Regularity in 4th order nonlinear eigenvalue problems [1]

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[1] Complete paper can be found at: http://www.birs.ca/ nassif/papers_list.html

The problem

$$\begin{cases} \Delta^2 u = \lambda f(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (N_\lambda)$$

where $\lambda \geq 0$ is a parameter, Ω is a bounded domain in $\mathbf{R}^N$, $N \geq 2$, and where f satisfies one of the following two conditions:

(R): f is smooth, increasing, convex on $\mathbf{R}$ with $f(0) = 1$ and f is superlinear at ∞ (i.e. $\lim_{t\to\infty} \frac{f(t)}{t} = \infty$);

(S): f is smooth, increasing, convex on $[0, 1)$ with $f(0) = 1$ and $\lim_{t\nearrow 1} f(t) = +\infty$.

Our main interest is in the regularity of the extremal solution u^* associated with (N_λ).

The second order case

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (Q_\lambda)$$

is relatively well understood (Joseph-Lundgren, Mignot-Puel, Brezis-Vasquez, Martel, Nedev, Cabre, Capella, Ghoussoub, Guo, Esposito, etc....)

- There exists a finite positive λ^* such that for all $0 < \lambda < \lambda^*$ there exists a **minimal solution** u_λ of (Q_λ).
- For each $0 < \lambda < \lambda^*$, u_λ is **semi-stable**:

$$\int_\Omega \lambda f'(u_\lambda)\psi^2 dx \leq \int_\Omega |\nabla\psi|^2 dx, \qquad \forall\psi \in H_0^1(\Omega),$$

  and is unique among all the weak semi-stable solutions.
- $\lambda \mapsto u_\lambda(x)$ is increasing on $(0, \lambda^*)$ for each $x \in \Omega$. This allows one to define $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$, the so-called **extremal solution**. It is the unique weak solution of (Q_{λ^*}).
- There are no solutions of (Q_λ) (even weak) for $\lambda > \lambda^*$.

Regularity of the extremal solution

Is u^* a regular solution ?

- Of interest since one can then apply Crandall-Rabinowitz to start a 2d branch of solutions emanating from (λ^*, u^*).
- If f satisfies (R) (resp. (S)) it is sufficient –in view of standard elliptic regularity theory– to show that u^* is bounded (resp. $\sup_\Omega u^* < 1$).
- This turned out to depend on the dimension, and so: Given a nonlinearity f, say that ***N* is the associated critical dimension** provided:
  - the extremal solution u^* for (Q_{λ^*}) is a classical solution for any bounded smooth domain $\Omega \subset \mathbb{R}^M$ for any $M \leq N - 1$, and
  - if there exists a domain $\Omega \subset \mathbb{R}^N$ such that the associated extremal solution u^* is not a classical solution.

What do we know?Still 2d order

- For $f(t) = e^t$, the critical dimension is $N = 10$.
  For $N \geq 10$, then on the unit ball the extremal solution is explicitly given by $u^*(x) = -2\log(|x|)$. (Joseph-Lundgren, Mignot-Puel, Brezis-Vasquez, etc... ?)
- For $f(t) = (t+1)^p$, $p > 1$, there exists $N(p)$ critical dimension that I cannot remember...
- For $\Omega = B$ the unit ball in $\mathbb{R}^N$, u^* is bounded for any f satisfying (R) provided $N \leq 9$, which –in view of the above– is optimal (Cabre-Capella).
- On general domains, and if f satisfies (R), then u^* is bounded for $N \leq 3$ (Nedev). Recently this has been improved to $N \leq 4$ provided the domain is convex (Cabre).
- For $f(t) = (1-t)^{-2}$ the critical dimension is $N = 8$ and $u^* = 1 - |x|^{\frac{2}{3}}$ is the extremal solution on the unit ball for $N \geq 8$. (Joseph-Lundgren, Mignot-Puel, Ghoussoub-Guo.)

The general approach

1. Use the semi-stability of the minimal solutions u_λ to obtain L^q-estimates estimates which translate to uniform L^∞ bounds and then passing to the limit. These estimates generally depend on the ambient space dimension.

2. On the other hand, to show the optimality of the regularity result one generally finds an explicit singular extremal solution u^* on a radial domain.
    - Here the crucial tool is the fact that a semi-stable singular solution in $H_0^1(\Omega)$, has to be the extremal solution.
    - In practice one considers an explicit singular solution on the unit ball and applies Hardy-type inequalities to show its semi-stability in the right dimension.

