand-waving

Conjectures Theories Plausible explanations Intuitions

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Diffusive processes

$$\partial_t \rho + \operatorname{div} \mathbf{j} = 0, \quad \mathbf{j} = -D\,\mathbf{grad}\,\rho \quad \Longrightarrow \quad \partial_t \rho = D\Delta\rho; \qquad \langle x^2 \rangle = 2Dt$$

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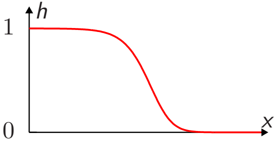
The mathematician's convention
$\langle x^2 \rangle = t$

The physicist's convention
$D = 1$

Outline

1 Deterministic Fronts



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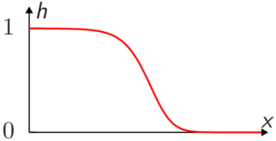

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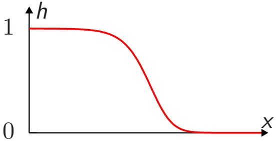

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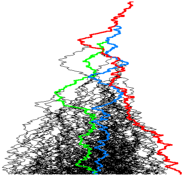
3. Fronts and Branching Brownian Motion



Deterministic fronts

$$\partial_t h = \partial_x^2 h + h - h^2$$

$h(x, t) = 0$ is an unstable solution

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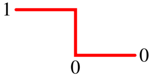
what if $h(x, 0) = \Bigg($ 

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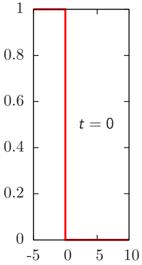

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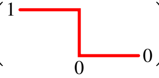
what if $h(x,0) = \Big($  $\Big)$?

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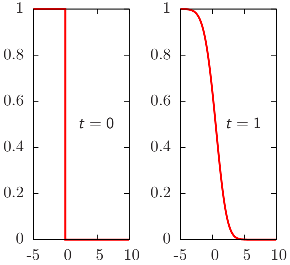
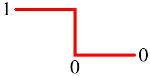


what if $h(x,0) = \Bigg($  $\Bigg)$?

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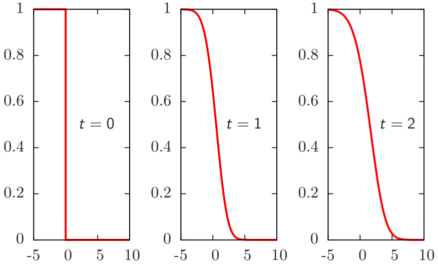

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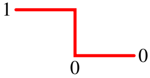
what if $h(x, 0) = \Bigg($  $\Bigg)$?

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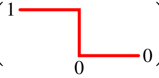
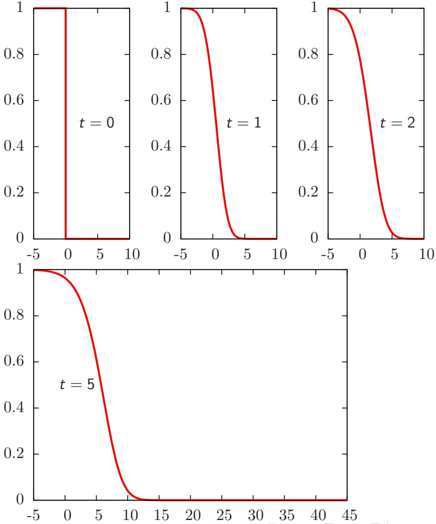
what if $h(x,0) = \Big($  $\Big)$?



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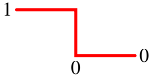
what if $h(x,0) = \Big($ 

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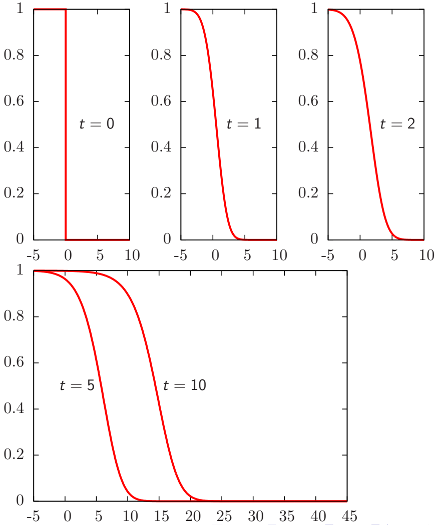

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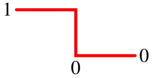
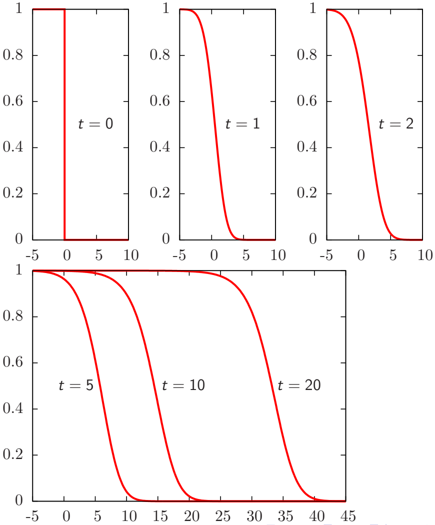
what if $h(x,0) = \Bigg($  $\Bigg)$?



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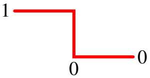
If $h(x,0) = \Bigg($ 

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$$h(X_t + z, t) \xrightarrow[t\to\infty]{} f_2(z)$$

with

$$X_t = (\text{position of the front})$$

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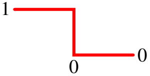
If $h(x,0) = \Big($ 

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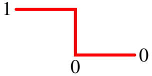
If $h(x,0) = \Bigg($ 

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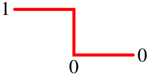
If $h(x,0) = \Big($ 

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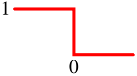
If $h(x,0) = \Bigg($ 

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$f_2(z) = \Bigg($  $\Bigg) = (\text{final shape of the front})$

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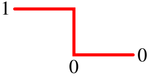
If $h(x,0) = \Big($ 

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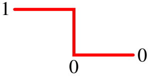
If $h(x,0) = \Bigg($ 

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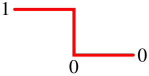
Answer:

$$X_t = 2t - \frac{3}{2}\ln t + a_0 - \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{a_1}{t} + \frac{a_{3/2}}{t^{3/2}} + \cdots \qquad \text{for large } t$$

Questions

If $h(x,0) = \Big($ 

$\Big)$, or if $h(x,0) = \Big($ 

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$f_2(z) = \Big($  $\Big) =$ (final shape of the front)

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Many equations, same kind of behavior

$$\partial_t h = \partial_x^2 h + h - h^2, \qquad X_t = 2t - \frac{3}{2}\ln t + a_0 - \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{a_1}{t} + \cdots$$

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$$h(x, t+\epsilon) = h(x,t) + \epsilon\left[\frac{h(x+s,t) + h(x-s,t) - 2h(x,t)}{s^2} + h - h^2\right]$$

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Fronts propagating into an unstable state

$h = 0$ and $h = 1$ are solutions

$h = 0$ is unstable (growth term), $h = 1$ is stable (saturation term)

First order equation in time, some mixing (diffusion) in space

Many velocities

$$\partial_t h = \partial_x^2 h + h - h^2$$

Uniformly translating front such that $h(x,t) = f_v(x - vt)$

$$\partial_z^2 f_v + v\partial_z f_v + f_v - f_v^2 = 0$$

Many velocities

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Look in the unstable region, where $f_v(z) \ll 1$

Many velocities

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Linear equation

$$f_v \approx e^{-\gamma z} \quad [\text{or } h \approx e^{-\gamma(x-vt)}]$$

Many velocities

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Look in the unstable region, where $f_v(z) \ll 1$
Linear equation

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. . . is solution if

$$\gamma^2 f_v - \gamma v f_v + f_v = 0$$

Many velocities

$$\partial_t h = \partial_x^2 h + h - h^2$$

Uniformly translating front such that $h(x,t) = f_v(x - vt)$

$$\partial_z^2 f_v + v\partial_z f_v + f_v \cancel{- f_v^2} = 0$$

Look in the unstable region, where $f_v(z) \ll 1$

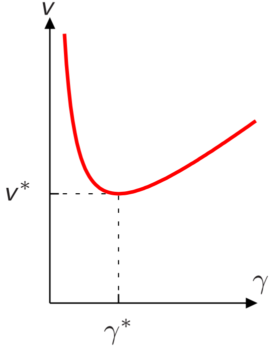
Linear equation

$$f_v \approx e^{-\gamma z} \quad [\text{or } h \approx e^{-\gamma(x-vt)}]$$

... is solution if

$$\gamma^2 f_v - \gamma v f_v + f_v = 0$$

$$v = \underbrace{\gamma + \frac{1}{\gamma}}_{v(\gamma)}$$



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$h(x,t) = f_v(x - vt) \ll 1$, linear equation, $f_v(z) \approx e^{-\gamma z}$ or $h \approx e^{-\gamma(x-vt)}$

$$\partial_t h = \partial_x^2 h + h - h^2 \quad \Longrightarrow \quad \gamma v h = \gamma^2 h + h \implies \quad v(\gamma) = \gamma + \frac{1}{\gamma}$$

$$h(x,t+1) = \min\left[1, 2\int_0^1 \mathrm{d}\epsilon\, h(x-\epsilon,t)\right] \implies e^{\gamma v} h = 2\int_0^1 \mathrm{d}\epsilon\, e^{\gamma\epsilon} h \implies v(\gamma) = \frac{1}{\gamma}\ln\left[2\int_0^1 \mathrm{d}\epsilon\, e^{\gamma\epsilon}\right]$$

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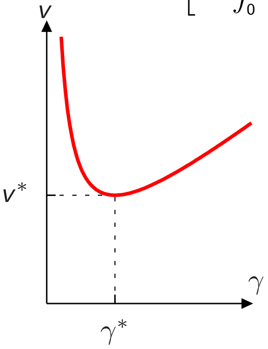

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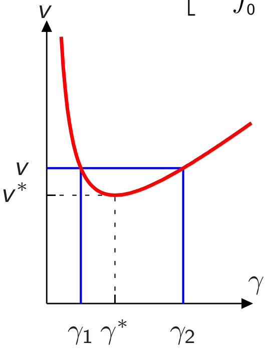
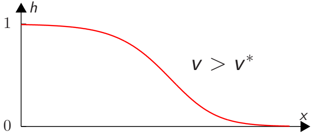


$$f_v \approx A_1 e^{-\gamma_1 z} + A_2 e^{-\gamma_2 z}$$



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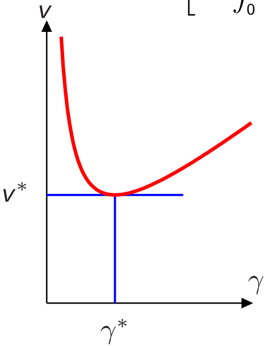

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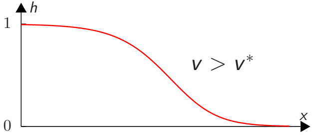
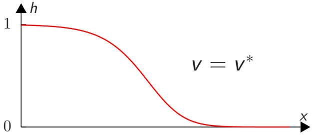


$$f_v \approx (Az + B)e^{-\gamma^* z}$$



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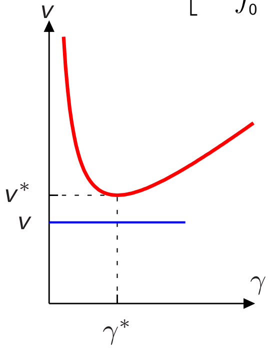

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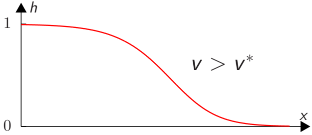

