Minimal surfaces and entire solutions of the Allen-Cahn equation

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The Allen-Cahn Equation

$$\text{(AC)} \qquad \Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N$$

Euler-Lagrange equation for the *energy functional*

$$J(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{4}\int (1-u^2)^2$$

$$F(u) \;=\; \frac{1}{4}(1-u^2)^2,$$

F ''double-well potential'':

$$F(u) > 0, \quad u \neq \pm 1; \quad F(+1) = 0 = F(-1).$$

ε-version: Ω bounded domain in $\mathbb{R}^N$, $\varepsilon > 0$ small.

$$(\mathrm{AC})_\varepsilon \qquad \varepsilon^2 \Delta u + u - u^3 = 0 \quad \text{in } \Omega$$

$$J_\varepsilon(u) = \frac{\varepsilon}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon} \int_\Omega (1 - u^2)^2$$

Critical points: continuous realization of *phase*.

Allen-Cahn: Gradient theory of phase transitions.

$$J_\varepsilon(u) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon}\int_\Omega (1-u^2)^2.$$

$u = \pm 1$ global minimizers: two phases of a material.

For $\Lambda \subset \Omega$, the function

$$u_\Gamma := \chi_\Lambda - \chi_{\Omega\setminus\Lambda} = \begin{cases} u = +1 & \text{in } \Lambda \\ u = -1 & \text{in } \Omega \setminus \Lambda \end{cases}$$

minimizes second integral in J_ε. The inclusion of the gradient term prevents the interface from being too wild:

The interface: $\Gamma = \partial\Lambda \cap \Omega$. Nature selects it approximately locally minimal.

Critical points of

$$J_\varepsilon(u) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon}\int_\Omega (1-u^2)^2.$$

in $H^1(\Omega)$, correspond to solutions of

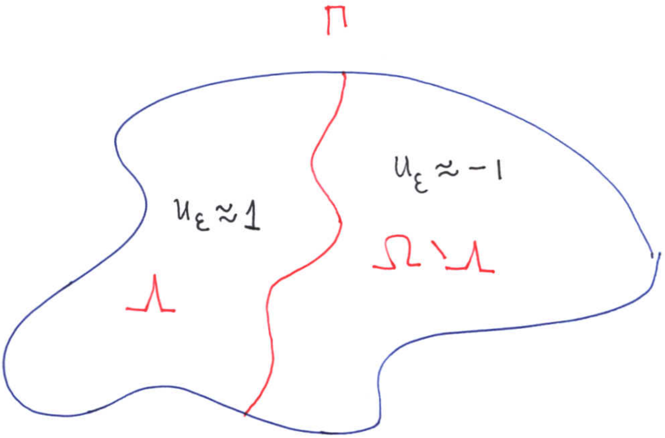
$$\varepsilon^2 \Delta u + u - u^3 = 0 \quad \text{in } \Omega, \quad \partial_\nu u = 0 \quad \text{on } \partial\Omega.$$

Modica and Mortola (1977): *u_ε family of local minimizers with $J_\varepsilon(u_\varepsilon) \leq C$. Then, up to subsequences:*

$$u_\varepsilon \to \chi_\Lambda - \chi_{\Omega\setminus\Lambda} \quad \textit{in } L^1, \quad J_\varepsilon(v_\varepsilon) \to \frac{2}{3}\sqrt{2}\mathcal{H}^{N-1}(\Gamma) \quad \textit{as } \varepsilon \to 0,$$

Perimeter $\mathcal{H}^{N-1}(\Gamma)$ is minimal: $\Gamma = \partial\Lambda \cap \Omega$ is a (generalized) minimal surface.

This result was the motivation of the *theory of Γ-convergence* in the calculus of variations.



$$\boxed{u_\varepsilon \approx \chi_\Lambda - \chi_{\Omega\setminus\Lambda}}$$

$$[u_\varepsilon = 0] \approx \Gamma$$

Γ is a minimal surface if and only if $H_\Gamma = 0$, $H_\Gamma =$ mean curvature of $\Gamma \iff \Gamma$ is stationary for surface area.

Formal asymptotic behavior of u_ε

Assume that Γ is a smooth hypersurface and let ν designate a choice of its unit normal. Local coordinates near Γ:

$$x = y + z\nu(y), \quad y \in \Gamma, \ |z| < \delta$$

Laplacian in these coordinates:

$$\boxed{\Delta_x = \partial_{zz} + \Delta_{\Gamma^z} - H_{\Gamma_z}(y)\,\partial_z}$$

$$\Gamma^z := \{y + z\nu(y) \ / \ y \in \Gamma\}.$$

Δ_{Γ^z} is the Laplace-Beltrami operator on Γ^z acting on functions of y, and $H_{\Gamma^z}(y)$ its mean curvature at the point $y + z\nu(y)$.

Let $k_1, \dots, k_N$ denote the principal curvatures of Γ. Then

$$\boxed{H_{\Gamma^z} = \sum_{i=1}^{N} \frac{k_i}{1 - z k_i}}$$

For later reference, we expand

$$H_{\Gamma^z}(y) = H_\Gamma(y) + z^2\, |A_\Gamma(y)|^2 + z^3 \sum_{i=1}^{N} k_i^3 + \cdots$$

where

$$\underbrace{H_\Gamma = \sum_{i=1}^{N} k_i\,,}_{\textit{mean curvature}} \qquad \textit{norm}\ \underbrace{|A_\Gamma|^2 = \sum_{i=1}^{N} k_i^2}_{\textit{second fundamental}}\ \textit{form} \qquad .$$

Equation for $u_\varepsilon = u_\varepsilon(y,z)$ near Γ:

$$\varepsilon^2(\partial_{zz} + \Delta_{\Gamma^z} - H_{\Gamma_z}(y)\,\partial_z) + u_\varepsilon - u_\varepsilon^3 = 0$$

Change of variable: $\zeta = \varepsilon^{-1} z$. Equation for $u_\varepsilon(y,\zeta)$:

$$\boxed{\varepsilon^2 \Delta_{\Gamma^{\varepsilon\zeta}} u_\varepsilon - \varepsilon H_{\Gamma^{\varepsilon\zeta}}(y)\,\partial_\zeta u_\varepsilon + \partial_{\zeta\zeta} u_\varepsilon + u_\varepsilon - u_\varepsilon^3 = 0}$$

$$u_\varepsilon(y,\zeta) = u_\varepsilon(x), \quad x = y + \varepsilon\zeta\nu(y), \quad y\in\Gamma, \quad |\zeta| < \delta\varepsilon^{-1}.$$

Two strong assumptions on $u_\varepsilon(y,\zeta)$:

- The level set $[u_\varepsilon = 0]$ lies in the region $|\zeta| = o(1)$ $\big($namely $\operatorname{dist}(x,\Gamma) = o(\varepsilon)\big)$ and $\partial_\tau u_\varepsilon > 0$ there.
- $u_\varepsilon(y,\zeta)$ can be expanded in powers of ε as

$$u_\varepsilon(y,\zeta) = u_0(y,\zeta) + \varepsilon u_1(y,\zeta) + \varepsilon^2 u_2(y,\zeta) + \cdots$$

for coefficients u_j with bounded derivatives.

Besides

$$\int_\Gamma \int_{-\delta/\varepsilon}^{\delta/\varepsilon} [\, \frac{1}{2} |\partial_\zeta u_\varepsilon|^2 + \frac{1}{4}(1 - u_\varepsilon^2)^2 \,] \, d\zeta \, d\sigma(y) \ \leq \ J_\varepsilon(u_\varepsilon) + o(1) \leq C$$

Substituting $u_\varepsilon = u_0 + \varepsilon u_1 + \cdots$ into

$$\varepsilon^2 \Delta_{\Gamma^{\varepsilon\zeta}} u_\varepsilon \; - \; \varepsilon H_{\Gamma^{\varepsilon\zeta}}(y)\, \partial_\zeta u_\varepsilon \; + \; \partial_{\zeta\zeta} u_\varepsilon \; + \; u_\varepsilon \; - \; u_\varepsilon^3 \; = \; 0$$

and letting formally $\varepsilon \to 0$ we get

$$\partial_{\zeta\zeta} u_0 + u_0 - u_0^3 = 0 \quad \text{for all} \quad (y, \zeta) \in \Gamma \times \mathbb{R},$$

$$u_0(0, y) = 0, \quad \partial_\zeta(0, y) \geq 0, \quad \text{for all} \quad y \in \Gamma,$$

$$\int_{\mathbb{R}} [\, \frac{1}{2} |\partial_\zeta u_0|^2 + \frac{1}{4}(1 - u_0^2)^2 \,]\, d\zeta \; < \; +\infty$$

This implies $u_0(y, \zeta) = w(\zeta)$ where w solves

$$w'' + w - w^3 = 0, \quad w(0) = 0, \quad w(\pm\infty) = \pm 1,$$

namely

$$\boxed{w(\zeta) \; := \; \tanh(\zeta/\sqrt{2})}.$$

Substitute $u_\varepsilon = w(\zeta) + \varepsilon v_1(y,\zeta) + \varepsilon^2 v_2 + \cdots$ into

$$\varepsilon^2 \Delta_{\Gamma\varepsilon\zeta} u_\varepsilon \;-\; \varepsilon H_{\Gamma\varepsilon\zeta}(y)\,\partial_\zeta u_\varepsilon \;+\; \partial_{\zeta\zeta} u_\varepsilon \;+\; u_\varepsilon \;-\; u_\varepsilon^3 \;=\; 0.$$

Using $\partial_{\zeta\zeta} w + w - w^3 = 0$ we get

$$\varepsilon[\,-H_{\Gamma\varepsilon\zeta}(y)\,\partial_\zeta w \;+\; \partial_{\zeta\zeta} v_1 \;+\; (1-3w^2)v_1\,] + O(\varepsilon^2) \;=\; 0$$

Letting $\varepsilon \longrightarrow 0$ we find

$$\boxed{\partial_{\zeta\zeta} v_1 + (1-3w(\zeta)^2)v_1 \;=\; H_\Gamma(y)\,w'(\zeta)}$$

The linear ODE problem

$$h'' + (1 - 3w^2)h = f(\zeta) \quad \texttt{for all} \quad \zeta \in (-\infty, \infty)$$

where f is bounded, has a bounded solution if and only if

$$\int_{-\infty}^{\infty} f\, w'\, d\zeta = 0.$$

$$h(\zeta) = -w'(\zeta) \left(A + \int_0^{\zeta} w'(t)^{-2} \int_t^{\infty} w'(s)\, f(s)\, ds \right)$$

Thus

$$\partial_{\zeta\zeta} v_1 + (1 - 3w(\zeta)^2) v_1 \ = \ H_\Gamma(y)\, w'(\zeta)$$

implies

$$H_\Gamma(y) \int_{-\infty}^{\infty} w'(\zeta)^2 \, d\zeta = 0$$

or

$$\boxed{H_\Gamma(y) = 0 \quad \texttt{for all} \quad y \in \Gamma}$$

namely Γ **must be a minimal surface**, as expected and

$$\boxed{v_1(y,\zeta) = -h_0(y) w'(\zeta)}$$

for a certain function $h_0(y)$.