The fourth order case–Dirichlet Boundary conditions

There are two obvious fourth order extensions of (Q_λ) namely the problem (N_λ) mentioned above, and its Dirichlet counterpart

$$\begin{cases} \Delta^2 u = \lambda f(u) & \text{in } \Omega \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{D_λ}$$

1. The bifurcation diagram for (Q_λ) is heavily dependent on the maximum principle, but for general domains there is no maximum principle for Δ^2 with Dirichlet boundary conditions. However, on the unit ball there is Boggio's maximum principle!
2. First real progress Davila-Dupaigne-Guerra-Montenegro, 2008): For (D_λ) on the ball with $f(t) = e^t$, u^* is bounded if and only if $N \leq 12$.
3. Followed by Cowan, Esposito, Ghoussoub, Moradifam, 2009, for (D_λ) on ball with $f(t) = (1-t)^{-2}$. Here $u^* < 1$ if and only if $N \leq 8$.

Both heavily dependent on the fact that Ω is the unit ball.

Even in this radial situation, two main hurdles

1. The standard energy estimate approach, so successful in the second order case, does not appear to work in the fourth order case. More later.
2. Not trivial to construct explicit unbounded u^* for $N \geq 13$.
   - An explicit singular, semi-stable solution which satisfies the first boundary condition is easy to guess. One then needs to perturb it enough to satisfy the second boundary condition but not too much so as to lose the semi-stability.
   - Davila et al. succeeded in doing so for $N \geq 32$, but they were forced to use a computer assisted proof to show that the extremal solution is unbounded for the intermediate dimensions $13 \leq N \leq 31$.
   - Using various improved Hardy-Rellich inequalities from Ghoussoub-Moradifam the need for the computer assisted proof was removed in Moradifam.

Navier type conditions

1. The problem (N_λ) with Navier boundary conditions does not suffer from the lack of a maximum principle and the existence of the minimal branch has been shown.

2. If the domain is the unit ball, then again one can use the methods of Davilla et al. for $f(t) = e^t$ and Cowan et al. for $f(t) = (1-t)^{-2}$ (in the Dirichlet case) to obtain optimal results in the Navier case on the ball (Moradifam).

3. However, the case of a general domain is only understood in dimensions $N \leq 4$ (Guo-Wei).

The following is a first attempt at giving energy estimates on general domains, which while they do improve known results, they still fall short of the conjectured critical dimensions established when the domain is a ball.

New results on general domain (Cowan-Esposito-Gh.)

u^* is smooth

- If f is any convex superlinear nonlinearity, provided $N \leq 5$.
- If $\liminf_{t\to+\infty} \frac{f(t)f''(t)}{(f')^2(t)} > 0$, and $N \leq 7$.
- If $\gamma := \limsup_{t\to+\infty} \frac{f(t)f''(t)}{(f')^2(t)} < +\infty$, and $N < \frac{8}{\gamma}$.

In particular,

- If $f(t) = e^t$ and $N \leq 8$;
- If $f(t) = (1+t)^p$ and $N < \frac{8p}{p-1}$.
- If $f(t) = (1-t)^{-p}$, $p > 1$, $p \neq 3$, and $N \leq \frac{8p}{p+1}$.

Major improvements on what is known for general domains, but still fall short of the expected optimal results as recently established on radial domains, e.g., u^* is smooth for $N \leq 12$ when $f(t) = e^t$ and for $N \leq 8$ when $f(t) = (1-t)^{-2}$

Sufficient L^q-estimates for regularity

It suffices to consider classical solutions $(u_n)_n$ of (N_{λ_n}), $(\lambda_n)_n$ uniformly bounded, and try to show that $\sup_n \|u_n\|_\infty < +\infty$.

By standard elliptic regularity theory follows by a uniform bound of $f(u_n)$ in $L^q(\Omega)$, for some $q > \frac{N}{4}$.

We can do better!!!

Suppose that for some $q \geq 1$ and $0 < \beta < \alpha$ we have

$$\sup_n \left\{ \int_\Omega \frac{f^\alpha(u_n)}{u_n^\beta + 1} + \int_\Omega f^q(u_n) \right\} < +\infty \tag{1}$$

Then:

1. If $1 \leq q \leq \frac{N}{4}$ and $\alpha \leq \frac{N}{4}$, then $\sup_n \|f(u_n)\|_s < +\infty$ for every $s < \max\{\frac{(\alpha-\beta)N}{N-4\beta}, q\}$.
2. If either $q > \frac{N}{4}$ or $\alpha > \frac{N}{4}$, then $\sup_n \|u_n\|_\infty < +\infty$.