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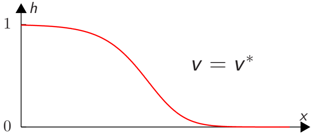
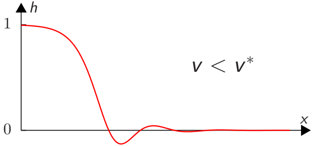


$$f_v \approx A\sin(\gamma_I z + \phi)e^{-\gamma_R z}$$



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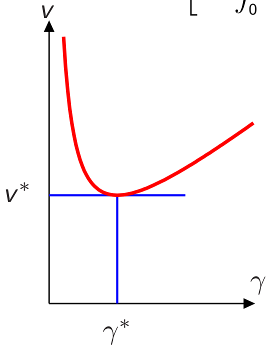

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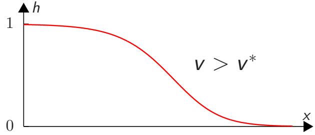

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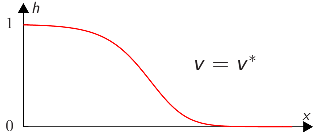
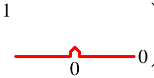


Fronts with $v < v^*$ are unstable

Linear perturbation

$$\partial_t h = \partial_x^2 h + h \cancel{- h^2}, \qquad h(x,0) = \left(\quad \right)$$



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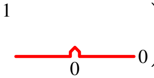

$$\Big)$$

$$h(x,t) = \frac{\epsilon}{\sqrt{4\pi t}} \exp\left[-\frac{x^2}{4t} + t \right]$$

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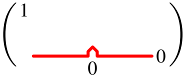

$$h(x,t) = \frac{\epsilon}{\sqrt{4\pi t}} \exp\left[-\frac{x^2}{4t} + t \right]$$

At a position $x = vt + y$

$$h(vt+y,t) = \frac{\epsilon}{\sqrt{4\pi t}} \exp\left[\left(1 - \frac{v^2}{4}\right) t - \frac{vy}{2} - \frac{y^2}{4t} \right]$$

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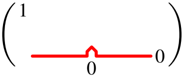

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A linear perturbation moves at velocity $v = v^* \ (= 2)$

$$h(2t+y,t) = \frac{\epsilon}{\sqrt{4\pi t}} \exp\left[-y - \frac{y^2}{4t} \right]$$

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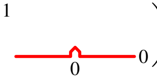

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$$h(2t - \frac{1}{2}\ln t + z) = \epsilon \frac{1}{\sqrt{4\pi t}} \exp\left[-z + \frac{1}{2}\ln t - \frac{z^2}{4t} + \cdots \right]$$

Linear perturbation

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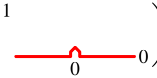

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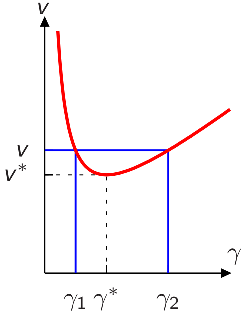
Shape and velocity

$$\partial_t h = \partial_x^2 h + h \cancel{-} \cancel{h^2},$$

For $v > v^*$, $\quad f_v \approx A_1 e^{-\gamma_1 z} + A_2 e^{-\gamma_2 z}$



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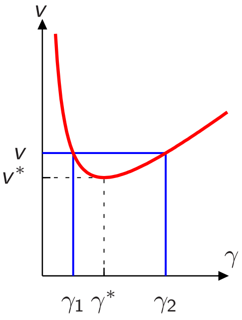
For $v > v^*$, $\quad f_v \approx A_1 e^{-\gamma_1 z} + A_2 e^{-\gamma_2 z} + A_{11} e^{-2\gamma_1 z} + A_{12} e^{-(\gamma_1+\gamma_2) z} + A_{22} e^{-2\gamma_2 z} + \cdots$



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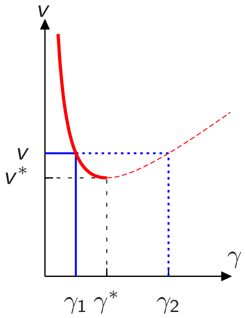

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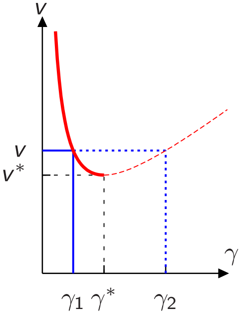
A fast front decays slowly in space
A slow front decays quickly in space



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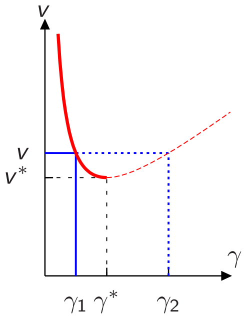

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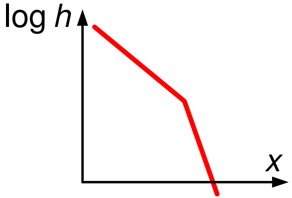

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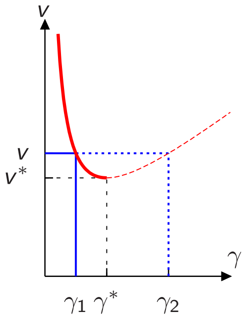

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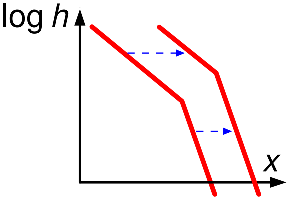

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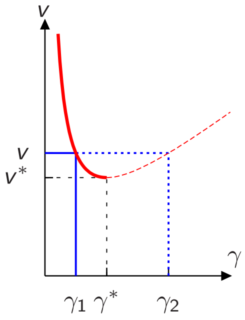

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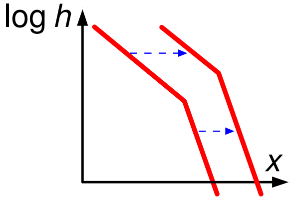

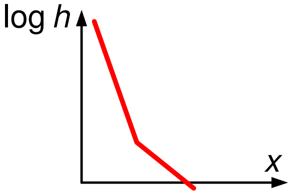

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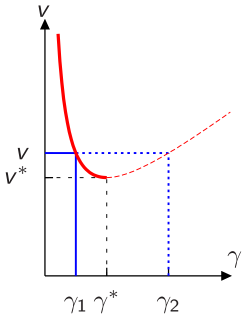

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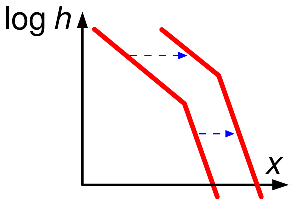

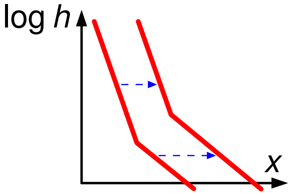

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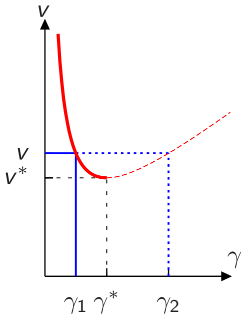

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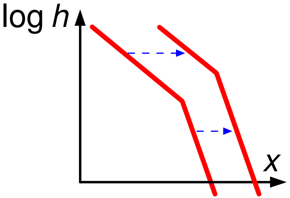

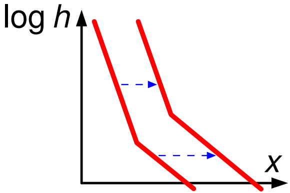

What is ahead wins

Shape and velocity

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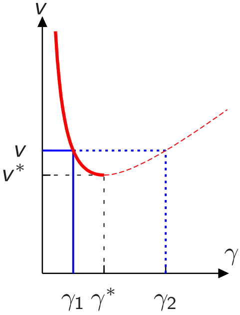

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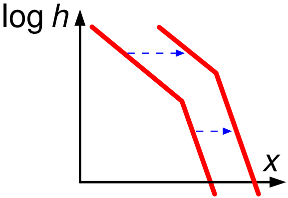

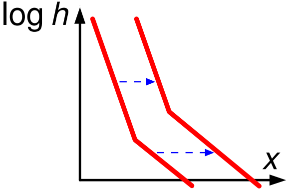

What is ahead wins

If $h(x,0) \sim e^{-\gamma x}$ with $\gamma \leq \gamma^*$, then $v = v(\gamma)$
If $h(x,0) \ll e^{-\gamma^* x}$ then $v = v^*$.

When it does not work — pushed fronts

For $v > v^*$ and z large

$$f_v \approx A_1 e^{-\gamma_1 z} + A_2 e^{-\gamma_2 z} + A_{11} e^{-2\gamma_1 z} + \cdots$$
$$\approx A_1 e^{-\gamma_1 z}$$

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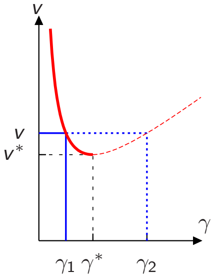
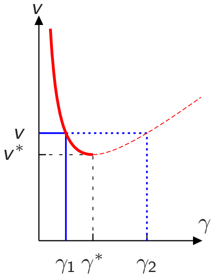


When it does not work — pushed fronts

For $v > v^*$ and z large

$$f_v \approx A_1 e^{-\gamma_1 z} + A_2 e^{-\gamma_2 z} + A_{11} e^{-2\gamma_1 z} + \cdots$$
$$\approx A_1 e^{-\gamma_1 z}$$



A fast front decays slowly in space
A slow front decays quickly in space

What if $A_1 < 0$?

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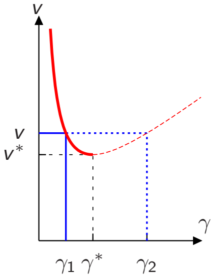

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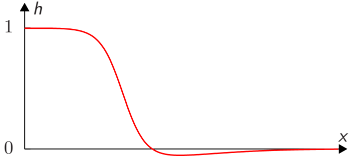

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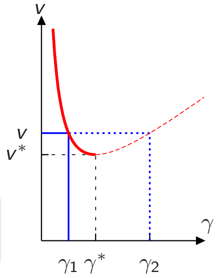

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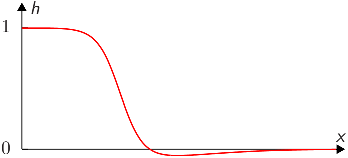

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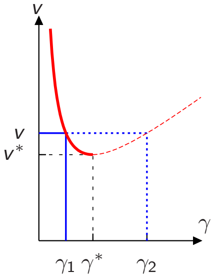

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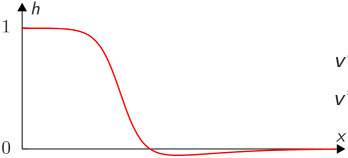

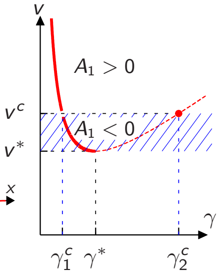

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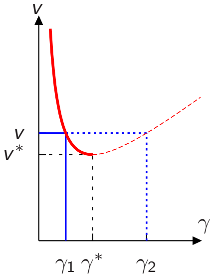

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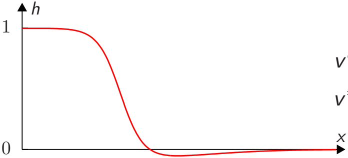
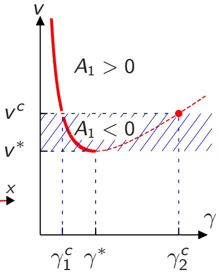


