The KPP minimal speed within large drift in two dimensions

Mohammad El Smaily
Joint work with Stéphane Kirsch

University of British Columbia &

Pacific Institute for the Mathematical Sciences

Banff, March-2010

Deterministic and Stochastic Front Propagation-BIRS

Introduction

• Traveling fronts in the homogenous case:
The equation is

$$u_t(t,x) = \Delta u + f(u) \quad t \in \mathbb{R},\ x \in \mathbb{R}^N. \tag{1}$$

•The Diffusion is the Id Matrix and the Reaction is $f = f(u)$ and no advection term $(q \cdot \nabla u)$.

• Given a unitary direction $e \in \mathbb{R}^N$, traveling fronts propagating in the direction of $-e$ and with a speed $c \in \mathbb{R}$ were introduced as solutions of (1) in the form $u(t,x) = \phi(x \cdot e + ct) = \phi(s)$ satisfying the limiting conditions $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$.

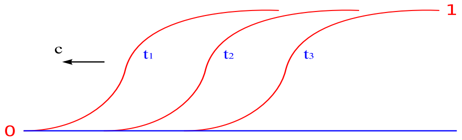


Figure: Traveling front, One dimensional case

Theorem (Kolmogorov, Petrovsky and Piskunov)

Having a KPP nonlinearity, a TF exists with a speed c iff $c \geq 2\sqrt{f'(0)}$. Moreover, this TF $u(t,x)$ is increasing in t.

$$c^* = 2\sqrt{f'(0)}$$

is the minimal speed in the homogeneous case where there is no advection.

The previous definition was extended to nonhomogeneous settings by Shigesada *et al* in 1986, H. Weinberger in 2002, J. Xin, and by Berestycki, Hamel in 2002:

• The domain is $\Omega \subset \mathbb{R}^d \times \mathbb{R}^{N-d}$ where $1 \leq d \leq N$ such that:

• Each $z \in \Omega$ can be written as $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}^{N-d}$.

• Ω is bounded in the y direction. That is, $\exists R > 0$ s.t $|y| \leq R$ for all $(x, y) \in \Omega$.

• There exist $L_1, \cdots, L_d > 0$ such that $\boxed{\Omega = \Omega + k}$ for all $k = (k_1, \cdots, k_d, 0, \cdots, 0) \in \prod_{i=1}^{d} L_i\mathbb{Z} \times \{0\}^{N-d}$.

• Notice that if $d = N$ then Ω is unbounded in all directions.

• Having such domains, we assume that $q = q(x, y)$ and $f = f(x, y, u)$ are $L-$periodic in x

$$q(x + L, y) = q(x, y), \quad f(x + L, y, u) = f(x, y, u)$$

s.t $L = (L_1, \cdots, L_d)$.

Definition of PTFs, Existence, Minimal speed...

Equation

$$\begin{cases} u_t = \Delta u \; + \; q(x,y)\cdot\nabla u + f(x,y,u), \; t\in\mathbb{R}, \; (x,y)\in\Omega, \\ \nu\cdot\nabla u = 0 \;\; \text{on } \mathbb{R}\times\partial\Omega, \end{cases} \tag{2}$$

- Let $e=(e^1,\cdots,e^d)\in\mathbb{R}^d$ be a unitary direction and denote by $\tilde{e}=(e,0,\cdots,0)\in\mathbb{R}^N$.

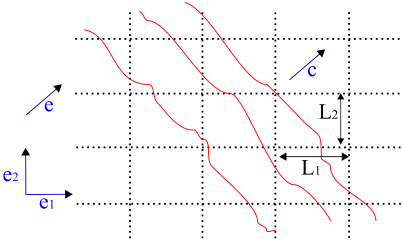
Definition

A PTF propagating in the direction of $-e$ with a speed c is a solution

$$u(t,x,y)=\phi(s,x,y)=\phi(x\cdot e+ct,x,y)$$

of (2) which is $L-$periodic in x and satisfies:

$$\phi(-\infty,\cdot,\cdot)=0, \;\; \phi(+\infty,\cdot,\cdot)=1 \text{ uniformly in } (x,y)\in\Omega.$$



- $u_t = \mathrm{div}(A(x)\nabla u) + f(x,u)$
- $\Omega = \mathbb{R}^2$, $d = N = 2$, $e \parallel (1,1)$.