Particular cases

Suppose $(u_n)_n$ is a sequence of solutions of (N_{λ_n}) such that

$$\sup_n \int_\Omega f^q(u_n) < +\infty \tag{2}$$

for $q \geq 1$. Then $\sup_n \|u_n\|_\infty < +\infty$, if:

1. $f(t) = e^t$ and $q \geq \frac{N}{4}$;
2. $f(t) = (t+1)^p$ and $q > \frac{N}{4}\left(1 - \frac{1}{p}\right)$.

Another criterium for regularity: Suppose

$$\sup_n \int_\Omega f^s(u_n) < +\infty \quad \text{for } 1 \leq s < \frac{N}{N-2} \tag{3}$$

and

$$\sup_n \int_\Omega (f')^q(u_n) < +\infty \text{ for some } q > \tfrac{N}{4}, \tag{4}$$

Then,

$$\sup_n \|u_n\|_\infty < +\infty. \tag{5}$$

Elementary use of stability

u is a semi-stable solution of (N_λ) if

$$\int_\Omega \lambda f'(u)\psi^2 dx \le \int_\Omega (\Delta\psi)^2 dx, \qquad \forall \psi \in H^2(\Omega) \cap H_0^1(\Omega). \quad (6)$$

To the best of my knowledge the only available energy estimates for smooth, semi-stable solutions so far, is:

$$\int_\Omega f'(u)u^2 dx \le \int_\Omega f(u)u dx. \quad (7)$$

To see this, just take $\psi = u$ in (**??**).This yields:

1. If $f(t) = e^t$, then $e^{u^*}(u^*)^2 \in L^1(\Omega)$ and u^* is then regular for $N \le 4$.
2. If $f(t) = (t+1)^p$, then $(u^*+1)^p \in L^{\frac{p+1}{p}}(\Omega)$, therefore u^* is regular for $N \le 4$ or if $1 \le p < \frac{N+4}{N-4}$ and $N > 4$
3. If $f(t) = (1-t)^{-2}$, then $(1-u^*)^{-2} \in L^{\frac{3}{2}}(\Omega)$ and u^* is regular for $N \le 4$.

A new idea

Suppose u is a semi-stable solution of (N_λ). Then

$$\int_\Omega f''(u)(-\Delta u)|\nabla u|^2 dx \leq \lambda \int_\Omega f(u)dx. \tag{8}$$

Proof: Set $\psi = \Delta u$ in the stability condition to arrive at

$$I := \int_\Omega f'(u)(\Delta u)^2 dx \leq \int_\Omega \Delta^2 u f(u) dx =: J.$$

Now an integration by parts shows that

$$\begin{aligned} I &= \int_\Omega f''(u)(-\Delta u)|\nabla u|^2 dx - \int_\Omega f'(u)\nabla u \cdot \nabla \Delta u dx \\ J &= \lambda \int_\Omega f(u)dx - \int_\Omega f'(u)\nabla u \cdot \nabla \Delta u dx. \end{aligned}$$

Since $I \leq J$ one obtains the result.

A new idea.bis

Suppose u is a solution of (N_λ) and g is a smooth function defined on the range of u with $f(t) \geq g(t)g'(t)$ and $g(t), g'(t), g''(t) \geq 0$ on the range of u with $g(0) = 0$. Then

$$-\Delta u \geq \sqrt{\lambda} g(u) \qquad \text{in } \Omega. \tag{9}$$

Proof: Let $v := -\Delta u - \sqrt{\lambda} g(u)$ and so $v = 0$ on $\partial\Omega$ and a computation shows that

$$-\Delta v + \sqrt{\lambda} g'(u) v = \lambda [f(u) - g(u)g'(u)] + \sqrt{\lambda} g''(u) |\nabla u|^2 \qquad \text{in } \Omega.$$

The assumptions on g allow one to apply the maximum principle and obtain that $v \geq 0$ in Ω.
Inspired by the proof of Souplet of the Lane-Emden conjecture in four space dimensions.