A_1 depends on v

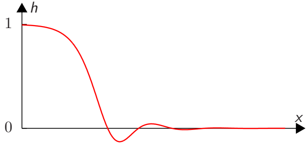


$$v = v(\gamma) \text{ if } \begin{cases} h(x,0) \sim e^{-\gamma x} \\ \text{with } \gamma \le \gamma_1^c \end{cases}$$

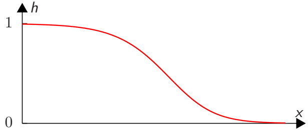
$$v = v^c \text{ if } h(x,0) \ll e^{-\gamma_1^c x}$$

Summary

Pulled fronts propagating into an unstable state

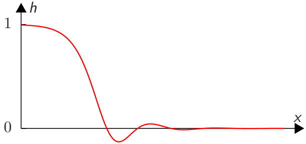


$v < v^*$, unstable



$v \geq v^*$, stable

Pushed fronts propagating into an unstable state



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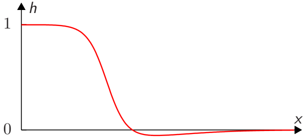
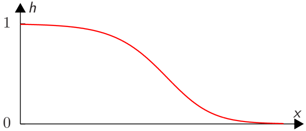


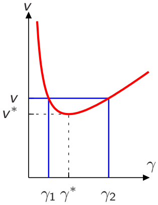
$v^* \leq v < v^c$, unstable



$v \geq v^c$, stable

An initial condition decaying fast enough leads to the slowest stable front
A pulled front goes at the same speed as a linear perturbation
A pushed front goes faster than a linear perturbation
A front can be pushed only if the non-linearities increase the growth rate

An example



$$\partial_t h = \partial_x^2 h + (h - h^2)(1 + \alpha h)$$

$h = 0$ unstable, $h = 1$ stable, $v(\gamma) = \gamma + \frac{1}{\gamma}$, $\gamma^* = 1$ and $v^* = 2$

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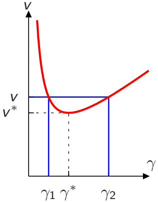

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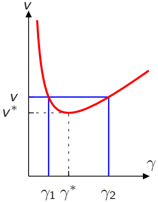
We look for uniformly translating solutions $\quad h(x,t) = f_v(x - vt)$

The solution is known for *one* value of v:

An example



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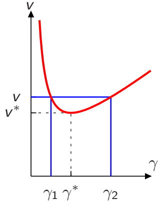

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But $\quad f_v(z) = e^{-\gamma z} - e^{-2\gamma z} + e^{-3\gamma z} - \cdots$

instead of $\quad f_v(z) = A_1 e^{-\gamma_1 z} + A_2 e^{-\gamma_2 z} + \cdots$

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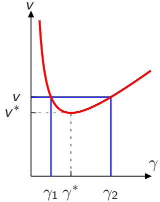

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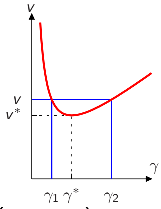

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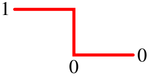
Either γ_1 or γ_2 is missing $(A_1 = 0$ or $A_2 = 0)$

If $\alpha > 2$, then $\gamma = \gamma_2$, and $A_1 = 0$, and the front is pushed with

$$\boxed{v_c = \sqrt{\frac{\alpha}{2}} + \sqrt{\frac{2}{\alpha}}}$$

Bramson's result

For $h(x,0) = \Big($ 

$\Big)$, $\dfrac{X_t}{t} \to v^*$ and $h(X_t + z, t) \xrightarrow[t\to\infty]{} f_{v^*}(z)$

with

$$f_{v^*}(z) \approx (Az + B)e^{-\gamma^* z} \qquad \text{for large } z$$

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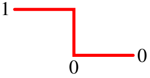
For $h(x,0) = \Big($ 

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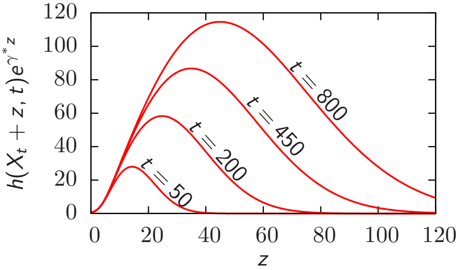

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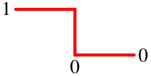
For $h(x,0) = \Big($  $\Big)$, $\dfrac{X_t}{t} \to v^*$ and $h(X_t + z, t) \xrightarrow[t\to\infty]{} f_{v^*}(z)$

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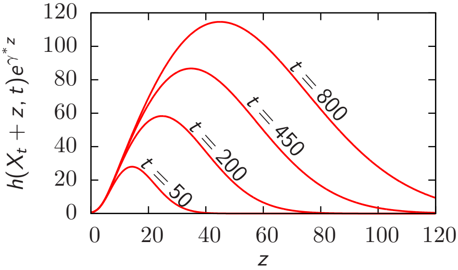

$$h(X_t + z) \approx \sqrt{t}\, S\left(\frac{z}{\sqrt{t}}\right) e^{-\gamma^* z}$$

$$S(u) \approx u \text{ for small } u \ (z \ll \sqrt{t})$$

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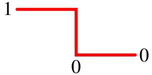
For $h(x,0) = \Big($ 

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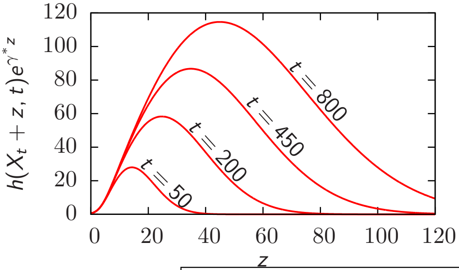

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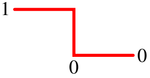
For $h(x,0) = \Big($ 

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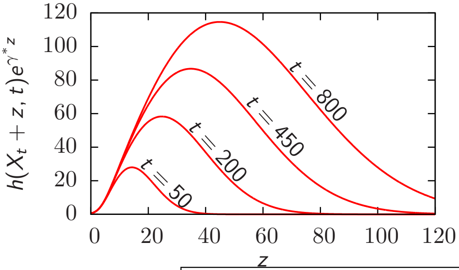

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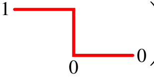
Solution of the linearized equation $\partial_t h = \partial_x^2 h + h$

$$h(2t + y, t) = \left\{ \frac{1}{\sqrt{4\pi t}} e^{-y - \frac{y^2}{4t}}, \right.$$

Bramson's result

For $h(x,0) = \Big($  $\Big)$, $\dfrac{X_t}{t} \to v^*$ and $h(X_t + z, t) \xrightarrow[t\to\infty]{} f_{v^*}(z)$

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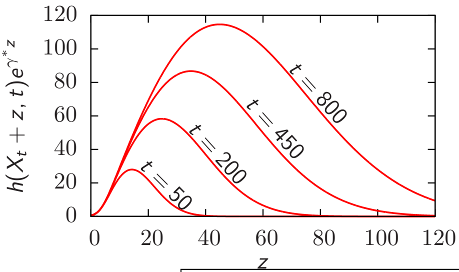

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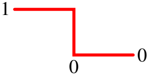
For $h(x,0) = \Big($  $\Big)$, $\dfrac{X_t}{t} \to v^*$ and $h(X_t + z, t) \xrightarrow[t\to\infty]{} f_{v^*}(z)$

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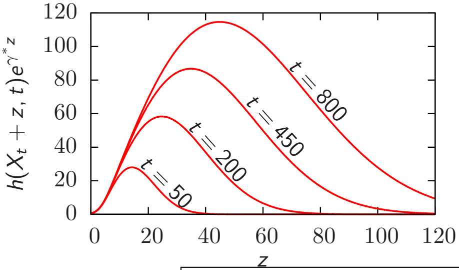

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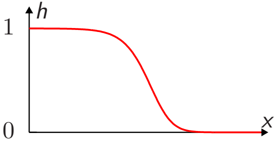
Outline

1. Deterministic Fronts



$$\partial_t h = \partial_x^2 h + h(1-h)$$

$$h(x,t+1) = \min\left[1, 2\int_0^1 \mathrm{d}\epsilon\ h(x-\epsilon,t)\right]$$

...

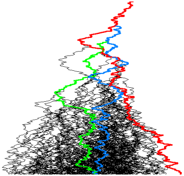
2. Stochastic Fronts

$$\partial_t h = \partial_x^2 h + h(1-h) + (\text{small noise term})$$

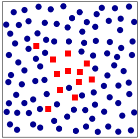
$$h(x,t+1) = \min\left[1, 2\int_0^1 \mathrm{d}\epsilon\ h(x-\epsilon,t) + \cdots\right]$$

...

3. Fronts and Branching Brownian Motion



Why the noise ?

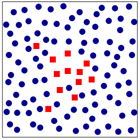


A and B diffuse, $A + B \to 2A$

Let $h(x,t)$ = proportion of A. In the limit of infinite concentration; $\partial_t h = \partial_x^2 h + h(1-h)$

What to write for a finite concentration ?

Why the noise ?



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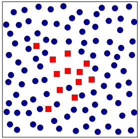
Let $h(x,t) =$ proportion of A. In the limit of infinite concentration; $\partial_t h = \partial_x^2 h + h(1-h)$

What to write for a finite concentration ?

N particles on one site, $n_t =$ number of A, and $N - n_t =$ number of B

Why the noise ?



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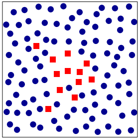

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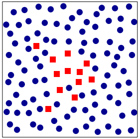

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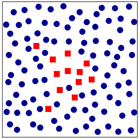

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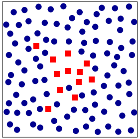

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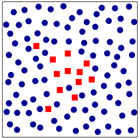

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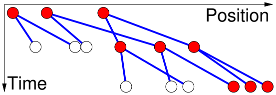
Other examples

$N\,(= 3)$ particles, at each time step a particle at x gives two offspring at positions $x + \epsilon_{1,2}$ with $\epsilon \in [0, 1]$ random. Keep only the N rightmost.



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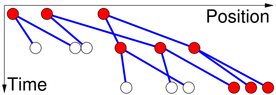

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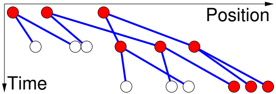

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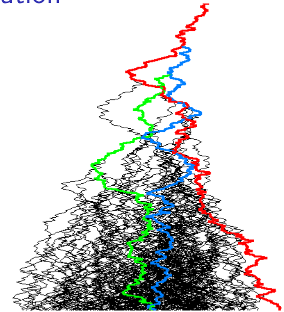
Branching Brownian Motion plus saturation

- Particles do a Brownian motion
- With rate 1, they split

plus



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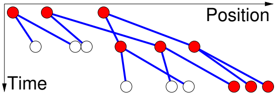

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Branching Brownian Motion plus saturation

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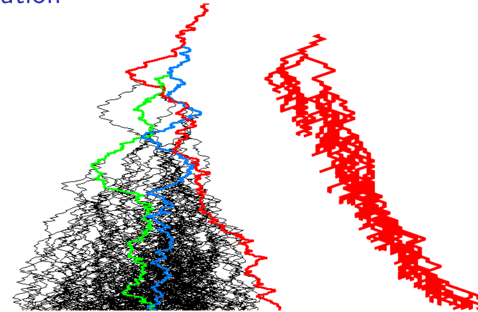

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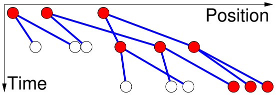

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Branching Brownian Motion plus saturation