Assumptions on The Advection and Reaction

- The advection $q(x,y) = (q_1(x,y), \cdots, q_N(x,y))$ is a $C^{1,\delta}(\overline{\Omega})$ (with $\delta > 0$) vector field satisfying

$$q \quad \text{is } L\text{- periodic with respect to} \;\; x, \quad \nabla \cdot q = 0 \quad \text{in } \overline{\Omega},$$

$$q \cdot \nu = 0 \quad \text{on } \partial\Omega \text{ (when } \partial\Omega \neq \emptyset), \quad \text{and } \int_C q \, dx \, dy = 0.$$

- Generalized KPP nonlinearity $f = f(x,y,u)$

$f \geq 0, f$ is L-periodic with respect to x, and of class $C^{1,\delta}(\overline{\Omega} \times [0,1])$,

$\forall (x,y) \in \overline{\Omega}, \quad f(x,y,0) = f(x,y,1) = 0,$

f is decreasing in u on $\Omega \times [1-\rho, 1]$ for some $\rho > 0$

- With the additional "KPP" assumption

$$\forall (x,y,s) \in \overline{\Omega} \times (0,1), \; 0 < f(x,y,s) \leq f'_u(x,y,0) \times s.$$

- Simple example: $(x,y,u) \mapsto u(1-u)h(x,y)$ defined on $\overline{\Omega} \times [0,1]$ where h is a positive $C^{1,\delta}(\overline{\Omega})$ L-periodic function.

Existence, Minimal Speed

Theorem (Berestycki and Hamel, CPAM 2002)

• *For any prefixed* $e \in \mathbb{R}^d$*, there exists a minimal speed* $c^* := c^*_{\Omega,q,f}(e) > 0$ *such that a PTF with a speed* c *exists* **if and only if** $c \geq c^*$.

• *Any PTF is increasing in time.*

• *Moreover, for any* $c \geq c^*$*, Hamel and Roques proved that the fronts* $u(t,x,y)$ *with a speed* c *are unique up to a translation in* t.

Variational formula for the minimal speed

• A variational formula of this minimal speed was given in 2005 (will be shown in the next slides...).

• This formula shows that this minimal speed depends strongly on the coefficients of the equation (Reaction, diffusion and advection) and on the geometry of the domain.

• Many asymptotic behaviors of c^* and many homogenization results have been studied by Berestycki-Hamel-Naderashvili, S. Heinze, Shigesada et al., J. Xin, A. Zlatoš, Zlatoš-Constantin-Kiselev-Ryzhik, E., and many others.

• In this talk, we will show a result about the asymptotic behavior of the minimal speed within large drift Mq ($M \to +\infty$) and we will give some details about the limit in the case N=2.

Variational Formula for the Parametric Minimal Speed

The equation that we study

$$\begin{cases} u_t = \Delta u + Mq(x,y)\cdot\nabla u + f(x,y,u), \ t\in\mathbb{R}, \ (x,y)\in\Omega, \\ \nu\cdot\nabla u = 0 \ \text{ on } \mathbb{R}\times\partial\Omega. \end{cases}$$

$$c^*(M,e) = \min_{\lambda>0} \frac{k(\lambda,M)}{\lambda};$$

• $k(\lambda,M)$ is the principal eigenvalue of the elliptic operator L_λ defined by

$$\boxed{L_\lambda\psi := \Delta\psi + 2\lambda\tilde{e}\cdot\nabla\psi + M\,q\cdot\nabla\psi + [\lambda^2 + \lambda M\,q\cdot\tilde{e} + \zeta]\psi \text{ in } \Omega,}$$

$$E_\lambda = \left\{\psi(x,y)\in C^2(\overline{\Omega}), \psi \text{ is } L\text{-periodic in } x,\ \nu\cdot\nabla\psi = -\lambda(\nu\cdot\tilde{e})\psi \text{ on } \partial\Omega\right\}.$$

• The principal eigenfunction $\psi^{\lambda,M}$ is positive in $\overline{\Omega}$. It is unique up to multiplication by a nonzero real number.

•$k(\lambda,M) > 0$ for all $(\lambda,M)\in(0,+\infty)\times(0,+\infty)$.