Hence

$$u_\varepsilon(y,\zeta) = w(\zeta) - \varepsilon h_0(y) w'(\zeta) + \varepsilon^2 v_2 + \cdots$$

or, Taylor expanding, redefining v_2,

$$u_\varepsilon(y,\zeta) = w(\zeta - \varepsilon h_0(y)) + \varepsilon^2 v_2 + \cdots$$

It is convenient to write this expansion in terms of the variable $t = \zeta - \varepsilon h_0(y)$ in the form

$$\boxed{u_\varepsilon(y,t) = w(t) + \varepsilon^2 u_2(t,y) + \varepsilon^3 u_3(t,y) + \cdots}$$

$$u_\varepsilon(y,t) = u_\varepsilon(x), \quad x = y + \varepsilon(t + \varepsilon h_0(y))\,\nu(y).$$

Laplacian in (y, t) coordinates

$$x = y + \varepsilon \zeta \nu(y), \quad \zeta = t + \varepsilon h_0(y)$$

Laplacian can be expanded as

$$\varepsilon^2 \Delta_x = \varepsilon^2 \Delta_{\Gamma\varepsilon\zeta} + \partial_{tt} - \varepsilon H_{\Gamma\varepsilon\zeta}(y)\,\partial_t$$

$$\varepsilon^4 |\nabla_{\Gamma\varepsilon\zeta} h_0|^2 \,\partial_{tt} - 2\varepsilon^2 \langle \nabla_{\Gamma\varepsilon\zeta} h_0, \nabla_{\Gamma\varepsilon\zeta} \partial_t \rangle - \varepsilon^3 \Delta_{\Gamma\varepsilon\zeta} h_0 \partial_t \quad .$$

Laplace beltrami

$$\varepsilon^2 \Delta_{\Gamma\varepsilon\zeta} = \varepsilon^2 \Delta_\Gamma + O(\varepsilon^3 (t + \varepsilon h_0)) D_\Gamma^2$$

$$\varepsilon H_{\Gamma\varepsilon\zeta} = 0 + \varepsilon^2 (t + \varepsilon h_0) |A_\Gamma|^2 + \varepsilon^3 (t + \varepsilon h_0)^2 \sum_{i=1}^{N-1} k_i^3 + O(\varepsilon^4 (t + \varepsilon h_0)^3)$$

Let us substitute into the equation the expansion

$$u_\varepsilon(y,t) = w(t) + \varepsilon^2 u_2(y,t) + \varepsilon^3 u_3(y,t) + \cdots$$

We get

$$0 = \Delta v_\varepsilon + v_\varepsilon + v_\varepsilon^3 = \left[\partial_{tt} + (1 - 3w(t)^2)\right](\varepsilon^2 v_2 + \varepsilon^3 v_3)$$

$$-w'(t)\left[\varepsilon^3 \Delta_\Gamma h_0 + \varepsilon^3 \sum_{i=1}^{N-1} k_i^3\, t^2 + \varepsilon^2 |A_\Gamma|^2 (t + \varepsilon h_0)\right] + O(\varepsilon^4).$$

Letting $\varepsilon \to 0$ we arrive to the following equations for u_2 and u_3.

$$u_\varepsilon(y,t) = w(t) + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots$$

$$\partial_{tt} u_2 + (1-3w^2)u_2 = |A_\Gamma|^2\, tw' =: f_2,$$

$$\partial_{tt} u_3 + (1-3w^2)u_3 = [\Delta_\Gamma h_0 + |A_\Gamma|^2 h_0 + \sum_{i=1}^{N-1} k_i^3\, t^2]\, w' =: f_3.$$

solvable in L^∞ iff

$$\int_{\mathbb{R}} f_2(y,t) w'(t)\, dt = 0 = \int_{\mathbb{R}} f_2(y,t) w'(t)\, dt0 \quad \textbf{for all} \quad y \in \Gamma.$$

- `Automatically` $\int_{\mathbb{R}} f_2(y,t) w'(t)\, dt = 0$.
- `Equation for` u_3 `is solvable if and only if`

$$\int_{\mathbb{R}} [\Delta_\Gamma h_0 + |A_\Gamma|^2 h_0 + \sum_{i=1}^{N-1} k_i^3\, t^2]\, w'^2\, dt = 0,$$

If and only if h_0 solves the equation

$$\boxed{\mathcal{J}_\Gamma[h_0](y) := \Delta_\Gamma h_0 + |A_\Gamma|^2 h_0 = c \sum_{i=1}^{N-1} k_i^3 \quad \text{in } \Gamma,}$$

where $c = -\int_{\mathbb{R}} t^2 w'^2\, dt / \int_{\mathbb{R}} w'^2\, dt$. $\mathcal{J}_\Gamma$ is by definition the **Jacobi operator** of the minimal surface Γ.

$$\boxed{u_\varepsilon(x) = w(\zeta - \varepsilon h_0(y)) + O(\varepsilon^2), \quad x = y + \varepsilon\, \zeta\, \nu(y), \ |\zeta| < \delta\varepsilon^{-1}.}$$

Problem I:

Given a minimal surface Γ find a solution u_ε to

$$\varepsilon^2 \Delta u + u - u^3 = 0 \quad \text{in } \Omega, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega$$

with

$$|u_\varepsilon(x)| \to 1 \quad x \text{ away from } \Gamma,$$

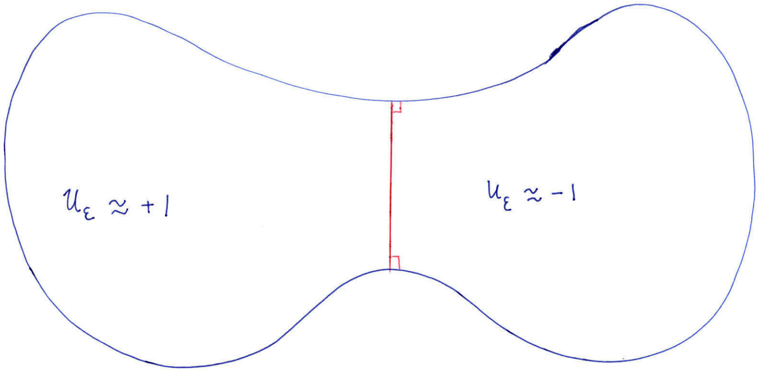
$$u_\varepsilon(x) = w(\zeta - \varepsilon h_0(y)) + O(\varepsilon^2) \quad \text{near } \Gamma.$$

Neumann boundary condition makes it necessary that Γ intersects orthogonally $\partial\Omega$.

Some known results:

• Kohn-Sternberg (1989): $N = 2$. If Γ is an *isolated minimizing segment, for curve length with endpoints on the boundary*, a solution u_ε with $|u_\varepsilon(x)| \to 1$ x away from Γ exists also with

$$J_\varepsilon(u_\varepsilon) \to \frac{2}{3}\sqrt{2}|\Gamma|$$



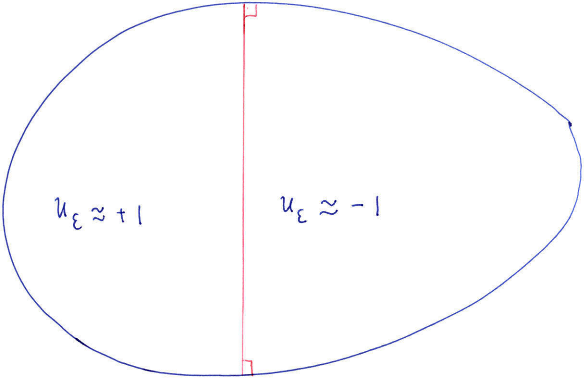
Γ locally minimizing segment, *Kohn and Sternberg 1989*

- Kowalzcyk (2002): $N = 2$. If Γ is an *nondegenerate critical segment, for curves with endpoints on the boundary* a solution as above, also with $u_\varepsilon(x) = w(\zeta - \varepsilon h_0(y)) + O(\varepsilon^2)$ exists.

Nondegeneracy means invertibility of second variation of arclength at Γ with suitable boundary conditions. $\{P_0, P_1\} = \Gamma \cap \partial\Omega$:

$$\kappa(P_0) + \kappa(P_1) - \kappa(P_0)\kappa(P_1)|\Gamma| \neq 0$$

This assumption has been relaxed by Jerrard and Sternberg.

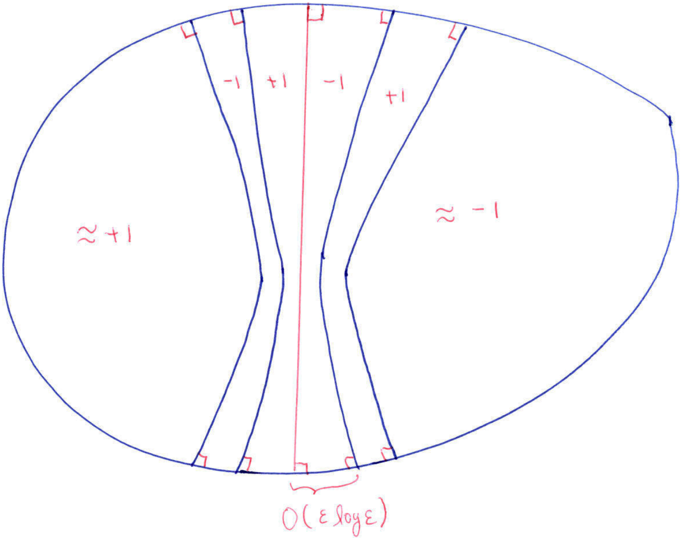


Γ nondegenerate critical segment, *Kowalczyk, 2002*.