- Particles do a Brownian motion
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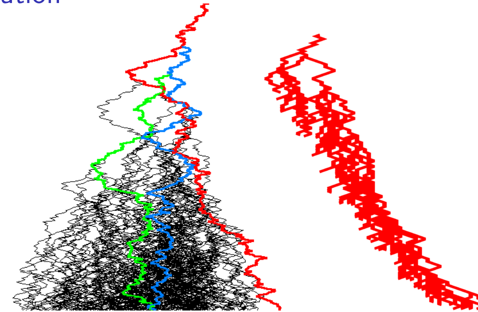
- When a $(N+1)^{\text{th}}$ particle appears, remove the leftmost to keep only N
- Or two particles crossing have a $1/N$ chance of coalescing



The noise term

$$\boxed{\text{growth term} \approx h \qquad\qquad \text{noise term} \approx \sqrt{\frac{h}{N}}}$$

$$\partial_t h = \partial_x^2 h + h(1-h) + \eta\sqrt{\frac{h(1-h)}{N}} \quad \text{with} \begin{cases} \langle \eta_{x,t} \rangle = 0 \\ \langle \eta_{x,t}\eta_{x',t'} \rangle = \delta(t-t')\delta(x-x') \end{cases}$$

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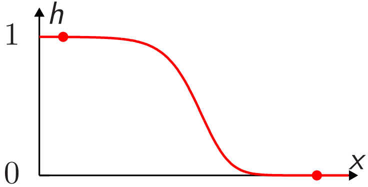
The front (almost surely) reaches 0 and 1



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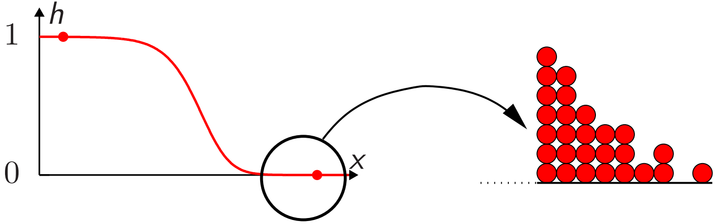

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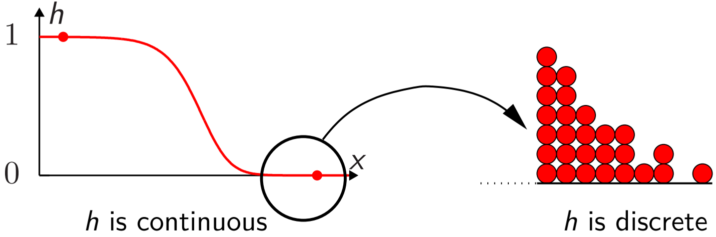

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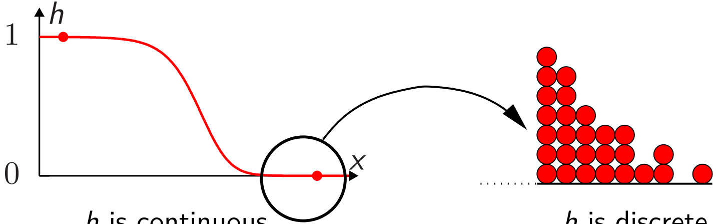

h is continuous

h is discrete

$Nh \simeq$ number of particles

if $h \neq 0$, then $h \geq 1/N$

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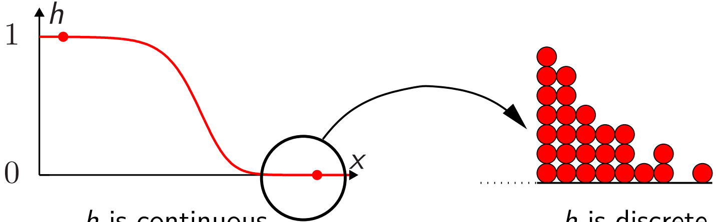

h is continuous

For $\partial_t h = h + \eta_t\sqrt{\frac{h}{N}}$, if $h(0) < 1/N$,

then, probably, $h(t) \to 0$

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The cutoff approximation

$$\begin{cases} \partial_t h = \partial_x^2 h + h - h^2 \cancel{+ \eta_{x,t}\sqrt{\frac{1}{N}(h - h^2)}} \\ \text{Wherever } h \text{ is of order } \frac{1}{N}\text{, it should go quickly to zero} \end{cases}$$

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Other example in the discrete

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It looks likely that $v_N^{\text{noise}} \approx v_N^{\text{cutoff}}$

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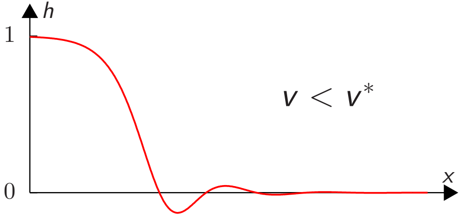

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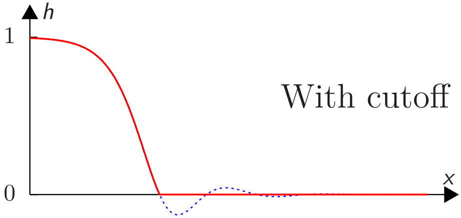

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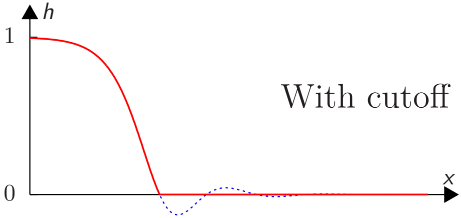

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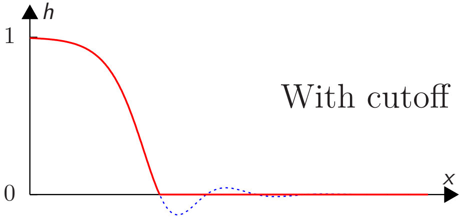

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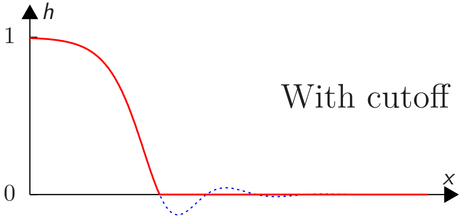

$$\gamma = \gamma_R + i\gamma_I \quad v = v(\gamma) \text{ (real)}$$

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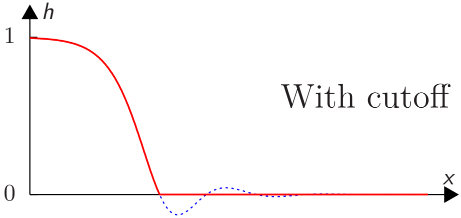

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$$\gamma_I \ll 1 \implies \gamma_R \approx \gamma^* \text{ to have } v(\gamma) \text{ real}$$

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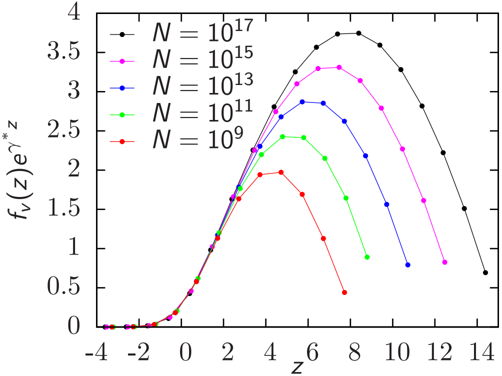

$$v_N^{\text{cutoff}} = v(\gamma) = v\left(\gamma^* + i\frac{\pi}{L}\right) = v^* - \frac{\pi^2 v''(\gamma^*)}{2L^2}$$

Beyond the cutoff approximation

Cutoff:

$$f_v(z) \approx \mathrm{Cste} \ln N \sin\left(\frac{\pi\gamma^* z}{\ln N}\right) e^{-\gamma^* z} \quad \text{and} \quad v_N^{\text{cutoff}} = v^* - \frac{a}{\ln^2 N} + \mathcal{O}\left(\frac{1}{\ln^3 N}\right)$$

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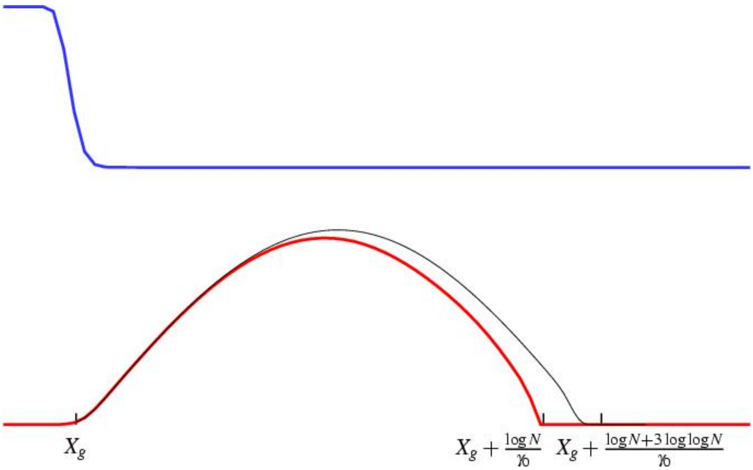
with $\qquad a = \dfrac{\pi^2 \gamma^{*2} v''(\gamma^*)}{2} \qquad b = \dfrac{\pi^4 \gamma^* v''(\gamma^*)}{3}$

Watching the fluctuations



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- A Sine shape. Cutoff approximation mostly correct

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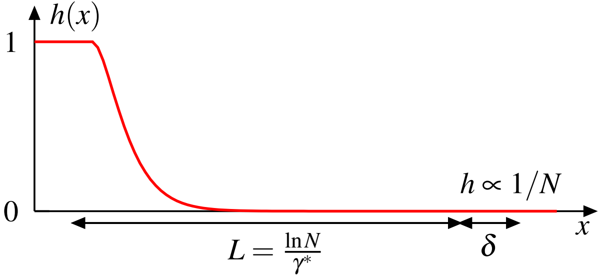

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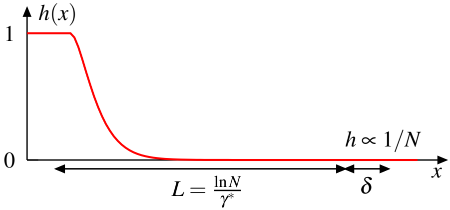

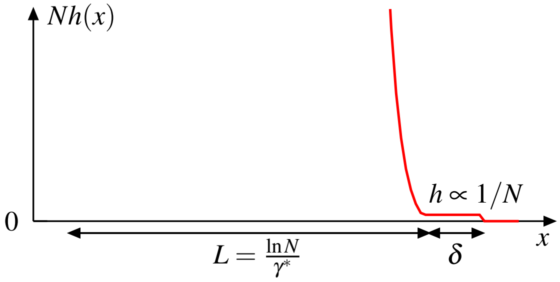

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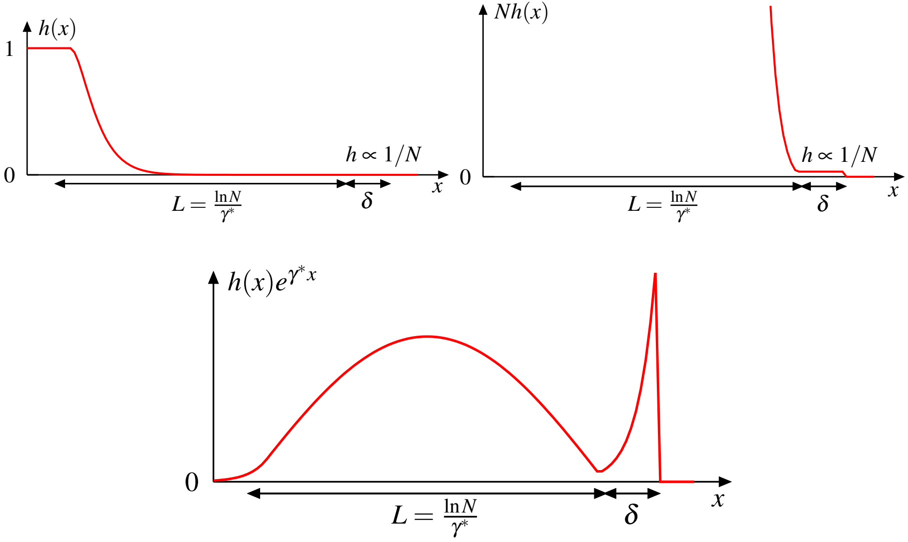