First integrals

Definition (First integrals)

The family of first integrals of q is defined by

$$\mathcal{I} := \left\{ w \in H^1_{loc}(\Omega),\ w \not\equiv 0,\ w \text{ is } L-\text{periodic in } x, \text{ and } q \cdot \nabla w = 0 \text{ almost everywhere in } \Omega \right\}.$$

We also define the two subsets $\mathcal{I}_1$ and $\mathcal{I}_2$:

$$\mathcal{I}_1 := \left\{ w \in \mathcal{I}, \text{ such that } \int_C \zeta w^2 \geq \int_C |\nabla w|^2 \right\}, \tag{3}$$

$$\mathcal{I}_2 := \left\{ w \in \mathcal{I}, \text{ such that } \int_C \zeta w^2 \leq \int_C |\nabla w|^2 \right\}.$$

$\zeta(x,y) := f'_u(x,y,0)$. $\zeta = f'(0)$ when $f = f(u)$.

About "first integrals" of a vector field q

- The set $\mathcal{I}$ is a closed subspace of $H^1_{loc}(\Omega)$.

Notice

One can see that if $w \in \mathcal{I}$ is a first integral of q and $\eta : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then $\eta \circ w \in \mathcal{I}$.

Asymptotics within large advection in any space dimension

Theorem (E.-Kirsch 2009)

We fix a unit direction $e \in \mathbb{R}^d$. Let q be an advection field which satisfies the previous assumptions. Then,

$$\lim_{M \to +\infty} \frac{c^*(Mq, e)}{M} = \max_{w \in \mathcal{I}_1} \frac{\displaystyle\int_C (q \cdot \tilde{e})\, w^2}{\displaystyle\int_C w^2}. \tag{4}$$

1. Berestycki, Hamel and Nadirashvili (2005) gave estimates showing that the limit exists, but exact limit was still unknown.

Asymptotics within large advection in any space dimension

Theorem (E.-Kirsch 2009)

We fix a unit direction $e \in \mathbb{R}^d$. Let q be an advection field which satisfies the previous assumptions. Then,

$$\lim_{M\to+\infty} \frac{c^*(Mq,e)}{M} = \max_{w \in \mathcal{I}_1} \frac{\displaystyle\int_C (q\cdot\tilde{e})\, w^2}{\displaystyle\int_C w^2}. \tag{4}$$

1. Berestycki, Hamel and Nadirashvili (2005) gave estimates showing that the limit exists, but exact limit was still unknown.
2. A. Zlatoš considered the same problem in any space dimension N.

Asymptotics within large advection in any space dimension

Theorem (E.-Kirsch 2009)

We fix a unit direction $e \in \mathbb{R}^d$*. Let* q *be an advection field which satisfies the previous assumptions. Then,*

$$\lim_{M \to +\infty} \frac{c^*(Mq, e)}{M} = \max_{w \in \mathcal{I}_1} \frac{\displaystyle\int_C (q \cdot \tilde{e})\, w^2}{\displaystyle\int_C w^2}. \tag{4}$$

1. Berestycki, Hamel and Nadirashvili (2005) gave estimates showing that the limit exists, but exact limit was still unknown.
2. A. Zlatoš considered the same problem in any space dimension N.
3. We did this study in any dimension N, and we gave details about the limit in the case $N = 2$.

Remarks and simple ideas

- $c^*(M) = \min_{\lambda > 0} \frac{k(\lambda, M)}{\lambda},$

Remarks and simple ideas

- $c^*(M) = \min_{\lambda>0} \dfrac{k(\lambda, M)}{\lambda}$,
- $L_\lambda \psi := \Delta\psi + 2\lambda\tilde{e}\cdot\nabla\psi + M\,q\cdot\nabla\psi + [\lambda^2 + \lambda M\,q\cdot\tilde{e} + \zeta]\psi$ in Ω,

Remarks and simple ideas

- $c^*(M) = \min_{\lambda>0} \dfrac{k(\lambda, M)}{\lambda}$,
- $L_\lambda \psi := \Delta\psi + 2\lambda\tilde{e}\cdot\nabla\psi + M\,q\cdot\nabla\psi + [\lambda^2 + \lambda M\,q\cdot\tilde{e} + \zeta]\psi$ in Ω,
- We call

$$\lambda' = \lambda \times M, \text{ and } \mu(\lambda', M) = k(\lambda, M) \text{ and } \psi^{\lambda', M} = \psi^{\lambda, M}.$$