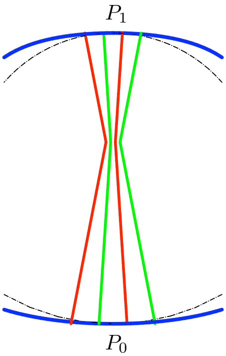
• del Pino, Kowalczyk, Wei (2005): $N = 2$. If $R_i := \kappa(P_i) > 0$, $R_1 + R_2 > |\Gamma|$, $|R_1 - R_2| < |\Gamma|$ (e.g. short axis of an ellipse), there is a solution with any given number $m \geq 1$ of interfaces $O(\varepsilon \log \varepsilon)$-distant one to each other:

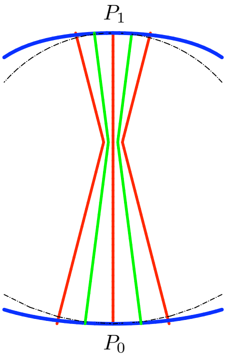
$$J_\varepsilon(u_\varepsilon) \to m\,\frac{2}{3}\sqrt{2}|\Gamma|.$$

Equilibria of interfaces is governed by an integrable system of ODEs along Γ: The *Toda system.*



Multiple interfaces, Γ "short-axis" type critical segment, *del Pino-Kowalczyk-Wei, 2005.*



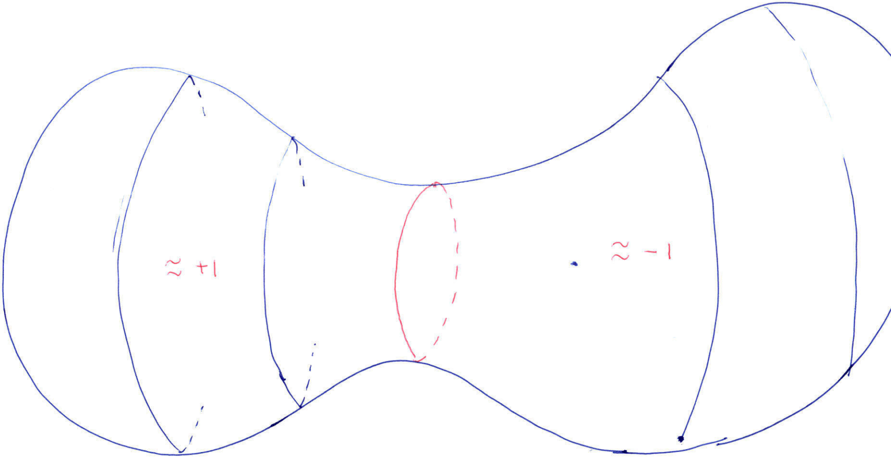


Configurations with 4 and 5 transitions.
Dotted lines indicate osculating circles at P_0 and P_1.

• Pacard-Ritoré (2002): Equation on a compact Riemannian manifold $\mathcal{M}$, $N \geq 2$.

$$\varepsilon^2 \Delta_g u + u - u^3 = 0 \quad \text{in } \mathcal{M}$$

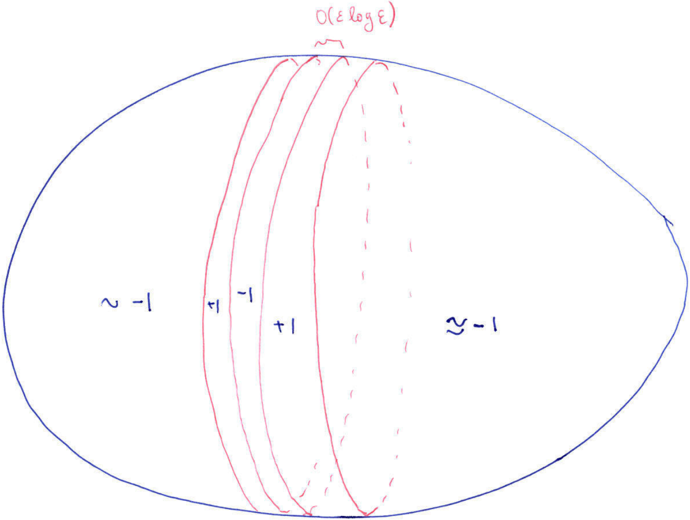
If Γ is an *nondegenerate minimal surface of codimension 1 that splits* $\mathcal{M}$ *into two components*, a solution u_ε as above, also with precise asymptotics near Γ exists.
Nondegeneracy means *invertibility of Jacobi operator of* Γ.



Γ nondegenerate minimal hypersurface in compact manifold. *Pacard-Ritoré, 2002*.

• del Pino-Kowalczyk-Wei (2009): Equation on compact manifold $\mathcal{M}$, $N \geq 2$. If Γ is a nondegenerate minimal surface of codimension 1, and $\mathcal{M}$ is *positively curved along* Γ *(Gauss curvature positive along geodesic* Γ *if* $N = 2$*)* then a solution u_ε with $m \geq 1$ given interfaces $O(\varepsilon \log \varepsilon)$-distant from Γ exists, at least along a sequence $\varepsilon = \varepsilon_j \to 0$.

Nondegeneracy + positive curvature $\implies$ *invertibility of the Jacobi-Toda operator of* Γ.



Multiple interfaces, Γ positively curved minimal hypersurface in compact manifold
del Pino-Kowalczyk-Wei, 2009.

Why just bounded domains or manifolds?

Why not entire space and a complete, embedded minimal hypersurface Γ in $\mathbb{R}^N$?

$$\varepsilon^2 \Delta u + u - u^3 = 0 \quad \texttt{in } \mathbb{R}^N \qquad (AC)_\varepsilon$$

Problem II:

Given a minimal surface Γ embedded in $\mathbb{R}^N$, that divides the space into two components, find a solution u_ε to

$$\varepsilon^2 \Delta u + u - u^3 = 0 \quad \texttt{in } \mathbb{R}^N$$

with

$$|u_\varepsilon(x)| \to 1 \quad x \texttt{ away from } \Gamma,$$

$$u_\varepsilon(x) = w(\zeta - \varepsilon h_0(y)) + O(\varepsilon^2) \quad \texttt{near } \Gamma.$$

$$x = y + \varepsilon \zeta \nu(y), \quad y \in \Gamma.$$

ε can be scaled out: replacing $u(x)$ by $u(x/\varepsilon)$ equation $(AC)_\varepsilon$ is equivalent to

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N \qquad (AC)$$

We express Problem II in terms of equation (AC) as follows.

Problem II

Given a minimal surface Γ embedded in $\mathbb{R}^N$ let

$$\Gamma_\varepsilon \ := \ \varepsilon^{-1}\Gamma,$$

find a solution u_ε to

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N \qquad (AC)$$

such that for a function h_ε defined on Γ with $\sup_{\varepsilon>0} \|h_\varepsilon\|_{L^\infty(\Gamma)} < +\infty$, we have

$$u_\varepsilon(x) = w(\zeta - \varepsilon h_\varepsilon(\varepsilon y)) + O(\varepsilon^2),$$

$$x = y + \zeta\nu(\varepsilon y), \quad |\zeta| \le \frac{\delta}{\varepsilon}, \quad y \in \Gamma_\varepsilon.$$

We solve Problem II in two important examples for Γ:

• A nontrivial minimal graph in $\mathbb{R}^9$. The solution found provides a negative answer to to a celebrated question by De Giorgi.

• A complete, embedded minimal surfaces in $\mathbb{R}^3$ with finite total curvature. This provides a large class of solutions with finite Morse index.

Problem II in a nontrivial minimal graph in $\mathbb{R}^9$.

The connection we have so far described, between minimal surfaces and solutions of (AC), motivated E.De Giorgi to conhecture that bounded entire solutions of (AC) that are monotone in one direction must have **one-dimensional symmetry**.

Case $N = 1$: the function

$$w(t) := \tanh\left(\frac{t}{\sqrt{2}}\right)$$

connects monotonically -1 and $+1$ and solves

$$w'' + w - w^3 = 0, \quad w(\pm\infty) = \pm 1, \quad w' > 0.$$

For any $p, \nu \in \mathbb{R}^N$, $|\nu| = 1$, the function

$$u(x) := w(\,(x - p) \cdot \nu)$$

solves equation (AC).

De Giorgi's conjecture (1978): *Let u be a bounded solution of equation*

$$(AC) \qquad \Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N,$$

which is monotone in one direction, say $\partial_{x_N} u > 0$. Then, **at least** *when $N \leq 8$, there exist p, ν such that*

$$u(x) = w(\,(x-p)\cdot\nu).$$

This statement is equivalent to:

At least when $N \leq 8$, all level sets of u, $[u = \lambda]$ must be hyperplanes.

Parallel to **Bernstein's problem** for minimal surfaces which are entire graphs.

Entire minimal graph in $\mathbb{R}^N$:

$$\Gamma = \{(x', F(x')) \in \mathbb{R}^{N-1} \times \mathbb{R} \ / \ x' \in \mathbb{R}^{N-1}\}$$

where F solves the minimal surface equation

$$(MS) \qquad H_\Gamma := \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}.$$

Bernstein's problem: *Is it true that all entire minimal graphs are hyperplanes, namely any entire solution of (MS) must be a linear affine function?* :

True for $N \leq 8$ (Bernstein, Fleming, Simons, Bombieri-De Giorgi-Giusti). **False** for $N \geq 9$: counterexample [BDG] in 1969.