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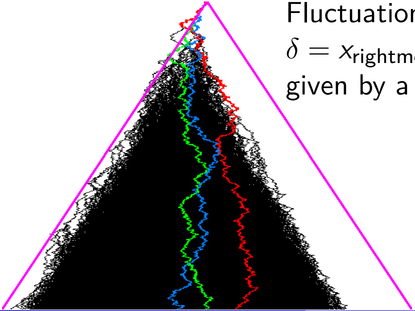
Branching Brownian Motion



Fluctuations of
$\delta = x_{\text{rightmost}} -$ (position of the tip of the BBM)
given by a Gumbel

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$$R(\delta) \approx \frac{1}{\gamma^*} \ln\left(1 + C_2 \frac{e^{\gamma^* \delta}}{\ln^3 N}\right)$$

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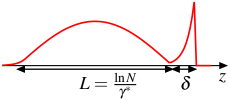
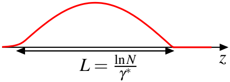
$h(X_0 + z, 0) \quad =$  $\times\, e^{-\gamma^* z}$

$h(X_t + z, t) \xrightarrow[t\to\infty]{}$  $\times\, e^{-\gamma^* z}$

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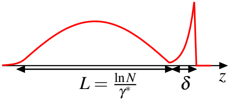

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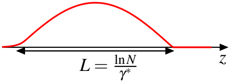

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$$h(X_t + z, t) = LG\left(\frac{z}{L}, \frac{t}{L^2}\right) e^{-\gamma^*(z + X_t - X_0 - v_{\text{cutoff}} t)}$$

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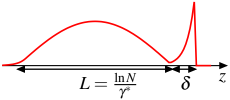

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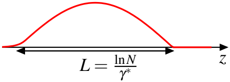

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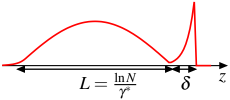
$h(X_0 + z, 0) \quad =$ 

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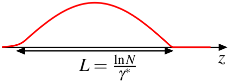
$h(X_t + z, t) \xrightarrow[t\to\infty]{}$ 

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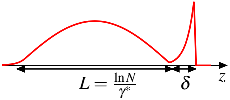
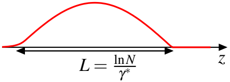
$h(X_0 + z, 0) \quad =$  $\times\, e^{-\gamma^* z} = LG\left(\frac{z}{L}, 0\right) e^{-\gamma^* z}$

$h(X_t + z, t) \xrightarrow[t\to\infty]{}$  $\times\, e^{-\gamma^* z} = LG\left(\frac{z}{L}, \infty\right) e^{-\gamma^* R(\delta)} e^{-\gamma^* z}$

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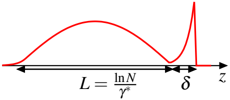
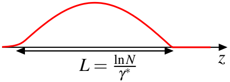
$h(X_0 + z, 0) \quad =$  $\times\, e^{-\gamma^* z} = LG\left(\frac{z}{L}, 0\right) e^{-\gamma^* z}$

$h(X_t + z, t) \xrightarrow[t\to\infty]{}$  $\times\, e^{-\gamma^* z} = LG\left(\frac{z}{L}, \infty\right) e^{-\gamma^* R(\delta)} e^{-\gamma^* z}$

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$C_1 C_2$?

- Cutoff theory gives $v = v^* - \dfrac{\pi^2 v''(\gamma^*)}{2L^2}$ with $L = \frac{1}{\gamma^*} \ln N$
- Use instead the effective length $L = \frac{1}{\gamma^*} [\ln N + 3 \ln \ln N]$

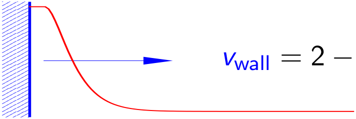
Relation between N and L

A diffuse, $A \to 2A$, ~~keep only the N rightmost~~

Relation between N and L

A diffuse, $A \to 2A$, a wall moving at velocity v absorbs the particles

replaced by



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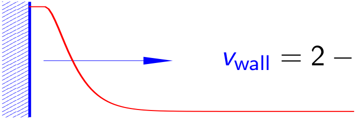


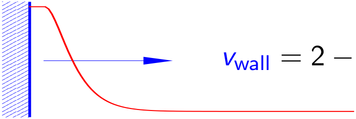
$$v_{\text{wall}} = 2 - \frac{\pi^2}{L^2}$$

- Start with one particle at $x > 0$
- Condition on the fact that there is one living particle at large time T
- How many particles at an intermediate time ?

Relation between N and L

A diffuse, $A \to 2A$, a wall moving at velocity v absorbs the particles

replaced by



$$v_{\text{wall}} = 2 - \frac{\pi^2}{L^2}$$

- Start with one particle at $x > 0$
- Condition on the fact that there is one living particle at large time T
- How many particles at an intermediate time ?

$$\boxed{L \approx \ln N + 3 \ln \ln N}$$

Relation between N and L

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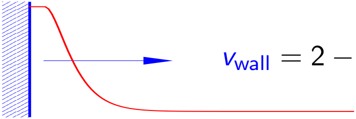

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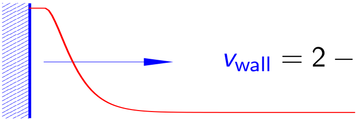

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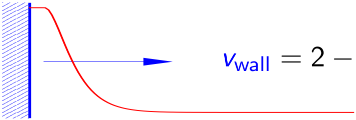

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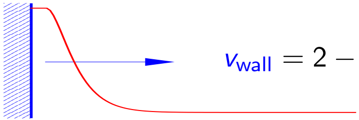

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- Populate with N particles
- Proba to survive $\sim 1 - e^{-KNL^3 e^{-L}}$

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Conclusion

$$v_N^{\text{noise}} = v^* - \frac{\pi^2 \gamma^{*2} v''(\gamma^*)}{2(\ln N + 3 \ln\ln N + \cdots)^2}$$

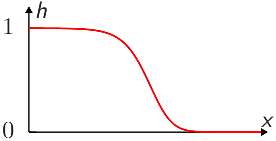
$$D_N^{\text{noise}} = \frac{\pi^4 \gamma^* v''(\gamma^*)}{3(\ln N + \cdots)^3}$$

- A phenomenological theory gives a prediction for v_N and D_N
- Agrees with simulations
- We still need a clean derivation

▸ Exponential model

Outline

1. Deterministic Fronts



$$\partial_t h = \partial_x^2 h + h(1-h)$$

$$h(x,t+1) = \min\left[1, 2\int_0^1 \mathrm{d}\epsilon\; h(x-\epsilon,t)\right]$$

...

2. Stochastic Fronts

$$\partial_t h = \partial_x^2 h + h(1-h) + (\text{small noise term})$$

$$h(x,t+1) = \min\left[1, 2\int_0^1 \mathrm{d}\epsilon\; h(x-\epsilon,t) + \cdots\right]$$

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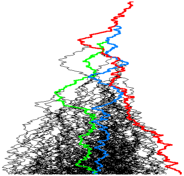
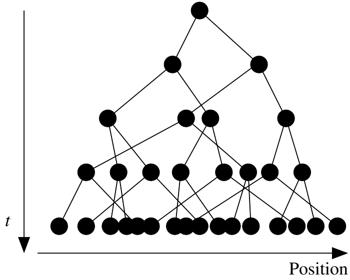
3. Fronts and Branching Brownian Motion



The models



Branching Random Walk

- At each **time step**, particles split into two
- The **positions** of the offspring are shifted by random **uncorrelated** amounts

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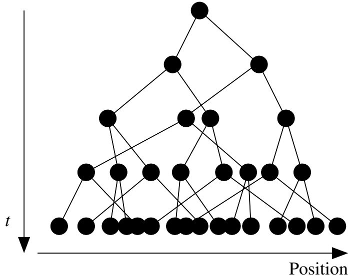

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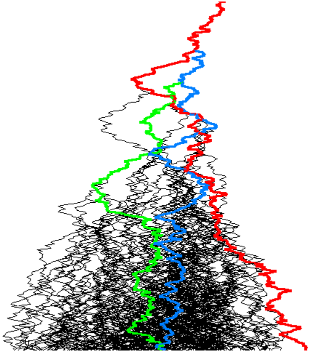

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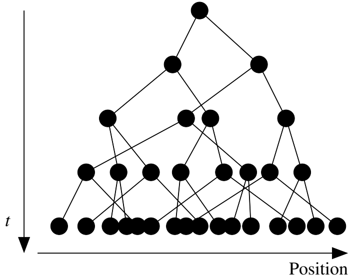

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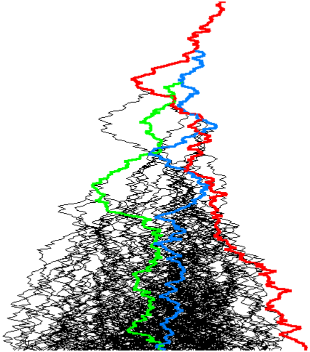

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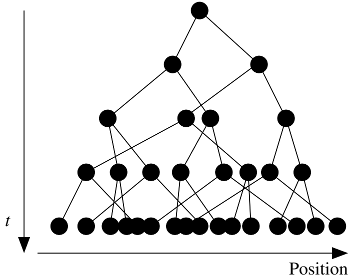

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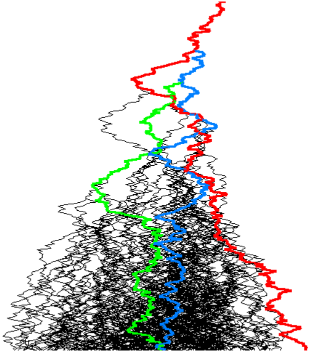

Positions of the rightmost particles ? (Energy spectrum ?)