Remarks and simple ideas

- $c^*(M) = \min_{\lambda>0} \dfrac{k(\lambda, M)}{\lambda}$,
- $L_\lambda \psi := \Delta\psi + 2\lambda\tilde{e}\cdot\nabla\psi + M\,q\cdot\nabla\psi + [\lambda^2 + \lambda M\,q\cdot\tilde{e} + \zeta]\psi$ in Ω,
- We call

$$\lambda' = \lambda \times M, \text{ and } \mu(\lambda', M) = k(\lambda, M) \text{ and } \psi^{\lambda', M} = \psi^{\lambda, M}.$$

- Then,

$$\forall\, M > 0, \quad \frac{c^*(M)}{M} = \min_{\lambda'>0} \frac{\mu(\lambda', M)}{\lambda'}.$$

Remarks and simple ideas

- $c^*(M) = \min_{\lambda>0} \frac{k(\lambda, M)}{\lambda}$,
- $L_\lambda \psi := \Delta\psi + 2\lambda\tilde{e}\cdot\nabla\psi + M\,q\cdot\nabla\psi + [\lambda^2 + \lambda M\,q\cdot\tilde{e} + \zeta]\psi$ in Ω,
- We call

$$\lambda' = \lambda \times M, \text{ and } \mu(\lambda', M) = k(\lambda, M) \text{ and } \psi^{\lambda',M} = \psi^{\lambda,M}.$$

- Then,

$$\forall\, M > 0, \quad \frac{c^*(M)}{M} = \min_{\lambda'>0} \frac{\mu(\lambda', M)}{\lambda'}.$$

-

$$(E)\begin{cases} \mu(\lambda', M)\psi^{\lambda',M} = & \Delta\psi^{\lambda',M} + 2\dfrac{\lambda'}{M}\tilde{e}\cdot\nabla\psi + M\,q\cdot\nabla\psi^{\lambda,M} \\ & + \left[\left(\dfrac{\lambda'}{M}\right)^2 + \lambda'\,q\cdot\tilde{e} + \zeta\right]\psi^{\lambda',M} \text{ in } \Omega, \\ \nu\cdot\nabla\psi^{\lambda',M} = & -\dfrac{\lambda'}{M}(\nu\cdot\tilde{e})\psi^{\lambda',M} \text{ on } \partial\Omega \text{ (whenever } \partial\Omega \neq \emptyset). \end{cases}$$

Remark: Eigenfunctions converge to first integrals

For a fixed λ', we take a sequence $\left\{\psi^{\lambda',M_n}\right\}_{n\in\mathbb{N}}$ such that $\int_C \left(\psi^{\lambda',M_n}\right)^2 = 1.$

We get $\left\{\psi^{\lambda',M_n}\right\}_{n\in\mathbb{N}}$ is bounded in $H^1(C)$.

Hence there exists $\psi^{\lambda',+\infty} \in H^1_{loc}(\Omega)$ s.t. $\psi^{\lambda',M_n} \to \psi^{\lambda',+\infty}$ in $H^1_{loc}(\Omega)$ weak, in $L^2_{loc}(\Omega)$ strong, and almost everywhere in Ω as $n \to +\infty$.

Elliptic eigenvalue problem implies that $\psi^{\lambda',+\infty}$ is a first integral.

In dimension $N = 2$, the domain Ω may be:
1- The whole space $\mathbb{R}^2$ $(d = N = 2)$.

2- $\mathbb{R}^2$ except a periodic array of holes $(d = N = 2)$

3- For $d = 1$, Ω can be an infinite cylinder with a uniform boundary or with an oscillating boundary.

4- For $d = 1$, the cylinder is connected but it may have a periodic array of holes.

A Question and Some Remarks

• We were interested in getting Necessary and Sufficient Conditions on the advection field for which the limit of $c^*(M)/M$ is positive.
• In dimension $N = 2$ the geometry helps to study the divergence free advection field q which appears explicitly in the limit.

Proposition (E.-Kirsch 2009)

Let $d = 1$ or 2 where d is defined before. Let $q = q(x, y) \in C^{1,\delta}(\overline{\Omega})$, L-periodic with respect to x and verifying the conditions

$$\int_C q = 0, \quad \nabla \cdot q = 0 \text{ in } \Omega, \quad q \cdot \nu = 0 \text{ on } \partial\Omega. \tag{5}$$

Then, there exists $\phi \in C^{2,\delta}(\overline{\Omega})$, L-periodic with respect to x, such that

$$q = \nabla^{\perp}\phi \ \text{ in } \Omega. \tag{6}$$

Moreover, ϕ is constant on every connected component of $\partial\Omega$.