De Giorgi's Conjecture: *u bounded solution of (AC),* $\partial_{x_N} u > 0$ *then level sets* $[u = \lambda]$ *are hyperplanes.*

- True for $N = 2$. Ghoussoub and Gui (1998).
- True for $N = 3$. Ambrosio and Cabré (1999).
- True for $4 \leq N \leq 8$ (Savin (2008), thesis (2003)) if in addition

$$(P) \qquad \lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \quad \text{for all} \quad x' \in \mathbb{R}^{N-1}.$$

The Bombieri-De Giorgi-Giusti minimal graph:

Explicit construction by super and sub-solutions.
$N = 9$:

$$\boxed{H(F) := \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8.}$$

$$F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}, \quad (\mathbf{u}, \mathbf{v}) \mapsto F(|\mathbf{u}|, |\mathbf{v}|).$$

In addition, $F(|\mathbf{u}|, |\mathbf{v}|) > 0$ for $|\mathbf{v}| > |\mathbf{u}|$ and

$$F(|\mathbf{u}|, |\mathbf{v}|) = -F(|\mathbf{v}|, |\mathbf{u}|).$$

Polar coordinates:

$$|\mathbf{u}| = r\cos\theta, \quad |\mathbf{v}| = r\sin\theta, \qquad \theta \in (0, \frac{\pi}{2})$$

Mean curvature operator at $F = F(r,\theta)$

$$\begin{aligned} H[F] \;&=\; \frac{1}{r^7\sin^3 2\theta}\partial_r\left(\frac{F_r r^7\sin^3 2\theta}{\sqrt{1+F_r^2+r^{-2}F_\theta^2}}\right) \\ &+\; \frac{1}{r^7\sin^3 2\theta}\partial_\theta\left(\frac{F_\theta r^5\sin^3 2\theta}{\sqrt{1+F_r^2+r^{-2}F_\theta^2}}\right). \end{aligned}$$

Separation of variables $F_0(r,\theta) = r^3 g(\theta)$.

$$
\begin{aligned}
H[F_0] &= \frac{1}{r^7 \sin^3 2\theta} \partial_r \left(\frac{3r^7 g \sin^3 2\theta}{\sqrt{r^{-4} + 9g^2 + g'^2}} \right) \\
&+ \frac{1}{r \sin^3 2\theta} \partial_\theta \left(\frac{g' \sin^3 2\theta}{\sqrt{r^{-4} + 9g^2 + g'^2}} \right) \quad .
\end{aligned}
$$

As $r \to \infty$ the equation $H(F_0) = 0$ becomes the ODE

$$
\frac{21 g \sin^3 2\theta}{\sqrt{9g^2 + g'^2}} + \left(\frac{g' \sin^3 2\theta}{\sqrt{9g^2 + g'^2}} \right)' = 0 \quad \text{in} \quad \left(\frac{\pi}{4}, \frac{\pi}{2} \right),
$$

$$
g\left(\frac{\pi}{4}\right) = 0 = g'\left(\frac{\pi}{2}\right).
$$

This problem has a solution g positive in $(\frac{\pi}{4}, \frac{\pi}{2})$.

We check directly that

- $F_0(r,\theta) = r^3 g(\theta)$ is a subsolution of the minimal surface equation $H(F) = 0$: $H(F_0) \geq 0$

- $F_0(r,\theta)$ accurate approximation to a solution of the minimal surface equation:

$$H(F_0) = O(r^{-5}) \quad \text{as } r \to +\infty.$$

The supersolution of Bombieri, De Giorgi and Giusti can be refined to yield that F_0 gives the precise asymptotic behavior of F.

Fact: $\big($Refinement of asymptotic behavior of BDG surface $x_9 = F(r,\theta)\big)$

For $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$ *we have, for* $0 < \sigma < 1$ *and all large* r*,*

$$\boxed{F_0(r,\theta) \;\leq\; F(r,\theta) \leq F_0(r,\theta) + Ar^{-\sigma} \quad \text{as } r \to +\infty.}$$

$$H(F) = H(F_0 + \varphi) \approx O(r^{-5}) + H'(F_0)[\varphi] = 0$$

Relation

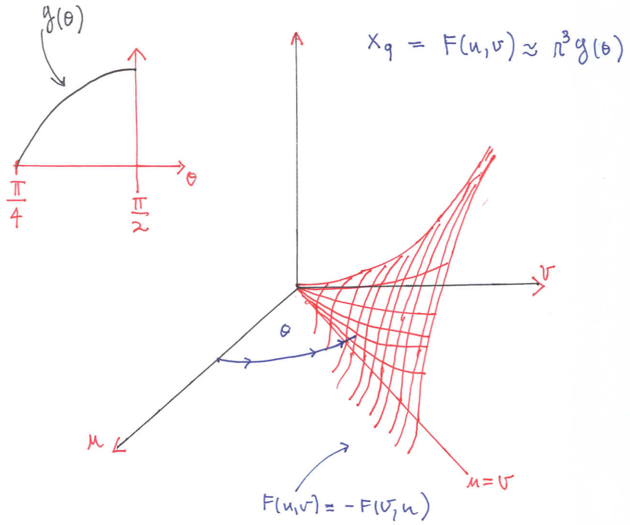
$$O(r^{-5}) + H'(F_0)[\varphi] \leq 0$$

has a positive supersolution $\varphi = O(r^{-\sigma})$. Here

$$H'(G)[\varphi] := \frac{d}{dt} H(G + t\varphi)\,|_{t=0} =$$

$$\nabla \cdot \left(\frac{\nabla \varphi}{\sqrt{1 + |\nabla G|^2}} - \frac{(\nabla G \cdot \nabla \varphi)}{(1 + |\nabla G|^2)^{\frac{3}{2}}} \nabla G \right).$$

The BDG surface:



An important characteristic of Γ is its *uniform flatness at infinity*:

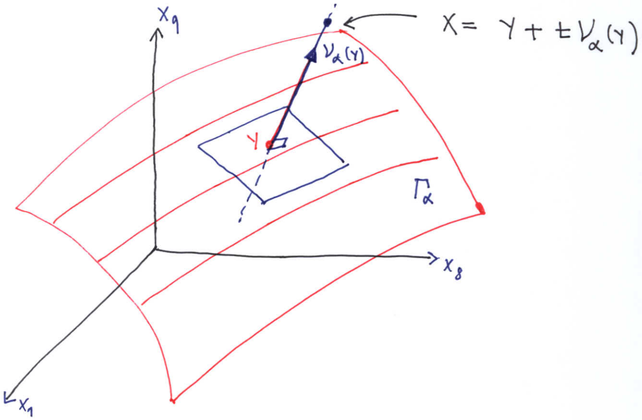
L. Simon, 1989: Curvatures decay along Γ: Let $k_i(y)$, $i = 1, \ldots, 8$ be principal curvatures of Γ. Then

$$k_j(y) = O(r(y)^{-1}) \quad \text{as} \ \ r(y) \to +\infty.$$

$$y = (y', y_9), \quad r(y) = |y'|.$$

Let $\nu_\varepsilon(y) := \nu(\varepsilon y), \quad y \in \Gamma_\varepsilon = \varepsilon^{-1}\Gamma$ be unit normal with $\nu_9 > 0$.
Local coordinates in in a tubular neighborhood of Γ_ε:

$$\boxed{x = y + \zeta \nu_\varepsilon(y), \quad y \in \Gamma_\varepsilon, \quad |\zeta| < \frac{\delta}{\varepsilon}}$$



Theorem (del Pino, Kowalczyk, Wei (2008))

Let Γ be a BDG minimal graph in $\mathbb{R}^9$ and $\Gamma_\varepsilon := \varepsilon^{-1}\Gamma$. Then for all small $\varepsilon > 0$, there exists a bounded solution u_ε of (AC), monotone in the x_9-direction, with

$$u_\varepsilon(x) = w(\zeta - \varepsilon h_\varepsilon(\varepsilon y)) + O(\varepsilon^2), \quad x = y + \zeta\nu(\varepsilon y), \quad y \in \Gamma_\varepsilon,\ |\zeta| < \frac{\delta}{\varepsilon},$$

$$\sup_{\varepsilon > 0} \|(1 + r)h_\varepsilon\|_{L^\infty(\Gamma)} < +\infty,$$

$$\lim_{x_9 \to \pm\infty} u(x', x_9) = \pm 1 \quad \text{for all} \quad x' \in \mathbb{R}^8.$$

u_ε is a ''counterexample'' to De Giorgi's conjecture in dimension 9 (hence in any dimension higher).

The Proof.

Letting $f(u) = u - u^3$ the equation

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^9$$

becomes, for

$$u(y, \zeta) := u(x), \quad x = y + \zeta\nu(\varepsilon y), \quad y \in \Gamma_\varepsilon, \ |\zeta| < \delta/\varepsilon,$$

ν unit normal to Γ with $\nu_N > 0$,

$$S(u) := \Delta u + f(u) \ =$$

$$\Delta_{\Gamma_\varepsilon^\zeta} u - \varepsilon H_{\Gamma^{\varepsilon\zeta}}(\varepsilon y)\, \partial_\zeta u \ + \partial_\zeta^2 u + f(u) \ = \ 0.$$

We look for a solution of the form (near Γ_ε)

$$u_\varepsilon(x) = w(\zeta - \varepsilon h(\varepsilon y)) + \phi, \quad x = y + \zeta\nu(\varepsilon y)$$

for a function h defined on Γ, left as a parameter to be adjusted.

$r(y', y_9) = |y'|$. We assume a priori on h that

$$\|(1+r^3)D_\Gamma^2 h\|_{L^\infty(\Gamma)} + \|(1+r^2)D_\Gamma h\|_{L^\infty(\Gamma)} + \|(1+r)h\|_{L^\infty(\Gamma)} \leq M$$

for some large, fixed number M.

Let us change variables to $t = \zeta - \varepsilon h(\varepsilon y)$, or

$$u(y,t) := u(x) \quad x = y + (t + \varepsilon h(\varepsilon y))\,\nu(\varepsilon y)$$

The equation becomes

$$S(u) = \partial_{tt} u + \Delta_{\Gamma^{\zeta}_{\varepsilon}} u - \varepsilon H_{\Gamma_{\varepsilon\zeta}}(\varepsilon y)\,\partial_t u \; +$$

$$+ \; \varepsilon^4 |\nabla_{\Gamma_{\varepsilon\zeta}} h(\varepsilon y)|^2 \partial_{tt} u - 2\varepsilon^3 \langle \nabla_{\Gamma_{\varepsilon\zeta}} h(\varepsilon y), \partial_t \nabla_{\Gamma_{\varepsilon\zeta}} u \rangle$$

$$- \; \varepsilon^3 \Delta_{\Gamma_{\varepsilon\zeta}} h(\varepsilon y)\,\partial_t u + f(u) \;=\; 0, \quad \zeta = t + \varepsilon h(\varepsilon y).$$

Look for solution u_ε of the form

$$u_\varepsilon(t,y) = w(t) + \phi(t,y)$$

for a small function ϕ.

$$u_\varepsilon(t,y) = w(t) + \phi(t,y)$$

The equation in terms of ϕ becomes

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E \,=\, 0.$$

where B is a small linear second order operator, and

$$E = S(w(t)), \quad N(\phi) = f(w+\phi) - f(w) - f'(w)\phi \approx f''(w)\phi^2.$$

The error of approximation.

$$E := S(w(t)) \;\; =$$

$$\varepsilon^4 |\nabla_{\Gamma_{\varepsilon\zeta}} h(\varepsilon y)|^2 w''(t) - [\varepsilon^3 \Delta_{\Gamma_{\varepsilon\zeta}} h(\varepsilon y) \; + \varepsilon H_{\Gamma_{\varepsilon\zeta}}(\varepsilon y)]\, w'(t),$$

and

$$\varepsilon H_{\Gamma_{\varepsilon\zeta}}(\varepsilon y) = \varepsilon^2 (t + \varepsilon h(\varepsilon y)) |A_\Gamma(\varepsilon y)|^2 \; +$$

$$\varepsilon^3 (t + \varepsilon h(\varepsilon y))^2 \sum_{i=1}^{N-1} k_i^3(\varepsilon y) + \cdots$$

We see in particular that

$$|E(y,t)| \;\; \le \;\; C\varepsilon^2 r(\varepsilon y)^{-2} e^{-|t|}.$$

Equation

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E \ = \ 0.$$

makes sense only for $|t| < \delta\varepsilon^{-1}$.