The rightmost particle

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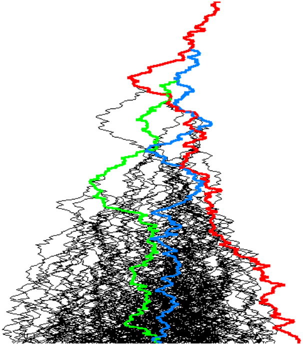

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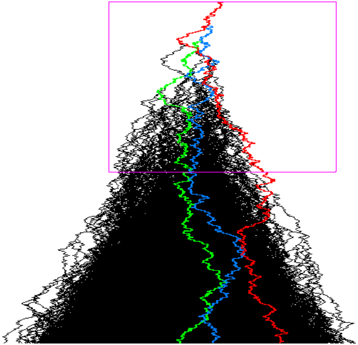

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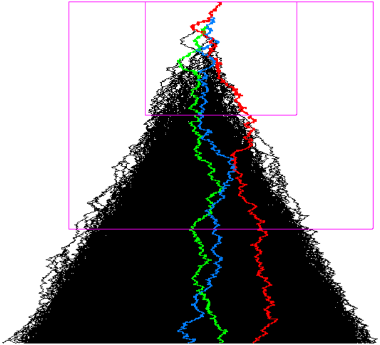

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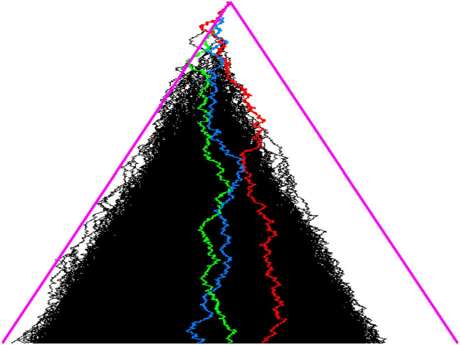

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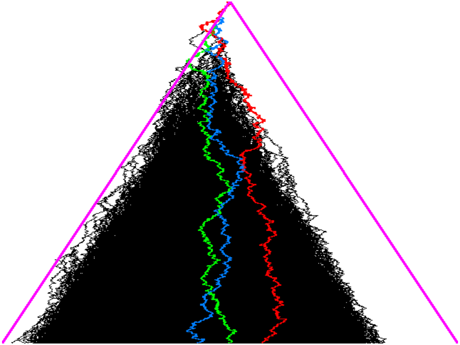

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$$Q_0(x, t) = \text{Proba}[X_1(t) < x]$$

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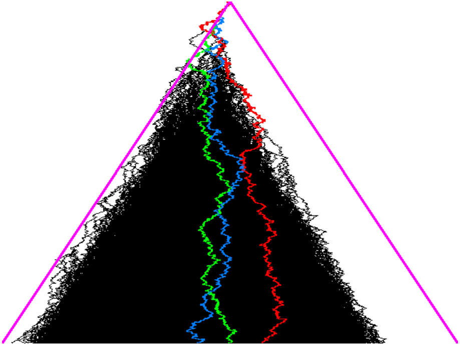

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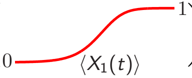

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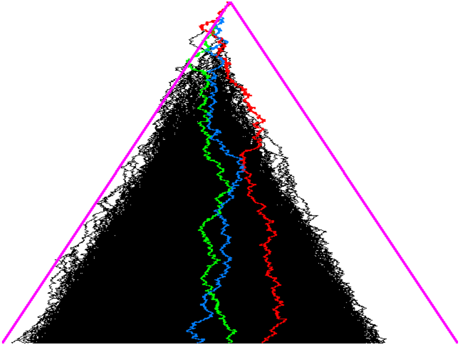

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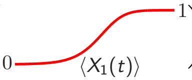
$Q_0(x,t) = \Big($  $\Big)$

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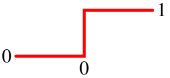
$Q_0(x,0) = \Big($  $\Big)$

FKPP equation! $(h = 1 - Q)$

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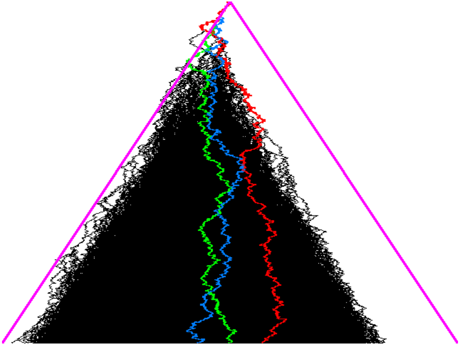

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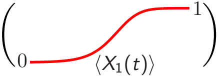
$Q_0(x,t) =$ 

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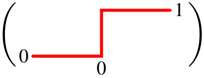
$Q_0(x,0) =$ 

FKPP equation! $(h = 1 - Q)$

- Position of the rightmost

$$X_1(t) = 2t - \frac{3}{2}\ln t + \mathcal{O}(1)$$

Why the FKPP equation ?

In general, for any well-behaved function ϕ, let

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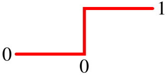
For $\phi(x) = \Big($ 

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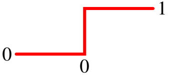
For $\phi(x) = \Big($ 

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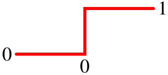
For $\phi(x) = \Big($ 

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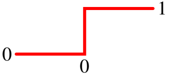
For $\phi(x) = \Bigg($  $\Bigg)$, H_ϕ is the probability Q_0 that all the particles are on the left of x

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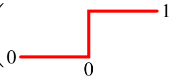
For $\phi(x) = \Big($  $\Big)$, H_ϕ is the probability Q_0 that all the particles are on the left of x

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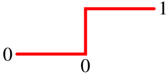
Position, shape and delay

For $\phi(x) = \Big($ 

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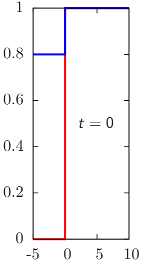
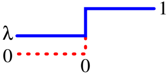


For $\phi(x) = \Big($ 

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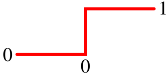
For $\phi(x) = \Bigg($  $\Bigg)$,

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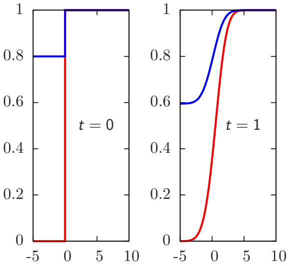
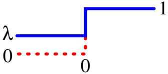


For $\phi(x) = \Bigg($  $\Bigg)$,

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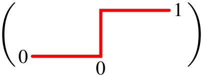
For $\phi(x) =$ ,

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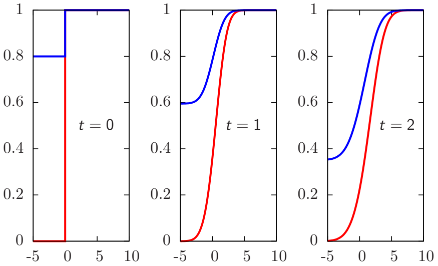
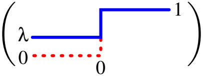


For $\phi(x) =$ ,

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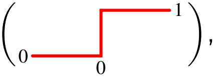
For $\phi(x) =$ 

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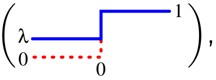
For $\phi(x) =$ 

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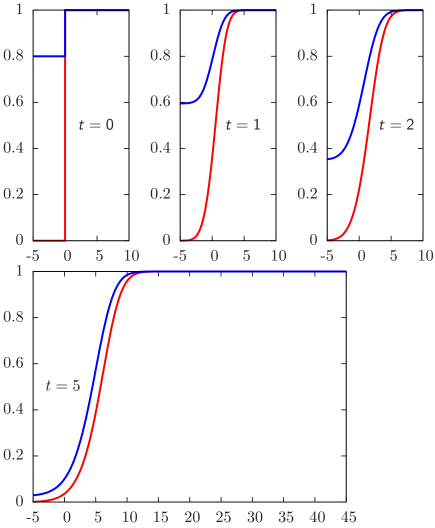

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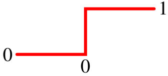
For $\phi(x) =$ (),

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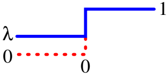
For $\phi(x) =$ (),

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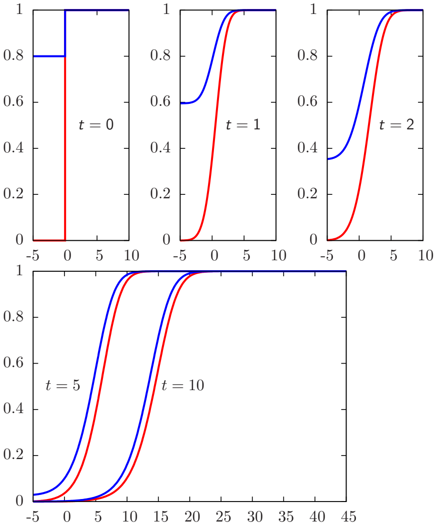

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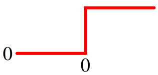
For $\phi(x) = \Big($  $\Big)$,

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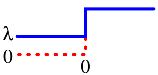
For $\phi(x) = \Big($  $\Big)$,

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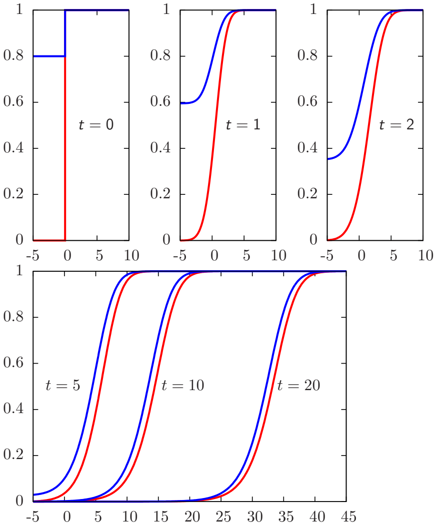

Universality

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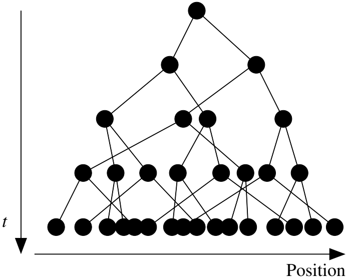

Branching Random Walk

- At each **time step**, particles split into two
- The **positions** of the offspring are shifted by random **uncorrelated** amounts ϵ

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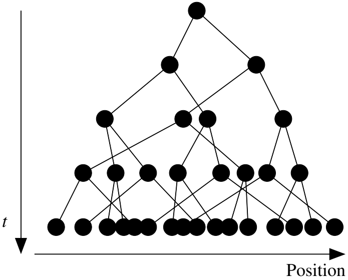

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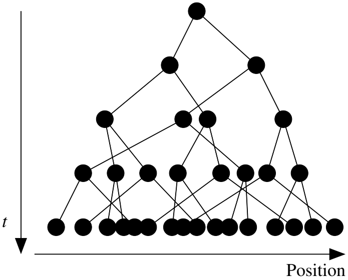

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Binary search tree

- During $\mathrm{d}t$, a particle at position x is replaced with probability $\mathrm{d}t$ by two particles at position $x+1$

$$\partial_t H_\phi(x,t) = -H_\phi(x,t) + H_\phi(x-1,t)^2$$

$$v = 4.31107\ldots \qquad \gamma = 0.768039\ldots$$

Average distances

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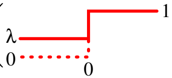
$$H_\phi(x,t) = \left\langle \prod_i \phi[x - X_i(t)] \right\rangle \qquad \text{with } \phi = \Bigg($$  $$\Bigg)$$

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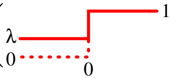
$$H_\phi(x,t) = \left\langle \prod_i \phi[x - X_i(t)] \right\rangle \qquad \text{with } \phi = \Bigg($$  $$\Bigg)$$

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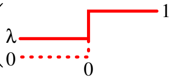
$$H_\phi(x,t) = \left\langle \prod_i \phi[x - X_i(t)] \right\rangle \qquad \text{with } \phi = \Bigg($$  $$\Bigg)$$

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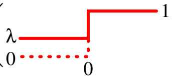

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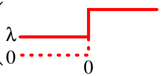

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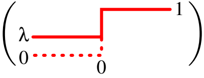
Then $\quad \partial_x Q_n(x,t) = p_{n+1}(x,t) - p_n(x,t)$

with $\quad p_n(x,t) = \text{proba to find the } n^{\text{th}} \text{ rightmost particle at } x$

Average distances

Summary

$\phi =$ 

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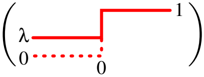
$\phi =$ 

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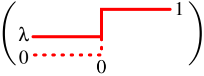
$\phi =$ 

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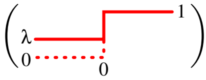
$\phi =$ 

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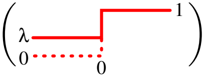
$\phi =$ 

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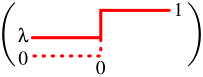
$\phi =$ 

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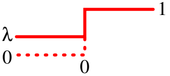
$\phi = \Big($ 

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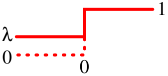
$\phi =$ ()

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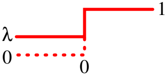
$\phi = \Big($ 