Remark

• *The representation $q = \nabla^{\perp}\phi$ is well-known in the case where the domain Ω is bounded and* **simply connected** *or equal to whole space $\mathbb{R}^2$.*
• *However, the above proposition applies for domains which are not simply connected.*

• $\nabla^{\perp}\phi \cdot \nu = q \cdot \nu = 0$ on $\partial\Omega \Rightarrow \phi$ is constant on every connected component of $\partial\Omega$.
• In the proof of existence of ϕ, ($d = 2$ let's say)

$$\hat{\Omega} := \Omega/(L_1\mathbb{Z} \times L_2\mathbb{Z}) \quad \text{and} \quad T := \mathbb{R}^2/(L_1\mathbb{Z} \times L_2\mathbb{Z}).$$

If $x \in \mathbb{R}^2$, we denote by $\hat{x}$ its class of equivalence in T, and if $\phi : \mathbb{R}^2 \to \mathbb{R}$ is L- periodic, we denote $\hat{\phi}$ the function $T \to \mathbb{R}^2$ verifying $\phi(x) = \hat{\phi}(\hat{x})$.

Define

$$\begin{array}{rcl} \tilde{q} : T & \longrightarrow & \mathbb{R}^2, \\ \hat{x} \in \overline{\hat{\Omega}} & \longmapsto & q(x), \\ \hat{x} \notin \overline{\hat{\Omega}} & \longmapsto & 0. \end{array}$$

• $\tilde{q}$ is a divergence free vector field on T in the sense of distributions:

$$\forall\, \psi \in C^\infty(T),$$

$$\begin{array}{rcl} < div(\tilde{q}), \psi > & := & - < \tilde{q}, \nabla\psi > = -\displaystyle\int_T \tilde{q} \cdot \nabla\psi \\ & = & -\displaystyle\int_{\hat{\Omega}} q \cdot \nabla\psi = -\int_{\partial\hat{\Omega}} \psi\, q \cdot \nu + \int_{\hat{\Omega}} \psi \nabla \cdot q \\ & = & 0 + 0 = 0, \end{array}$$

We first get $\tilde{\phi}$ solution of

$$\Delta\tilde{\phi} = \nabla \cdot R\tilde{q} \text{ in } T$$

in the weak sense.
We then have $\tilde{\phi} \in H^1(T)$ such that in the sense of distributions

$$\begin{array}{rcll} \nabla \cdot R(\tilde{q} - \nabla^{\perp}\tilde{\phi}) & = & 0 \text{ in } T & \text{and} \\ \nabla \cdot (\tilde{q} - \nabla^{\perp}\tilde{\phi}) & = & 0 \text{ in } T & \text{since } \nabla \cdot \tilde{q} = 0 \text{ in } \mathcal{D}'(T) \text{ and } div(\nabla^{\perp}\cdot) = 0. \end{array}$$

This implies that $\tilde{q} - \nabla^{\perp}\tilde{\phi}$ is a harmonic distribution on T. Using Weyl's theorem, we conclude that $\tilde{q} - \nabla^{\perp}\tilde{\phi}$ is a harmonic function on the torus T and therefore is constant.

- Then we define $\hat{\phi} = \tilde{\phi}|_{\hat{\Omega}}$ and we take ϕ the corresponding $L-$ periodic function on Ω.

- Also we get $\nabla^{\perp}\tilde{\phi} = \tilde{q} = 0$ on $T \setminus \hat{\Omega}$.

- Hence $\tilde{\phi}$ =Constant on $T \setminus \hat{\Omega}$.

Corollary (Now we know more about first integrals...)