A **gluing procedure** reduces the full problem to

$$\boxed{\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w)\phi + N(\phi) + E = 0 \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon,}$$

where E and B are the same as before, but cut-off far away. N is modified by the addition of a small nonlocal operator of ϕ.

We find a small solution to this problem in **two steps**.

Infinite dimensional Lyapunov-Schmidt reduction:

Step 1: Given the parameter function h, find a a solution $\phi = \Phi(h)$ to the problem

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E = c(y)w'(t) \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon,$$

$$\int_{\mathbb{R}} \phi(t,y)w'(t)\,dt = 0 \quad \text{for all} \quad y \in \Gamma_\varepsilon.$$

Step 2: Find a function h such that for all $y \in \Gamma_\varepsilon$,

$$c(y)\int_{\mathbb{R}} w'^2\,dt := \int_{\mathbb{R}} (E + B\Phi(h) + N(\Phi(h)))\,w'dt = 0.$$

For **Step 1** we solve first the linear problem

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + f'(w(t))\phi = g(t,y) - c(y)w'(t) \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon$$

$$\int_{\mathbb{R}} \phi(y,t)w'(t)\,dt = 0 \quad \text{in } \Gamma_\varepsilon, \quad c(y) := \frac{\int_{\mathbb{R}} g(y,t)w'(t)\,dt}{\int_{\mathbb{R}} w'^2\,dt}.$$

There is a unique bounded solution ϕ if g is bounded, and

$$\|\phi\|_\infty \;\le\; C\,\|g\|_\infty.$$

$\Gamma_\varepsilon \approx \mathbb{R}^{N-1}$ around each of its points as $\varepsilon\to 0$, in uniform way. The problem is qualitatively similar to Γ_ε replaced with $\mathbb{R}^{N-1}$.

Fact: *The linear model problem*

$$\partial_{tt}\phi + \Delta_y \phi + f'(w(t))\phi = g(t,y) - c(y)w'(t) \quad \text{in } \mathbb{R}^N$$

$$\int_{\mathbb{R}} \phi(y,t)w'(t)\,dt = 0 \quad \text{in } \mathbb{R}^{N-1}, \quad c(y) := \frac{\int_{\mathbb{R}} g(y,t)w'(t)\,dt}{\int_{\mathbb{R}} w'^2\,dt}$$

has a unique bounded solution ϕ *if* g *is bounded, and*

$$\|\phi\|_\infty \ \le \ C\,\|g\|_\infty .$$

Let us prove first the a priori estimate:

If the a priori estimate did not hold, there would exist

$$\|\phi_n\|_\infty = 1, \quad \|g_n\|_\infty \to 0,$$

$$\partial_{tt}\phi_n + \Delta_y \phi_n + f'(w(t))\phi_n = g_n(t,y), \quad \int_{\mathbb{R}} \phi_n(y,t) w'(t)\, dt = 0.$$

Using maximum principle and local elliptic estimates, we may assume that $\phi_n \to \phi \neq 0$ uniformly over compact sets where

$$\partial_{tt}\phi + \Delta_y \phi + f'(w(t))\phi = 0, \quad \int_{\mathbb{R}} \phi(y,t) w'(t)\, dt = 0.$$

Claim: $\phi = 0$, which is a contradiction

A key one-dimensional fact: The spectral gap estimate.

$$L_0(p) := p'' + f'(w(t))p$$

There is a $\gamma > 0$ such that if $p \in H^1(\mathbb{R})$ and $\int_{\mathbb{R}} p\, w'\, dt = 0$ then

$$-\int_{\mathbb{R}} L_0(p)\, p\, dt = \int_{\mathbb{R}} (|p'|^2 - f'(w)p^2)\, dt \geq \gamma \int_{\mathbb{R}} p^2\, dt\, .$$

Using maximum principle we find $|\phi(y,t)| \le Ce^{-|t|}$. Set $\varphi(y) = \int_{\mathbb{R}} \phi^2(y,t)\,dt$. Then

$$\Delta_y \varphi(y) = 2\int_{\mathbb{R}} \phi\Delta\phi(y,t)\,dt + 2\int_{\mathbb{R}} |\nabla_y \phi(y,t)|^2\,dt \ge$$

$$-2\int_{\mathbb{R}} \phi\partial_{tt}\phi + f'(w)\phi^2\,dt \;\; =$$

$$2\int_{\mathbb{R}} (|\phi_t|^2 - f'(w)\phi^2)\,dt \ge \gamma\varphi(y).$$

$$-\Delta_y \varphi(y) + \gamma\varphi(y) \le 0$$

and $\varphi \ge 0$ bounded, implies $\varphi \equiv 0$, hence $\phi = 0$, a contradiction. This proves the a priori estimate.

Existence: take g compactly supported. Set H be the space of all $\phi \in H^1(\mathbb{R}^N)$ with

$$\int_{\mathbb{R}} \phi(y,t) w'(t)\, dt = 0 \quad \text{for all} \quad y \in \mathbb{R}^{N-1}.$$

H is a closed subspace of $H^1(\mathbb{R}^N)$.

The problem: $\phi \in H$ and

$$\partial_{tt}\phi + \Delta_y \phi + f'(w(t))\phi = g(t,y) - w'(t)\frac{\int_{\mathbb{R}} g(y,\tau) w'(\tau)\, d\tau}{\int_{\mathbb{R}} w'^2\, d\tau},$$

can be written variationally as that of minimizing in H the energy

$$I(\phi) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_y \phi|^2 + |\phi_t|^2 - f'(w)\phi^2 + \int_{\mathbb{R}^N} g\phi$$

I is coercive in H thanks to the 1d spectral gap estimate. Existence in the general case follows by the L^∞-a priori estimate and approximations.

A similar proof works to yield existence of a solution $\phi := T(g)$ for the problem

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + f'(w(t))\phi = g(t,y) - c(y)w'(t) \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon$$

$$\int_{\mathbb{R}} \phi(y,t)w'(t)\,dt = 0 \quad \text{in } \Gamma_\varepsilon, \;\; c(y) := \frac{\int_{\mathbb{R}} g(y,t)w'(t)\,dt}{\int_{\mathbb{R}} w'^2\,dt}.$$

We need control for ϕ in other norms that adapt to the error size: $g = O(\varepsilon^2 e^{-\sigma|t|} r^{-2}(\varepsilon y)))$.

We want to consider right hand sides g that have

- Exponential decay in t-variable
- Algebraic decay in $r(\varepsilon y)$ in y-variable

We consider the norms

$$\|g\|_{p,\nu,\sigma} \;:=\; \sup_{(y,t)\in\Gamma_\varepsilon\times\mathbb{R}} e^{\sigma|t|} r_\varepsilon^{\nu}(y)\,\|g\|_{L^p(B((y,t),1))}.$$

Let $0 < \sigma < \sqrt{2}$, $\nu \geq 0$, $p > 1$. *For the solution* $\phi := T(g)$ *to the above problem we have*

$$\|D_\Gamma^2\phi\|_{p,\nu,\sigma} + \|\phi\|_{p,\nu,\sigma} \;\leq\; C\,\|g\|_{p,\nu,\sigma}.$$

We use this norm and contraction mapping principle to solve the nonlinear projected problem of **Step 1**.

We write the problem of **Step 1**,

$$\begin{aligned} \partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E &= \\ c(y)w'(t) \quad &\text{in } \mathbb{R}\times\Gamma_\varepsilon, \\ \int_{\mathbb{R}} \phi(t,y)w'(t)\,dt &= 0 \quad \text{for all} \quad y\in\Gamma_\varepsilon, \end{aligned}$$

in fixed point form

$$\phi = T(B\phi + N(\phi) + E).$$

Let $p > 9$. *Since* $\|E\|_{p,\sigma,2} = O(\varepsilon^2)$, *contraction mapping principle implies the existence of a unique solution* $\phi := \Phi(h)$ *with*

$$\|D_\Gamma^2\phi\|_{p,\nu,\sigma} + \|\phi\|_{p,\sigma,2} = O(\varepsilon^2).$$

Finally, we carry out **Step 2**. We need to find h such that

$$\int_{\mathbb{R}} [E + B\Phi(h) + N(\Phi(h))] (\varepsilon^{-1}y, t)\, w'(t)\, dt = 0 \;\; \forall\, y \in \Gamma.$$

Since

$$-E(\varepsilon^{-1}y, t) = \varepsilon^2 t w'(t)\, |A_\Gamma(y)|^2 + \varepsilon^3 [\Delta_\Gamma h(y) + |A_\Gamma(y)|^2 h(y)\,]\, w'(t)$$

$$+\;\varepsilon^3 t^2 w'(t) \sum_{j=1}^{N-1} k_j(y)^3 + \;\textit{smaller terms}$$

the problem becomes (as in the formal derivation)

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h \;=\; c \sum_{i=1}^{8} k_i^3 + \mathcal{N}(h) \quad \text{in } \Gamma,$$

where $\mathcal{N}(h)$ is a small operator.

The problem

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h = c \sum_{i=1}^{8} k_i^3 + \mathcal{N}(h) \quad \text{in } \Gamma,$$

is solved by a contraction mapping argument after we invert the Jacobi operator of Γ. We explain this procedure next.