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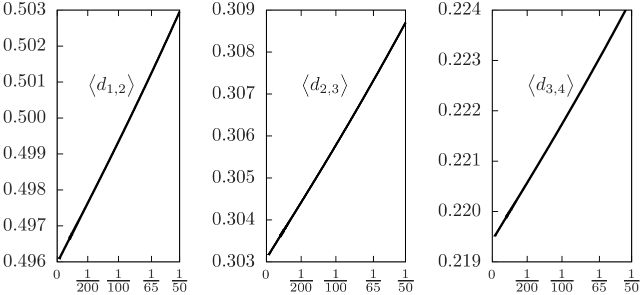
Numerical results: average distances

The results: average distances as a function of $1/t$



In the long time limit

$$\langle d_{1,2}\rangle_{\rm st} \simeq 0.496 \quad \langle d_{2,3}\rangle_{\rm st} \simeq 0.303 \quad \langle d_{3,4}\rangle_{\rm st} \simeq 0.219$$
$$\langle d_{4,5}\rangle_{\rm st} \simeq 0.172 \quad \langle d_{5,6}\rangle_{\rm st} \simeq 0.142 \quad \langle d_{6,7}\rangle_{\rm st} \simeq 0.121$$

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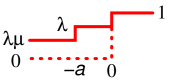
$$H_\phi(x,t) = \left\langle \prod_i \phi[x - X_i(t)] \right\rangle \qquad \text{with } \phi = \Bigg($$  $$\Bigg)$$

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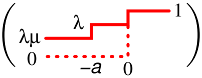

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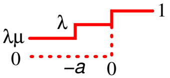
$$H_\phi(x,t) = \left\langle \prod_i \phi[x - X_i(t)] \right\rangle \qquad \text{with } \phi = \Bigg($$ 

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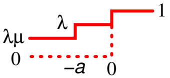
$$\Rightarrow \; Q_{mn}(x,a,t) = \text{Proba}\left[N(x,t) = n \quad \text{and} \quad N(x+a,t) = m\right]$$

P.d.f. of the distances between two particles

$$H_\phi(x,t) = \left\langle \prod_i \phi[x - X_i(t)] \right\rangle \qquad \text{with } \phi = \left(\right)$$



$$H_\phi(x,t) = \left\langle \lambda^{N(x,t)} \mu^{N(x+a,t)} \right\rangle \quad \text{with } N(x,t) = \begin{bmatrix} \text{Number of particles on the} \\ \text{right of } x \text{ at time } t \end{bmatrix}$$

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$$\Rightarrow\ R_{mn}(x,a,t) = \text{Proba}\left[N(x,t) < n \quad \text{and} \quad N(x+a,t) < m \right]$$

P.d.f. of the distances between two particles

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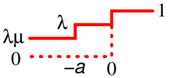

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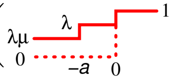
$$\Rightarrow\ (\partial_x - \partial_a) R_{mn}(x,a,t)\mathrm{d}x = \text{Proba}\left[X_n(t) \in \mathrm{d}x \quad \text{and} \quad N(x+a,t) < m\right]$$

P.d.f. of the distances between two particles

$$H_\phi(x,t) = \left\langle \prod_i \phi[x - X_i(t)] \right\rangle \qquad \text{with } \phi = \Bigg($$  $$\Bigg)$$

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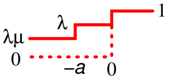
$$\Rightarrow \; (\partial_x - \partial_a) R_{mn}(x,a,t)\mathrm{d}x = \text{Proba}\left[X_n(t) \in \mathrm{d}x \quad \text{and} \quad X_m(t) < x + a\right]$$

P.d.f. of the distances between two particles

$$H_\phi(x,t) = \left\langle \prod_i \phi[x - X_i(t)] \right\rangle \qquad \text{with } \phi = \Bigg($$ 

$$\Bigg)$$

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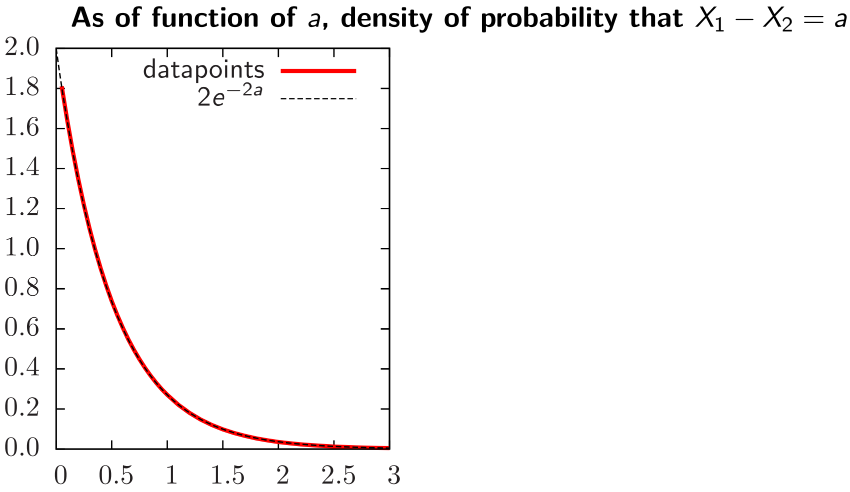
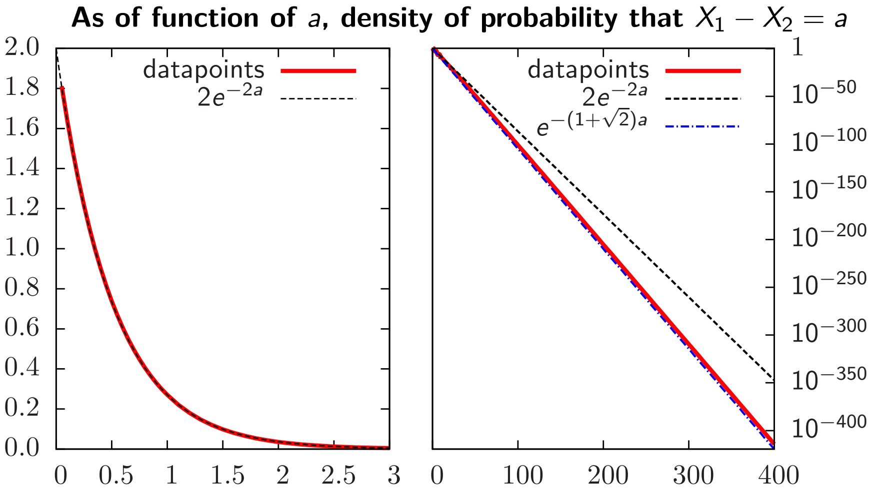
$$\Rightarrow \; \int \mathrm{d}x\, (\partial_x - \partial_a) R_{mn}(x,a,t) = \text{Proba}\left[X_m(t) - X_n(t) < a\right]$$

H_ϕ gives the p.d.f. of the distance between m^{th} and n^{th} particles

Numerical results: p.d.f. of the distances



Numerical results: p.d.f. of the distances

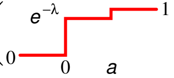


Density at a distance *a*

$$H_\phi(x,t) = \left\langle \prod_i \phi[x - X_i(t)] \right\rangle$$

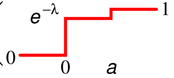
Density at a distance a

$$H_\phi(x,t) = \left\langle \prod_i \phi[x - X_i(t)] \right\rangle \qquad \text{with } \phi = \Bigg(\quad\Bigg)$$



Density at a distance a

$$H_\phi(x,t) = \left\langle \prod_i \phi[x - X_i(t)] \right\rangle \qquad \text{with } \phi = \left(\quad \right)$$



$$H_\phi(x,t) = \left\langle e^{-\lambda N(x-a,t)} \mathbb{1}_{N(x,t)=0} \right\rangle$$

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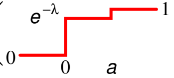

$$\Bigg)$$

$$H_\phi(x,t) = \left\langle e^{-\lambda N(x-a,t)} \mathbb{1}_{N(x,t)=0} \right\rangle$$

$$\Rightarrow \; (\partial_x + \partial_a) H_\phi(x,t)\mathrm{d}x = \left\langle e^{-\lambda N(x-a,t)} \mathbb{1}_{X_1(t)\in \mathrm{d}x} \right\rangle$$

Density at a distance a

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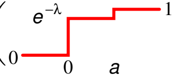


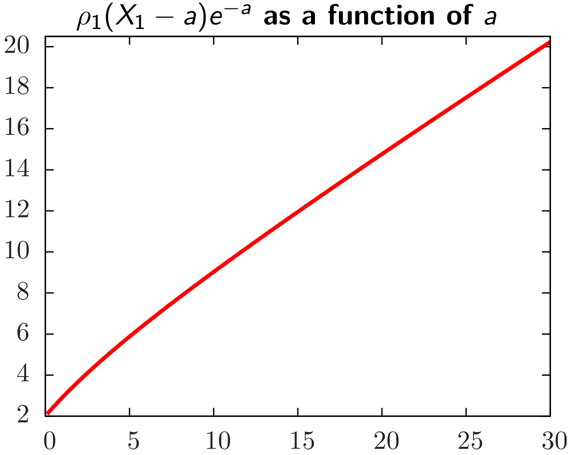
$$H_\phi(x,t) = \left\langle e^{-\lambda N(x-a,t)} \mathbb{1}_{N(x,t)=0} \right\rangle$$

$$\Rightarrow \ (\partial_x + \partial_a) H_\phi(x,t) \mathrm{d}x = \left\langle e^{-\lambda N(x-a,t)} \mathbb{1}_{X_1(t) \in \mathrm{d}x} \right\rangle$$

$$\Rightarrow \ 1 + \int \mathrm{d}x \, \partial_a H_\phi(x,t) = \left\langle e^{-\lambda N(X_1(t)-a,t)} \right\rangle$$

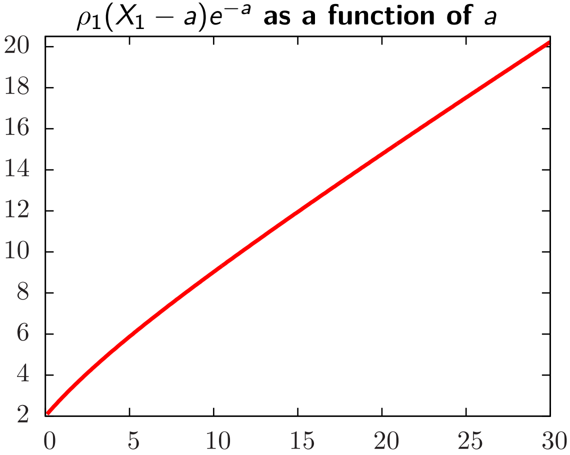
H_ϕ gives the generating function of the number of particles within a distance a of the rightmost particle

Numerical results: density at a distance a



$$\rho_1(X_1 - a) = \frac{1}{\mathrm{d}a}\begin{pmatrix}\text{Average number of particles in an interval } \mathrm{d}a \\ \text{at a distance } a \text{ of the rightmost particle}\end{pmatrix}$$

Numerical results: density at a distance a

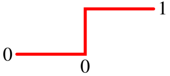


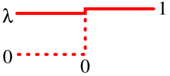
$$\rho_1(X_1 - a) = \frac{1}{\mathrm{d}a}\begin{pmatrix}\text{Average number of particles in an interval } \mathrm{d}a \\ \text{at a distance } a \text{ of the rightmost particle}\end{pmatrix}$$

$$\boxed{\rho_1(X_1 - a) \simeq a e^{a}}$$

Analytical result: average distances

$$\partial_t H_\phi = \partial_x^2 H_\phi - H_\phi + H_\phi^2, \qquad H_\phi(x,0) = \phi(x)$$