Let

$$\mathcal{J} := \{\eta \circ \phi, \text{ such that } \eta : \mathbb{R} \to \mathbb{R} \text{ is Lipschitz}\}, \tag{7}$$

where ϕ, *such that* $q = \nabla^{\perp}\phi$, *is given by Proposition 6. Then,*

$$\mathcal{J} \subset \mathcal{I}.$$

The first integrals of the form $w = \eta \circ \phi$, $\mathcal{J}$

$$\forall\, w \in \mathcal{J}, \text{ we have } \int_C (q\cdot\tilde{e})w^2 = 0.$$

- Indeed, $w = \eta \circ \phi$ and $q = \nabla^\perp \phi$. This gives

$$\begin{aligned}\int_C (q\cdot\tilde{e})w^2 &= \tilde{e}\cdot\int_C \left(\nabla^\perp\phi\right)\, \eta^2(\phi)\\ &= \tilde{e}\cdot R\int_C \nabla\left(F\circ\phi\right) = \tilde{e}\cdot R\int_{\hat{\Omega}} \nabla(F\circ\tilde{\phi}),\end{aligned}$$

where R the matrix of a direct rotation of angle $\pi/2$, $F' = \eta^2$,

and where

$$\begin{aligned} T &:= \mathbb{R}^2/(L_1\mathbb{Z}\times L_2\mathbb{Z}) \text{ and } \hat{\Omega} := \Omega/(L_1\mathbb{Z}\times L_2\mathbb{Z}) \text{ if } d=2,\\ T &:= \mathbb{R}^2/\left(L_1\mathbb{Z}\times\{0\}\right) \text{ and } \hat{\Omega} := \Omega/\left(L_1\mathbb{Z}\times\{0\}\right) \text{ if } d=1.\end{aligned}$$

• $\tilde{\phi}$ is constant on every connected component of $T\backslash\hat{\Omega}$, and so is $F \circ \tilde{\phi}$. We then have

$$\int_{T\backslash\hat{\Omega}} \nabla \left(F \circ \tilde{\phi}\right) = 0.$$

• Hence, $\int_C (q \cdot \tilde{e}) w^2 = \tilde{e} \cdot R \int_T \nabla \left(F \circ \tilde{\phi}\right) = 0$, because T has no boundary. □

After studying the quantities of the form $\int_C q \cdot \tilde{e} w^2$, where $w \in \mathcal{I}$, it turned out that the limit of $c^*(M)/M$ depends strongly on the trajectories (stream lines) of the advection field q.

Trajectories of an $L-$periodic vector field, Periodicity of trajectories?

Definition (Trajectory of a vector field)

Assume that $N = 2$. Let $x \in \Omega$ such that $q(x) \neq 0$. The trajectory of q at x is the largest (in the sense of inclusion) connected differentiable curve $T(x)$ in Ω verifying:

(i) $x \in T(x)$,

(ii) $\forall y \in T(x)$, $q(y) \neq 0$,

(iii) $\forall y \in T(x)$, $q(y)$ is tangent to $T(x)$ at the point y.

The decision about the limit (null or positive) will depend on the existence of periodic unbounded trajs. for q!

Lemma (unbounded periodic trajectories)

Let $T(x)$ be an unbounded periodic trajectory of q in Ω, that is:

- *there exists $\mathbf{a} \in L_1\mathbb{Z} \times L_2\mathbb{Z} \setminus \{0\}$ (resp. $L_1\mathbb{Z} \times \{0\} \setminus \{0\}$) when $d = 2$ (resp. $d = 1$) such that $T(x) = T(x) + \mathbf{a}$.*
- *In this case, we say that $T(x)$ is $\mathbf{a}-$periodic.*

Then,

if $T(y)$ is another unbounded periodic trajectory of q, $T(y)$ is also $\mathbf{a}-$periodic.

Moreover,

in the case $d = 1$, $\mathbf{a} = L_1 e_1$. That is, all the unbounded periodic trajectories of q in Ω are $L_1 e_1-$periodic.

• There may exist **unbounded trajectories which are not periodic**, even though the vector field q **is periodic**.

A periodic vector field whose unbounded trajectories are not periodic!

Let

$$\phi(x,y) := \begin{cases} e^{-\frac{1}{\sin^2(\pi y)}} \sin(2\pi(x + \ln(y - [y]))) & \text{if } y \notin \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

• ϕ is C^∞ on $\mathbb{R}^2$, and 1-periodic in x and y.

• Hence the vector field $q = \nabla^\perp \phi$ is also C^∞, 1-periodic in x and y, and $\int_{[0,1]\times[0,1]} q = 0$ with $\nabla \cdot q \equiv 0$.

• The part of the graph of $x \mapsto e^{-x}$ lying between $y = 0$ and $y = 1$ is a trajectory of q, and is obviously unbounded and not periodic.