The Jacobi operator

$$\mathcal{J}_\Gamma[h] = \Delta_\Gamma h + |A_\Gamma(y)|^2 h,$$

is the linearization of the mean curvature, when normal perturbations are considered. In the case of a minimal graph $x_9 = F(x')$, if we linearize along vertical perturbations we get

$$H'(F)[\phi] \;=\; \nabla \cdot \left(\frac{\nabla \phi}{\sqrt{1+|\nabla F|^2}} - \frac{(\nabla F \cdot \nabla \phi)}{(1+|\nabla F|^2)^{\frac{3}{2}}} \nabla F \right).$$

These two operators are linked through the relation

$$\mathcal{J}_\Gamma[h] = H'(F)[\phi], \quad \text{where} \quad \phi(x') = \sqrt{1+|\nabla F(x')|^2}\, h(x', F(x')).$$

The relation $\mathcal{J}_{\Gamma_0}[h] = H'(F_0)[\sqrt{1+|\nabla F_0|^2} h]$ also holds.

The closeness between $\mathcal{J}_{\Gamma_0}$ and $\mathcal{J}_{\Gamma}$.

Let $p \in \Gamma$ with $r(p) \gg 1$. There is a unique $\pi(p) \in \Gamma_0$ such that $\pi(p) = p + t_p \nu(p)$.

Let us assume

$$\tilde{h}(\pi(y)) = h(y), \quad \text{for all} \quad y \in \Gamma, \quad r(y) > r_0.$$

Then

$$\mathcal{J}_{\Gamma}[h](y) \ =$$

$$[\mathcal{J}_{\Gamma_0}[h_0] + O(r^{-2-\sigma}) D^2_{\Gamma_0} h_0 + O(r^{-3-\sigma}) D_{\Gamma_0} h_0 + O(r^{-4-\sigma}) h_0\,]\,(\pi(y)) \ .$$

We keep in mind that $\mathcal{J}_{\Gamma_0}[h] = H'(F_0)[\sqrt{1 + |\nabla F_0|^2}\, h]$ and make explicit computations.

We compute explicitly

$$
\begin{aligned}
H'(F_0)[\phi] \; = \; & \frac{1}{r^7 \sin^3(2\theta)} \Big\{ (9g^2\, \tilde{w} r^3 \phi_\theta)_\theta + (r^5 g'^2\, \tilde{w} \phi_r)_r \\
& \qquad - 3(gg'\, \tilde{w} r^4 \phi_r)_\theta - 3(gg'\, \tilde{w} r^4 \phi_\theta)_r \Big\} \\
& + \frac{1}{r^7 \sin^3(2\theta)} \Big\{ (r^{-1}\, \tilde{w} \phi_\theta)_\theta + (r \tilde{w} \phi_r)_r \Big\},
\end{aligned}
$$

where

$$
\tilde{w}(r,\theta) := \frac{\sin^3 2\theta}{(r^{-4} + 9g^2 + g'^2)^{\frac{3}{2}}}.
$$

Further expand

$$L[\phi] := H'(F_0)[\phi] := L_0 + L_1,$$

with

$$\begin{aligned} L_0[\phi] \;=\; & \frac{1}{r^7 \sin^3(2\theta)} \Big\{ (9g^2\, \tilde{w}_0 r^3 \phi_\theta)_\theta \;+\; (r^5 g'^2\, \tilde{w}_0 \phi_r)_r \\ & \qquad - 3(gg'\, \tilde{w}_0 r^4 \phi_r)_\theta - 3(gg'\, \tilde{w}_0 r^4 \phi_\theta)_r \Big\} \\ & + \frac{1}{r^7 \sin^3(2\theta)} \Big\{ (r^{-1}\, \tilde{w}_0 \phi_\theta)_\theta + (r \tilde{w}_0 \phi_r)_r \Big\}, \end{aligned}$$

where

$$\tilde{w}_0(\theta) := \frac{\sin^3 2\theta}{(9g^2 + g'^2)^{\frac{3}{2}}}.$$

An important fact: If $0 < \sigma < 1$ there is a positive supersolution $\bar{\phi} = O(r^{-\sigma})$ to

$$-L[\bar{\phi}] \geq \frac{1}{r^{4+\sigma}} \quad \text{in } \Gamma$$

We have that

$$L_0[r^{-\sigma} q(\theta)] = \frac{1}{r^{4+\sigma}} \frac{9g^{\frac{4-\sigma}{3}}}{\sin^3 2\theta} \left[\frac{g^{\frac{2}{3}} \sin^3 2\theta}{(9g^2 + g'^2)^{\frac{3}{2}}} \left(g^{\frac{\sigma}{3}} q \right)' \right]' = \frac{1}{r^{4+\sigma}}.$$

if and only if $q(\theta)$ solves the ODE

$$\left[\frac{g^{\frac{2}{3}} \sin^3 2\theta}{(9g^2 + g'^2)^{\frac{3}{2}}} \left(g^{\frac{\sigma}{3}} q \right)' \right]' = \frac{1}{9} \sin^3 2\theta g(\theta)^{-\frac{4-\sigma}{3}}, \quad .$$

A solution in $(\frac{\pi}{4}, \frac{\pi}{2})$:

$$q(\theta) = \frac{1}{9} g^{-\frac{\sigma}{3}}(\theta) \int_{\frac{\pi}{4}}^{\theta} \frac{(9g^2 + g'^2)^{\frac{3}{2}}}{g^{\frac{2}{3}} \sin^3(2s)} \, ds \int_s^{\frac{\pi}{2}} g^{-\frac{4-\sigma}{3}}(\tau) \sin^3(2\tau) \, d\tau \, .$$

Since $g'(\frac{\pi}{4}) > 0$, q is defined up to $\frac{\pi}{4}$ and can be extended smoothly (evenly) to $(0, \frac{\pi}{4})$. Thus and $\bar{\phi} := q(\theta) r^{-\sigma}$ satisfies

$$-L_0(\bar{\phi}) = r^{-4-\mu} \quad \text{in } \mathbb{R}^8.$$

We can show that also $-L(\bar{\phi}) \geq r^{-4-\sigma}$ for all large r. Thus

$$-\mathcal{J}_{\Gamma_0}[\bar{h}] \geq r^{-4-\sigma}, \quad \bar{h} = \frac{\phi}{\sqrt{1 + |\nabla F_0|^2}} \sim r^{-2-\sigma}$$

The closeness of $\mathcal{J}_\Gamma$ and $\mathcal{J}_{\Gamma_0}$ makes $\bar{h}$ to induce a positive supersolution $\hat{h} \sim r^{-2-\sigma}$ to

$$-\mathcal{J}_\Gamma[\hat{h}] \geq r^{-4-\sigma} \quad \text{in } \Gamma.$$

Conclusion: *Let* $0 < \sigma < 1$. *Then if*

$$\|(1 + r^{4+\sigma})\, g\|_{L^\infty(\Gamma)} < +\infty$$

there is a unique solution $h = T(\mathrm{g})$ *to the problem*

$$\mathcal{J}_\Gamma[h] := \Delta_\Gamma h + |A_\Gamma(y)|^2 h = \mathrm{g}(y) \quad \textit{in } \Gamma.$$

with

$$\|(1 + r)^{2+\sigma}\, h\|_{L^\infty(\Gamma)} \;\leq\; C\, \|(1 + r)^{4+\sigma}\, \mathrm{g}\|_{L^\infty(\Gamma)}\,.$$

We want to solve

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h = c\sum_{i=1}^{8} k_i^3 + \mathcal{N}(h) \quad \text{in } \Gamma,$$

using a fixed point formulation for the operator T above.

In $\mathcal{N}(h)$ everything decays $O(r^{-4-\sigma})$, but we only have

$$\sum_{i=1}^{8} k_i^3 = O(r^{-3}).$$

Let $k_i^0(y)$ be the principal curvatures of Γ_0.

Facts:

- $$\sum_{i=1}^{8} k_i(y)^3 = \sum_{i=1}^{8} k_i^0(\pi(y))^3 + O(r^{-4-\sigma})$$
- $$\sum_{i=1}^{8} k_i^0(y)^3 = \frac{p(\theta)}{r^3} + O(r^{-4-\sigma})$$

p smooth, $p(\frac{\pi}{2} - \theta) = -p(\theta)$ for all $\theta \in (0, \frac{\pi}{4})$.

We claim: *there exists a smooth function* $h_*(r, \theta)$ *such that* $h_* = O(r^{-1})$ *and for some* $\sigma > 0$,

$$\mathcal{J}_{\Gamma_0}[h_*] = \frac{p(\theta)}{r^3} + O(r^{-4-\sigma}) \quad \textit{as } r \to +\infty.$$

Setting $h_0(y) = h_*(\pi(y))$ we then get $h_0 = O(r^{-1})$ and

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h \;=\; c \sum_{i=1}^{8} k_i^3 + O(r^{-4-\sigma}) \quad \text{in } \Gamma.$$

Our final problem then becomes $h = h_0 + h_1$ where

$$h_1 = T(O(r^{-4-\sigma}) + \mathcal{N}(h_0 + h_1))$$

which we can solve for $h_1 = O(r^{-2-\sigma})$, using contraction mapping principle, keeping track of Lipschitz dependence in h of the objects involved in in $\mathcal{N}(h)$.

Construction of h_*.

We argue as before (separation of variables) to find $q(\theta)$ solution of

$$L_0(r\, q(\theta)) = \frac{p(\theta)}{r^3}, \theta \in (\frac{\pi}{4}, \frac{\pi}{2}).$$

$$q(\theta) = -\frac{1}{9} g^{\frac{1}{3}}(\theta) \int_{\frac{\pi}{4}}^{\theta} (\, 9g^2 + g'^2\,)^{\frac{3}{2}} \, \frac{g^{-\frac{2}{3}} ds}{\sin^3(2s)} \int_s^{\frac{\pi}{2}} p(\tau) g^{-\frac{5}{3}}(\tau) \sin^3(2\tau)\, d\tau\,.$$

Let $\eta(s) = 1$ for $s < 1$, $= 0$ for $s > 2$ be a smooth cut-off function. Then

$$\phi_0(r,\theta) \;:=\; (1-\eta(s))\, r\, q(\theta) \quad \text{in } (\frac{\pi}{4}, \frac{\pi}{2}), \quad s = r^2 g(\theta).$$

satisfies

$$L(\phi_0) = \frac{p(\theta)}{r^3} + O(r^{-4-\frac{1}{3}}).$$

Finally, the function

$$h_* = \frac{\phi_0}{\sqrt{1+|\nabla F_0|^2}} = O(r^{-1})$$

extended oddly through $\theta = \frac{\pi}{4}$ satisfies

$$J_{\Gamma_0}[h_*] = \frac{p(\theta)}{r^3} + O(r^{-4-\frac{1}{3}})$$

as desired.