X_t is the position for $\phi(x) = \Bigg($  $\Bigg)$

$\tilde{X}_t$ is the position for $\phi(x) = \Bigg($  $\Bigg)$

$$\sum_{n\geq 1} \lambda^n \langle d_{n,n+1}(t) \rangle = X_t - \tilde{X}_t \xrightarrow[t\to\infty]{} \delta[\phi]$$

Analytical result: average distances

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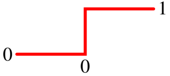
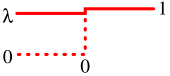
X_t is the position for $\phi(x) =$ 

$\tilde{X}_t$ is the position for $\phi(x) =$ 

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For λ close to 1, $\qquad \delta[\phi] = \tau_\lambda - \ln \tau_\lambda + \mathcal{O}(1) \qquad$ with $\tau_\lambda = -\ln(1-\lambda)$

Analytical result: average distances

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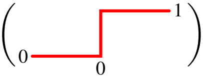

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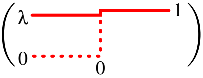


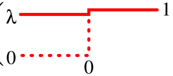
$$\sum_{n\geq 1} \lambda^n \langle d_{n,n+1}(t)\rangle = X_t - \tilde{X}_t \xrightarrow[t\to\infty]{} \delta[\phi]$$

For λ close to 1, $\delta[\phi] = \tau_\lambda - \ln \tau_\lambda + \mathcal{O}(1)$ with $\tau_\lambda = -\ln(1-\lambda)$

$$\langle d_{n,n+1}\rangle_{\text{st}} = \frac{1}{n} - \frac{1}{n \ln n} + \cdots$$ for large n

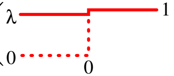
Analytical result: average distances

$$\partial_t H_\phi = \partial_x^2 H_\phi - H_\phi + H_\phi^2$$

with $H_\phi(x,0) = \Bigg($  $\Bigg)$

$$\boxed{\text{For } \lambda \simeq 1, \qquad \delta[\phi] = \tau_\lambda - \ln \tau_\lambda + \mathcal{O}(1) \quad \text{with } \tau_\lambda = -\ln(1-\lambda)}$$

Analytical result: average distances

$$\partial_t H_\phi = \partial_x^2 H_\phi - H_\phi + H_\phi^2 \qquad \text{with } H_\phi(x,0) = \Bigg($$  $$\Bigg)$$

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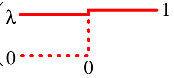
- $\tau_\lambda =$ time needed for $H_\phi(-\infty, 0)$ to "reach" 0
- As long as $t \ll \tau_\lambda$, one has $1 - H_\phi \ll 1$

Analytical result: average distances

$$\partial_t H_\phi = \partial_x^2 H_\phi - H_\phi + H_\phi^2 \qquad \text{with } H_\phi(x,0) = \left(\right)$$



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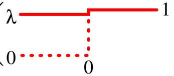
Analytical result: average distances

$$\partial_t H_\phi = \partial_x^2 H_\phi - H_\phi + H_\phi^2 \qquad \text{with } H_\phi(x,0) = \left(\vphantom{\int} \right.$$



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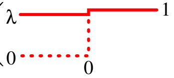

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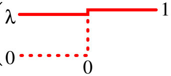
- $\tilde{X}_t$ is the position, let $v_t = \partial_t \tilde{X}_t$ be the velocity. For t large enough, $H_\phi(x,t) \simeq F_{v_t}(x - \tilde{X}_t)$

$$\partial_x^2 F_v + v\partial_x F_v - F_v + F_v^2 = 0$$

Analytical result: average distances

$$\partial_t H_\phi = \partial_x^2 H_\phi - H_\phi + H_\phi^2 \qquad \text{with } H_\phi(x,0) = \Bigg($$



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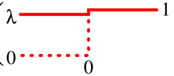
$$v_t = \gamma_t + \frac{1}{\gamma_t}, \quad \gamma_t < 1, \qquad 1 - F_v(z) = A_1(v_t) e^{-\gamma_t z} + A_2(v_t) e^{-\frac{1}{\gamma_t} z}$$

Analytical result: average distances

$$\partial_t H_\phi = \partial_x^2 H_\phi - H_\phi + H_\phi^2$$

with $H_\phi(x,0) =$



$$\boxed{\text{For } \lambda \simeq 1, \qquad \delta[\phi] = \tau_\lambda - \ln \tau_\lambda + \mathcal{O}(1) \quad \text{with } \tau_\lambda = -\ln(1-\lambda)}$$

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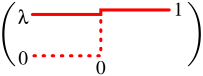
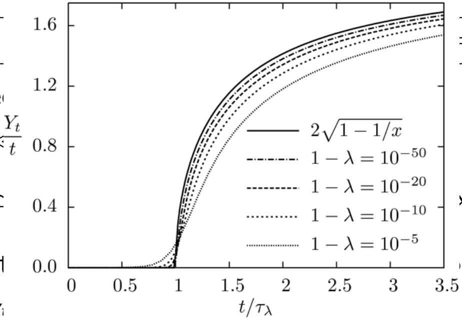
- $1 - H_\phi(x,t) \simeq \dfrac{1-\lambda}{2} e^t \operatorname{erfc}\left(\dfrac{x}{\sqrt{4t}}\right)$ for $t \ll \tau_\lambda$ or x large enough
- $\tilde{X}_t$ is the position, let $v_t = \partial_t \tilde{X}_t$ be the velocity. For t large enough, $H_\phi(x,t) \simeq F_{v_t}(x - \tilde{X}_t) \simeq 1 - A_1(v_t) e^{-\gamma_t (x - \tilde{X}_t)}$
- Matching in the range $1 \ll x - \tilde{X}_t \ll \sqrt{t}$ gives the result
- As a bonus: $\tilde{X}_t \approx 2t\sqrt{1 - \tau_\lambda / t}$.

Analytical result: average distances

$\partial_t H_\phi = \partial_x^2 H_\phi - H_\phi + H_\phi^2$ with $H_\phi(x,0) =$ 

For $\lambda \simeq 1$, … $= -\ln(1-\lambda)$

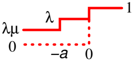
- $\tau_\lambda =$ time nee…
- As long as $t <$ …
- $1 - H_\phi(x,t) \simeq$ … x large enough
- $\tilde{X}_t$ is the posi… …r t large enough, $H_\phi(x,t) \simeq F_{v_i}$…



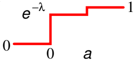
- Matching in the range $1 \ll x - \tilde{X}_t \ll \sqrt{t}$ gives the result
- As a bonus: $\tilde{X}_t \approx 2t\sqrt{1 - \tau_\lambda/t}$.

Analytical result: distance and density

P.d.f. of the distances:

$\phi = \Big($ 

$\Big) \implies Q_{mn}(x,a,t) \implies R_{mn}(x,a,t) \implies \ldots$

Number of particles on the right of $X_1(t) - a$: $\phi = \Big($ 

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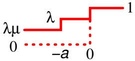
Analytical result: distance and density

P.d.f. of the distances:

$\phi = \Big($ 

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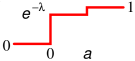
Number of particles on the right of $X_1(t) - a$: $\phi = \Big($ 

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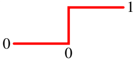
$\partial_t Q = \partial_x^2 Q - Q + Q^2$ with $Q(x, 0) = \Big($ 

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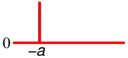
$\partial_t R_a = \partial_x^2 R_a - R_a + 2QR_a$ with $R_a(x, 0) = \delta(x + a) = \Big($ 

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$\Big)$

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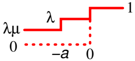
$\phi = \Big($ 

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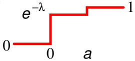
Number of particles on the right of $X_1(t) - a$: $\phi = \Big($ 

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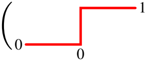
$\partial_t Q = \partial_x^2 Q - Q + Q^2$ with $Q(x,0) = \Big($ 

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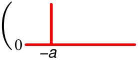
$\partial_t R_a = \partial_x^2 R_a - R_a + 2QR_a$ with $R_a(x,0) = \delta(x+a) = \Big($ 

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$$\text{Proba}[X_1(t) - X_2(t) > a] = \int \mathrm{d}x\; R_a(x,t)$$

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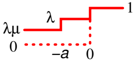
$\phi = \Big($ 

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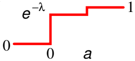
Number of particles on the right of $X_1(t) - a$: $\phi = \Big($ 

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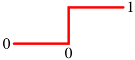
$\partial_t Q = \partial_x^2 Q - Q + Q^2$ with $Q(x,0) = \Big($ 

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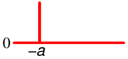
$\partial_t R_a = \partial_x^2 R_a - R_a + 2QR_a$ with $R_a(x,0) = \delta(x+a) = \Big($ 

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$$\text{Proba}[X_1(t) - X_2(t) > a] = \int \mathrm{d}x\ R_a(x,t) \quad \approx e^{-(1+\sqrt{2})a} \ ?$$

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Analytical result: distance and density

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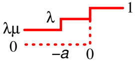
$\phi = \Big($ 

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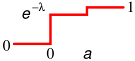
Number of particles on the right of $X_1(t)-a$: $\phi = \Big($ 

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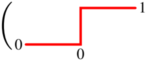
$\partial_t Q = \partial_x^2 Q - Q + Q^2$ with $Q(x,0) = \Big($ 

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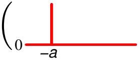
$\partial_t R_a = \partial_x^2 R_a - R_a + 2QR_a$ with $R_a(x,0) = \delta(x+a) = \Big($ 

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$$\text{Proba}[X_1(t)-X_2(t)>a] = \int \mathrm{d}x\ R_a(x,t) \approx e^{-(1+\sqrt{2})a}\ ?$$

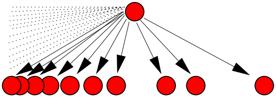
$$\langle N(X_1(t)-a,t)\rangle = \int \mathrm{d}x\ \tilde R_a(x,t) \approx a e^{a}\ ?$$

$$R_a(x,t) \to \lambda_a Q'(x,t) \qquad \text{for } t \text{ large}$$

Thank you !

Exponential model

- N particles, discrete time
- Each particle has infinitely many offspring given by a Poisson process of density ψ: for each $\mathrm{d}x$, there is an offspring with probability $\psi(x - x_{\text{parent}})\,\mathrm{d}x$



- One only keep the N rightmost particles of a given generation

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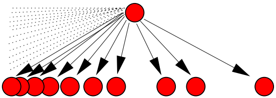

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Usually a Fisher equation

$$v(\gamma) = \frac{1}{\gamma} \ln \left(\int \mathrm{d}\epsilon\, \psi(\epsilon) e^{\gamma\epsilon} \right)$$

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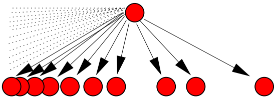

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But not always: $\psi(\epsilon) = e^{-\epsilon}$

Exponential model *vs* Fisher

	Exponential model	Fisher case
v_N	$\ln(\ln N + \ln\ln N) + \mathcal{O}\left(\frac{1}{\ln N}\right)$	$v^* - \frac{A}{(\ln N + 3\ln\ln N)^2}$
D_N	$\frac{\pi^2}{3(\ln N + \ln\ln N)} + \mathcal{O}\left(\frac{1}{\ln^2 N}\right)$	$\frac{B}{(\ln N + ???)^3}$
$p(\delta)$	$e^{-\delta}$	$C_1 e^{-\gamma^*\delta}$
$R(\delta)$	$\ln\left(1 + \frac{e^{\delta}}{\ln N}\right)$	$\frac{1}{\gamma^*}\ln\left(1 + C_2\frac{e^{\gamma^*\delta}}{\ln^3 N}\right)$
Relaxation time	1	$\ln^2 N$
Fluctuation size	$\ln\ln N$	$\frac{1}{\gamma^*}3\ln\ln N$

Conclusion