• There exist no periodic unbounded trajectory for this vector field, so the theorem asserts that for all $w \in \mathcal{I}$ we have

$$\int_C q w^2 = 0.$$

Theorem (E.-Kirsch 2009)

Assume that $N = 2$ and that Ω and q satisfy the assumptions. The two following statements are equivalent:

(i) There exists $w \in \mathcal{I}$, such that $\int_C qw^2 \neq 0$.

(ii) There exists a periodic unbounded trajectory $T(x)$ of q in Ω.

Moreover, if (ii) is verified and $T(x)$ is $\mathbf{a}-$periodic, then for any $w \in \mathcal{I}$ we have $\int_C q\, w^2 \in \mathbb{R}\mathbf{a}$.

Consequences

As a direct consequence of the previous Theorems, we get the following about the asymptotic behavior of the minimal speed within large drift:

Assume that $N = 2$. Then,

(i) If there exists no periodic unbounded trajectory of q in Ω, then

$$\lim_{M\to+\infty} \frac{c^*_{\Omega, M\,q, f}(e)}{M} = 0,$$

for any unit direction e.

(ii) If there exists a periodic unbounded trajectory $T(x)$ of q in Ω (which will be $\mathbf{a}-$periodic for some vector $\mathbf{a} \in \mathbb{R}^2$) then

$$\lim_{M\to+\infty} \frac{c^*_{\Omega, M\,q, f}(e)}{M} > 0 \iff \tilde{e}\cdot\mathbf{a} \neq 0. \tag{8}$$

Notice that in the case where $d = 1$, we have $\tilde{e} = \pm e_1$. Lemma 10 yields that $\tilde{e} \cdot \mathbf{a} = \pm L_1 \neq 0$.

Thus, for $d = 1$,

$$\lim_{M \to +\infty} \frac{c^*_{M\,q}(e)}{M} > 0 \Longleftrightarrow \exists \text{ a periodic unbounded traj. } T(x) \text{ of } q \text{ in } \Omega.$$

Proof of the Theorem

Definition

We define here the set of "regular trajectories" in $\hat{\Omega}$. Let
$\hat{U} := \left\{ \hat{x} \in \hat{\Omega} \text{ such that } T(\hat{x}) \text{ is well defined and closed in } \overline{\hat{\Omega}} \right\}$.

- We denote by $\hat{U}_i$ the connected components of $\hat{U}$.

Proposition

The set $\hat{U}$ is exactly the union of the trajectories which are simple closed curves in $\hat{\Omega}$.

Proof of the Theorem.

- $$\int_C qw^2 = R\int_C (\nabla\phi)w^2 = R\int_{\hat{\Omega}} (\nabla\hat{\phi})\hat{w}^2.$$

- Let $W := \{\hat{x} \in \hat{\Omega} \text{ such that } \hat{\phi}(\hat{x}) \text{ is a critical value of } \hat{\phi}\}$.

• $\boxed{\text{Co-area} \Rightarrow \left|\int_W \hat{w}^2 \nabla\hat{\phi}\right| \leq \int_W \hat{w}^2 |\nabla\hat{\phi}| = \int_{\hat{\phi}(W)} \left(\int_{\hat{\phi}^{-1}(t)} \hat{w}^2(x)\right) dt.}$

• From Sard's theorem, since $\hat{\phi}$ is C^2, $\mathcal{L}^1(\hat{\phi}(W)) = 0$, where $\mathcal{L}^1$ denotes the Lebesgue measure on $\mathbb{R}$.
One then gets

$$\int_W \hat{w}^2 \nabla\hat{\phi} = 0.$$

• $\hat{\Omega}\backslash W \subset \hat{U} \subset \hat{\Omega}$, we get

$$\int_C q w^2 = R \int_{\hat{\Omega}} (\nabla\hat{\phi})\hat{w}^2 = R \int_{\hat{U}} (\nabla\hat{\phi})\hat{w}^2 = R \sum_i \int_{\hat{U}_i} (\nabla\hat{\phi})\hat{w}^2. \tag{9}$$

We need the following preliminary lemma in order to give details about the limit when $N = 2$:

Lemma

Let $\hat{\Omega}$ be the set defined before, $\hat{V}$ be an open subset of $\hat{\Omega}$, and $\hat{\phi}$ given by (6). Suppose that:

(i) $\hat{q}(\hat{x}) \neq 0$ *for all* $\hat{x} \in \hat{V}$,

(ii) the level sets of $\hat{\phi}$ in $\hat{V}$ are all connected.