Problem II in $\mathbb{R}^3$.

For which minimal hypersurfaces Γ in $\mathbb{R}^N$, that split the space into two components, can one find an entire solution u_ε to $\Delta u + u - u^3 = 0$ with transition set near $\Gamma_\varepsilon = \varepsilon^{-1}\Gamma$?

$$\boxed{u_\varepsilon(x) \approx \tanh\left(\frac{\zeta}{\sqrt{2}}\right), \quad x = y + \zeta\nu_\varepsilon(y), \quad y \in \Gamma_\varepsilon}$$

The question for $N = 3$

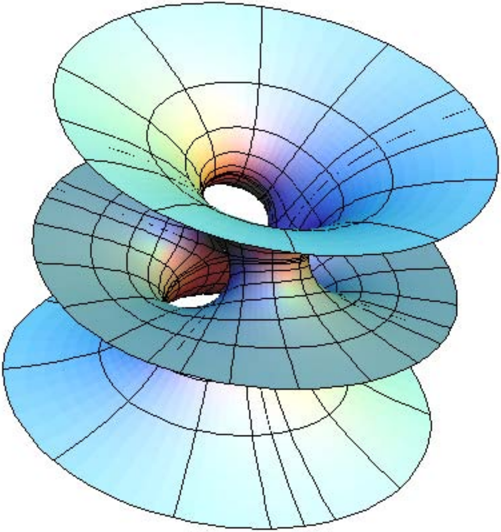
Embedded, complete minimal surfaces Γ:

• Plane; Catenoid (Euler, 1744); Helicoid (Meusnier, 1776), Riemann (1865).

• Costa, 1982: a minimal surface with genus 1 with two catenoidal and one planar ends.

• Hoffman-Meeks, 1985, 1989: embeddedness of Costa surface and a 3-end example for any genus $\ell \geq 1$. (CHM surface).

• Intense research on construction and classification of minimal surfaces since then.



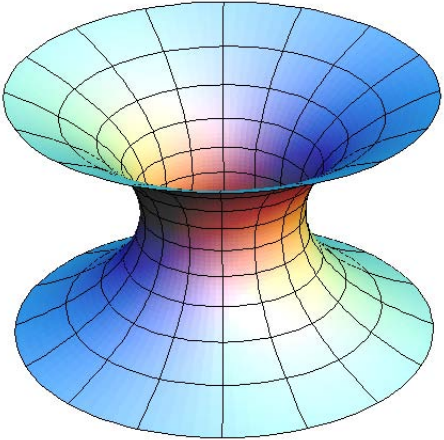
The Costa Surface

Existence of u_ε with transition near $\Gamma_\varepsilon = \varepsilon^{-1}\Gamma$?

Yes: for the catenoid and the CHM surfaces. (del Pino, Kowalczyk, Wei, 2009).

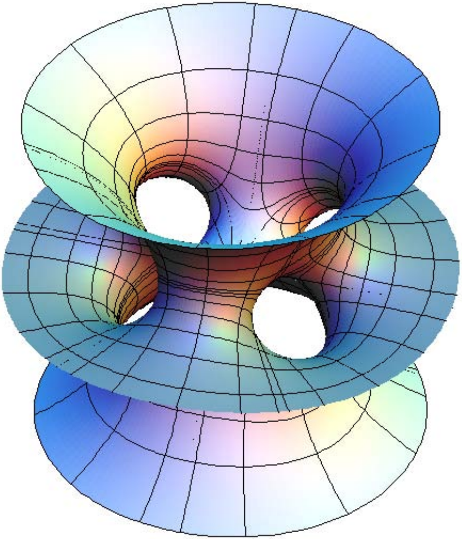
$$u_\varepsilon(x) \approx w(\zeta) = \tanh\left(\frac{\zeta}{\sqrt{2}}\right), \quad x = y + \zeta\nu_\varepsilon(y), \quad y \in \Gamma_\varepsilon$$

$\Gamma =$ a catenoid: $\quad \exists\ u_\varepsilon(x) = w(\zeta) + o(1),\ \ x = y + \zeta\nu_\varepsilon(y).$



u_ε axially symmetric: $\ u_\varepsilon(x) = u_\varepsilon(\sqrt{x_1^2 + x_2^2}, x_3),\ \ x_3$ rotation axis coordinate.

$\Gamma =$ CHM surface genus $\ell \geq 1$:



$\exists\ \ u_\varepsilon(x) = w(\zeta) + o(1),\ \ x = y + \zeta\nu_\varepsilon(y).$

How far can we go?

Theorem (del Pino, Kowalczyk, Wei (2009))

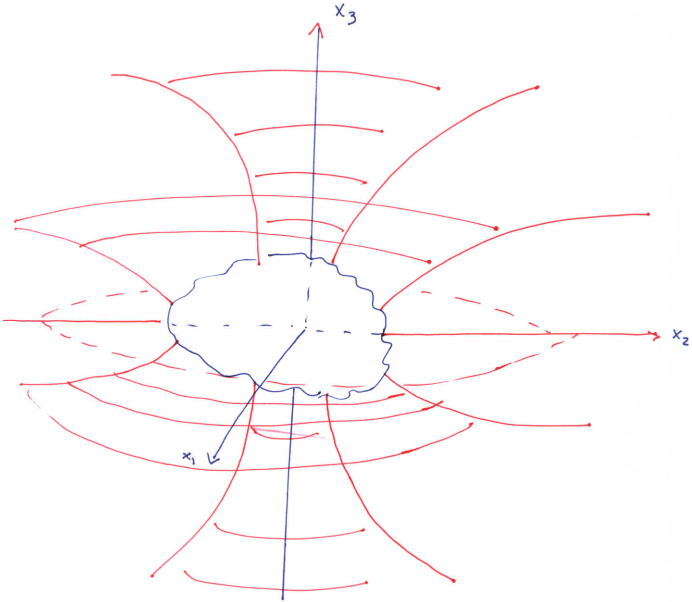
Let Γ *be a complete, embedded minimal surface in* $\mathbb{R}^3$ *with finite total curvature:* $\int_\Gamma |K| < \infty$, K *Gauss curvature, and non-parallel ends.*

If Γ *is non-degenerate, namely its bounded Jacobi fields originate only from rigid motions, then for small* $\varepsilon > 0$ *there is a solution* u_ε *to (AC) with*

$$u_\varepsilon(x) = w(\zeta - \varepsilon h(\varepsilon y)), \quad x = y + \zeta \nu_\varepsilon(y),$$

h_ε *uniformly bounded. In addition* $i(u_\varepsilon) = i(\Gamma)$ *where* i *denotes Morse index.*

Nondegeneracy and Morse index are known for the catenoid and CHM surfaces (Nayatani (1990), Morabito, (2008)).



General look

- Nondegeneracy: The only nontrivial bounded solutions of

$$\mathcal{J}_\Gamma(\phi) = \Delta_\Gamma \phi - 2K\phi = 0$$

arise from translations and rotation about the common symmetry axis (x_3) of the ends: $\nu_i(x)$ $i = 1, 2, 3$, $x_2\nu_1(x) - x_1\nu_2(x)$.

- $i(\Gamma)$, the Morse index of Γ, is the number of negative eigenvalues of J_Γ in $L^\infty(\Gamma)$. This number is finite $\Longleftrightarrow$ Γ has finite total curvature.

- $i(\Gamma) =$ 0 for the plane, $= 1$ for the catenoid and $= 2\ell + 3$ for the CHM surface genus ℓ.

Morse index of a solution u of (AC), $i(u)$: roughly, the number of negative eigenvalues of the linearized operator, namely those of the problem

$$\Delta\phi + (1 - 3u^2)\phi + \lambda\phi = 0 \quad \phi \in L^\infty(\mathbb{R}^N).$$

De Giorgi solution: ‘‘stable’’, $i(u) = 0$ since $\lambda = 0$ is an eigenvalue with eigenfunction $\partial_{x_N} u > 0$.

$i(u) = 0 \implies$ DG statement for $N = 2$ (Farina). Open if $N \geq 3$.

A next step from DG conjecture: Construction and classification of *finite-Morse index solutions*.

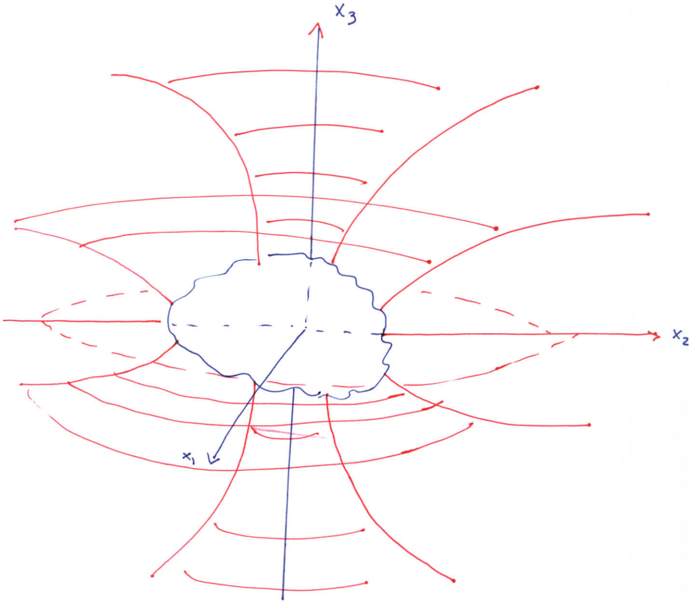
A DG-type conjecture for Morse index 1 in 3d: *A bounded solution u of (AC) in $\mathbb{R}^3$ with $i(u) = 1$, and $\nabla u \neq 0$ outside a bounded set, must be axially symmetric, namely radially symmetric in two variables.*

The solution we found, associated to the dilated catenoid, has this property. Schoen, Pérez and Ros, proved that if $i(\Gamma) = 1$ and Γ has embedded ends, then it must be a catenoid.

The 3d Case: Qualitative properties of embedded minimal surfaces with finite Morse index should hold for the asymptotic behavior of nodal sets of finite Morse index solutions of (AC) provided that this nodal set is embedded outside a compact set:

Speculation: *Any finite Morse index solution* u *with* $\nabla u \neq 0$ *outside a compact set should have a finite, even number of catenoidal or planar ends with a common axis.*

The latter fact does hold for minimal surfaces with finite total curvature and embedded ends (Ossermann, Schoen).