Then, for every $w \in \mathcal{I}$, there exists a continuous function $\eta : \hat{\phi}(\hat{V}) \to \mathbb{R}$ such that

$$\hat{w} = \eta \circ \hat{\phi} \text{ on } \hat{V}. \tag{10}$$

We now use Lemma 14 to get η_i continuous such that

$$\int_{\hat{U}_i} (\nabla \hat{\phi}) \hat{w}^2 = \int_{\hat{U}_i} (\nabla \hat{\phi}) \eta_i^2(\hat{\phi}).$$

We define the function F_i by $F_i' = \eta_i^2$ and $F_i(0) = 0$, and we obtain

$$\int_{\hat{U}_i} (\nabla \hat{\phi}) \hat{w}^2 = \int_{\hat{U}_i} \nabla F_i(\hat{\phi}).$$

Lemma

Let $\hat{U}_i$ as in the previous definition. Then,
(i) all the level sets of $\hat{\phi}$ in $\hat{U}_i$ are connected,
(ii) all the level sets of $\hat{\phi}$ in $\hat{U}_i$ are homeomorphic,
(iii) $\partial \hat{U}_i$ has exactly two connected components $\hat{\gamma}_1$ and $\hat{\gamma}_2$ such that

$$\hat{\phi}(\hat{\gamma}_1) = \sup_{\hat{x} \in \hat{U}_i} \hat{\phi}(\hat{x}) \text{ and } \hat{\phi}(\hat{\gamma}_2) = \inf_{\hat{x} \in \hat{U}_i} \hat{\phi}(\hat{x}).$$

Due to the condition $q \cdot \nu = 0$ on $\partial\Omega$, we have

Trajs of q follow the boundary, and this led us to: γ_1 (resp. γ_2) is either a connected component of $\partial\hat{\Omega}$ or contains a critical point of $\hat{\phi}$.

If we define

$$\hat{U}_i^{\varepsilon} := \{\hat{x} \in \hat{U}_i \text{ such that } \inf_{\hat{U}_i} \hat{\phi} + \varepsilon < \hat{\phi}(x) < \sup_{\hat{U}_i} \hat{\phi} - \varepsilon\},$$

then it follows from dominated convergence theorem that

$$\int_{\hat{U}_i^{\varepsilon}} (\nabla \hat{\phi}) \hat{w}^2 \xrightarrow[\varepsilon \to 0]{} \int_{\hat{U}_i} (\nabla \hat{\phi}) \hat{w}^2. \tag{11}$$

• $\neg ii) \Longrightarrow \neg i)$ We suppose that there exist no periodic unbounded trajectories of q. In $\hat{U}_i$, the trajectories of q are exactly the level sets of $\hat{\phi}$. We consider the following set

$$U_i^{\varepsilon} := \Pi^{-1}(\hat{U}_i^{\varepsilon}).$$

Let $x_0 \in U_i^{\varepsilon}$ and let $U_{i,0}^{\varepsilon}$ be the connected component of U_i^{ε} containing x_0.

• We proved that Π is a measure preserving bijection from $U_{i,0}^{\varepsilon}$ to $\hat{U}_i^{\varepsilon}$.

• Thus $\int_{\hat{U}_i^\varepsilon}(\nabla\hat{\phi})\hat{w}^2 = \int_{U_{i,0}^\varepsilon}(\nabla\phi)w^2 = \int_{U_{i,0}^\varepsilon}\nabla F_i(\phi) = \int_{\partial U_{i,0}^\varepsilon} F_i(\phi)\mathbf{n}$,

• $\partial U_{i,0}^\varepsilon$ is the union of two level sets C_1 and C_2 of ϕ in Ω, which are both simple closed curves!

• So we can write

$$\int_{U_{i,0}^\varepsilon}(\nabla\phi)w^2 = F(\phi(C_1))\int_{C_1}\mathbf{n} + F(\phi(C_2))\int_{C_2}\mathbf{n},$$

with

$$\int_{C_1}\mathbf{n} = \int_{C_2}\mathbf{n} = 0,$$

because the integral of the unit normal on a C^1 closed curve in $\mathbb{R}^2$ is zero.
□

$ii) \Longrightarrow i)$ was proved using the same technics.

Thank You