The possible picture in 3d for nodal set

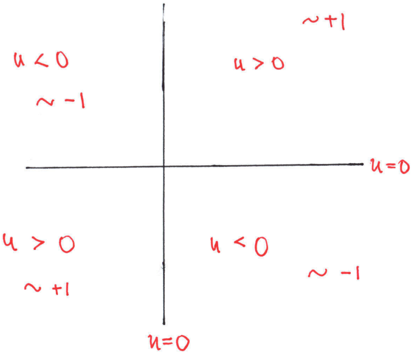
The case $N = 2$: Very few solutions known with $1 \leq i(u) < +\infty$.

• Dang, Fife, Peletier (1992). The cross saddle solution: $u(x_1, x_2) > 0$ for $x_1, x_2 > 0$,

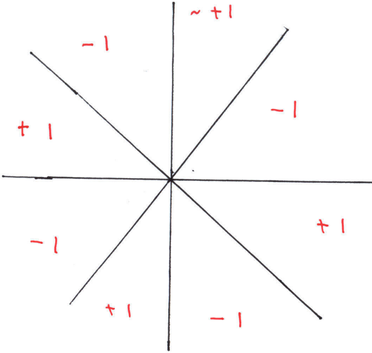
$$u(x_1, x_2) = -u(-x_1, x_2) = -u(x_1, -x_2).$$

Nodal set two lines (4 ends). Super-subsolutions in first quadrant. Schatzmann (1995): $i(u) = 1$.

• Alessio, Calamai, Montecchiari (2007). Extension: saddle solution with dihedral symmetry. Nodal set k lines ($2k$ ends), $k \geq 2$. Presumably $i(u) = k - 1$.



DANG-FIFE-PELETIER 92
(4-END)



ALESSIO-CALAMAI-MONTECCHIARI 07
(2K-END)

The saddle solutions

A result: Existence of entire solutions with embedded level set and finite number of transition lines of $\Delta u + u - u^3 = 0$ in $\mathbb{R}^2$:

Solutions with k ‘‘nearly parallel’’ transition lines are found for any $k \geq 1$.

Theorem (del Pino, Kowalczyk, Pacard, Wei (2007))

If f satisfies

$$\frac{\sqrt{2}}{24} f''(z) = e^{-2\sqrt{2} f(z)}, \quad f'(0) = 0,$$

and $f_\varepsilon(z) := \sqrt{2} \log \frac{1}{\varepsilon} + f(\varepsilon z)$*, then there exists a solution* u_ε *to (AC) in* $\mathbb{R}^2$ *with*

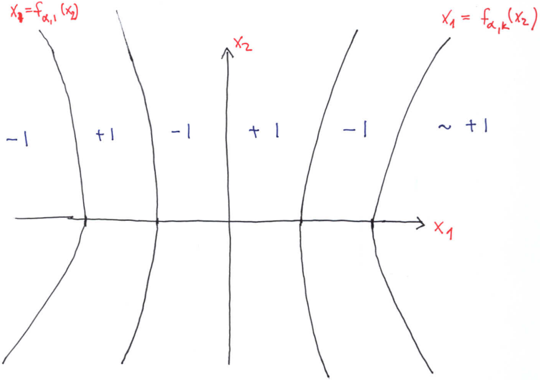
$$\boxed{u_\varepsilon(x_1, x_2) = w(x_1 + f_\varepsilon(x_2)\,) + w(x_1 - f_\varepsilon(x_2)\ - 1\ + o(1)}$$

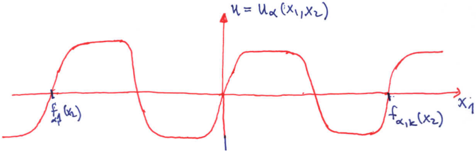
as $\varepsilon \to 0^+$*. Here* $w(s) = \tanh(s/\sqrt{2})$.

`This solution has 2 transition lines.`

$$f(z) = A|z| + B + o(1) \quad \texttt{as} \ \ z \to \pm\infty.$$

More in general: the equilibrium of k far-apart, embedded transition lines is governed by the **Toda system,** a classical integrable model for scattering of particles on a line under the action of a repulsive exponential potential:





$$u_\varepsilon(x_1, x_2) = \sum_{j=1}^{k} (-1)^{j-1} w(x_1 - f_{\varepsilon,j}(x_2)) - \frac{1}{2}(1 + (-1)^k) + o(1)$$

The Toda system:

$$\frac{\sqrt{2}}{24} f_j'' = e^{-\sqrt{2}(f_j - f_{j-1})} - e^{-\sqrt{2}(f_{j+1} - f_j)}, \quad j = 1, \dots k,$$

$f_0 \equiv -\infty, \;\; f_{k+1} \equiv +\infty\,.$

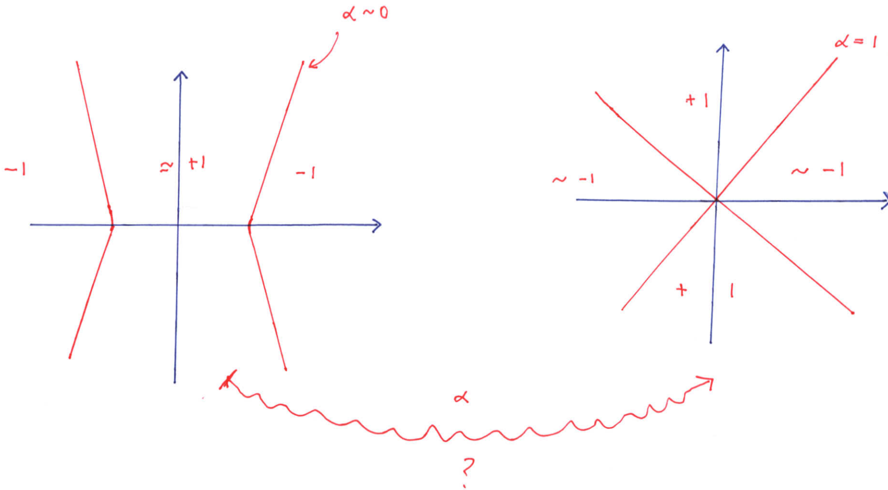
Given a solution f (with asymptotically linear components), if we scale

$$f_{\varepsilon,j}(z) \; := \; \sqrt{2}\,(j - \frac{k+1}{2})\log\frac{1}{\varepsilon} + f_j(\varepsilon z),$$

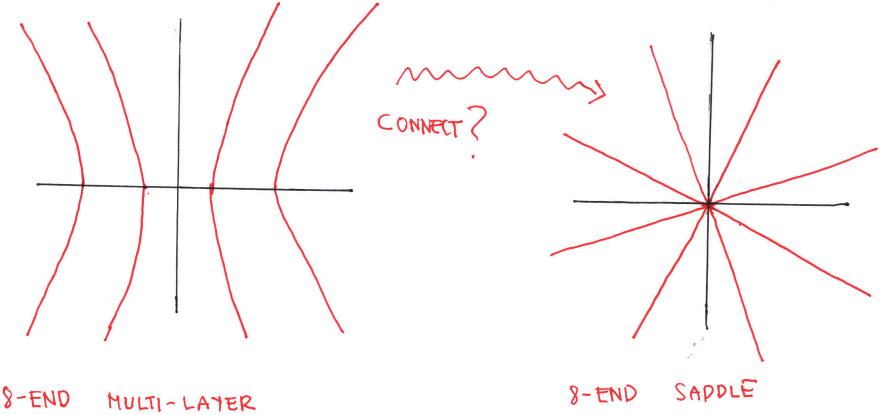
then there is a solution with k transitions:

$$\boxed{u_\varepsilon(x_1, x_2) = \sum_{j=1}^{k} (-1)^{j-1} w(x_1 - f_{\varepsilon,j}(x_2)\,) \; - \; \frac{1}{2}(1 + (-1)^k) \; + \; o(1)}$$

We conjecture: The 4-end (two-line) solution is a limit case of a continuum of solutions with Morse index 1 that has the cross saddle as the other endpoint *All intermediate slopes missing*. This is also the case for $k > 2$.

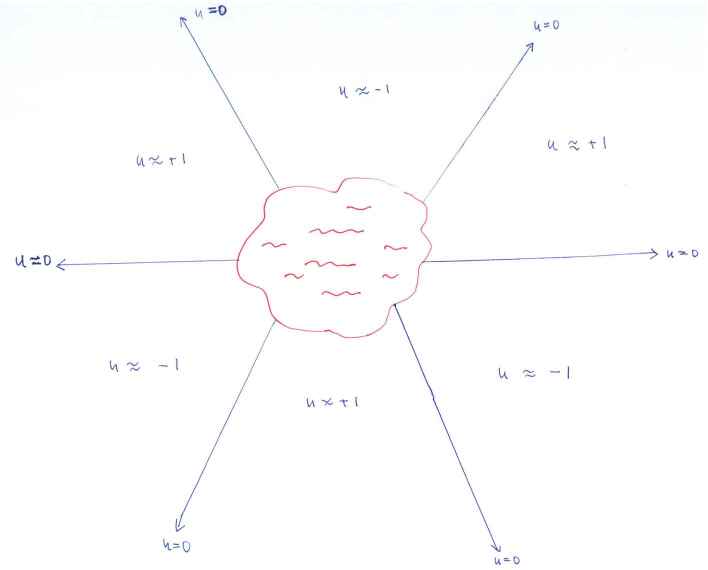


2-line transition layer and 4 end saddle: Do they connect?



Do they connect?

Further speculation: Any finite Morse index solution should have *a finite, even number of ends*



General $2k$-end

- Gui (2007) proved that a $2k$-end solution satisfies a *balancing formula* for its ends: if e_j, $j=1,\ldots,2k$ are the asymptotic directions for the nodal set then $\sum_{j=1}^{2k} e_j = 0$

- **We prove:** given a *nondegenerate* $2k$-end solution u, the class of $2k$-end solutions nearby constitute a $2k$-dimensional manifold. (This is the case for the solution with k nearly parallel transition lines and the cross saddle). For 2 transition lines we thus have one parameter (ε) besides translations and rotations.

The discovery in the case of $\mathbb{R}^N$:

• Internal mechanisms of the equation rather than the geometry of the ambient space or spacial dependence of the equation, govern the transition layer solutions.

• The structure of the set of bounded solutions of the simple PDE (AC) is highly complex: not only the entire universe of minimal surfaces is presumably embedded in it: disjoint interfaces ‘‘interact’’ giving rise to other type of phenomena (e.g. the role of the Toda system).