Main estimate

Suppose u is a semi-stable solution of (N_λ) and that g chosen as above. If $H(u) := \int_0^u f''(\tau)g(\tau)d\tau$, then

$$\int_\Omega g(u)H(u)dx \leq \int_\Omega f(u)dx. \tag{10}$$

Proof: Rewrite the result from previous Lemma as

$$\begin{aligned}
\lambda \int_\Omega g(u)H(u)dx &\leq \sqrt{\lambda}\int_\Omega (-\Delta u)H(u)dx = \sqrt{\lambda}\int_\Omega \nabla H(u)\cdot\nabla u dx \\
&= \sqrt{\lambda}\int_\Omega H'(u)|\nabla u|^2 dx \\
&\leq \lambda\int_\Omega f''(u)(-\Delta u)|\nabla u|^2 dx \leq \lambda\int_\Omega f(u)dx.
\end{aligned}$$

An example

Take $f(u) = e^u$ and set $g(u) := \sqrt{2}\left(e^{\frac{u}{2}} - 1\right)$.
Then $-\Delta u \geq g(u)$ in Ω.
Now $H(u) = \sqrt{2}\left(\frac{2}{3}e^{\frac{3u}{2}} - e^u + \frac{1}{3}\right)$, and by above lemma:

$$2\int_\Omega \left(e^{\frac{u}{2}} - 1\right)\left(\frac{2}{3}e^{\frac{3u}{2}} - e^u + \frac{1}{3}\right)dx \leq \int_\Omega e^u dx,$$

hence

$$\int_\Omega e^{2u} dx \leq \frac{5}{2}\int_\Omega e^{\frac{3u}{2}} dx,$$

and by Holder we get $\int_\Omega e^{2u} dx \leq (\frac{5}{2})^4 |\Omega|$.
Actually for most explicit nonlinearities f, the method yields that for any stable solution u_λ

$$\|f(u_\lambda)\|_2 \leq C < +\infty$$

where C does not depend on λ and u.

For a general superlinear nonlinearity ...

we can take $g(u) := \sqrt{2}\left(\int_0^u (f(t)-1)dt\right)^{\frac{1}{2}}$, use the superlinearity of f at ∞ to prove that

$$\frac{g(u)H(u)}{f(u)} \to +\infty \qquad \text{as } u \to +\infty,$$

and re-state above as $\int_\Omega g(u)H(u) \le C$, for every semi-stable solution u of (N_λ), where C is independent of λ and u.
One then proves that for $u \ge 0$ is a semi-stable solution of (N_λ),

$$\int_\Omega \frac{f(u)^{\frac{3}{2}}}{\sqrt{u}+1}dx \le C \quad \text{and} \qquad \int_\Omega f(u)dx \le C, \qquad (11)$$

for some constant $C > 0$ independent of λ and u.
The extremal solution u^* of (N_λ) is then regular for $N \le 5$, while $f(u^*) \in L^q(\Omega)$ for all $q < \frac{N}{N-2}$ if $N \ge 6$.

Singular nonlinearities

Theorem 1: Suppose $f(t) = (1-t)^{-p}$ with $p > 1$ and $p \neq 3$. Then u^* is regular (i.e. $\sup_\Omega u^* < 1$) provided $N \leq \frac{8p}{p+1}$.

This will follow immediately from the following two theorems.

Theorem 2: Let u_n be solutions of (N_{λ_n}) such that

$$\sup_n \|f(u_n)\|_q < \infty \text{ for some } q > 1 \text{ and } q \geq \frac{(p+1)N}{4p}.$$

Then $\sup_n \|u_n\|_\infty < 1$.

Theorem 3: Suppose $p > 1$ and $u \geq 0$ is a semi-stable solution of (N_λ). Then

$$\|f(u)\|_2 \leq C, \text{ where } C \text{ is independent of } u \text{ and } \lambda.$$

Proof: Let

$$g(u) := \sqrt{\frac{2}{p-1}}\left(\frac{1}{(1-u)^{\frac{p-1}{2}}} - 1\right).$$

It does verify the conditions of Lemma above and therefore one has $-\Delta u \geq g(u)$ a.e. in Ω, and hence $\int_\Omega g(u)H(u)dx \leq \int_\Omega f(u)dx$, where $H(u) := \int_0^u f''(\tau)g(\tau)d\tau$. A computation shows that

$$H(u) = C_p\left(\frac{1}{(1-u)^{\frac{3p+1}{2}}} - 1\right) + \tilde{C}_p\left(1 - \frac{1}{(1-u)^{p+1}}\right)$$

where $C_p, \tilde{C}_p > 0$. It follows that

$$\int_\Omega \frac{1}{(1-u)^{2p}}dx \leq C(p)\int_\Omega \frac{1}{(1-u)^{\frac{3p+1}{2}}}dx + C(p)\int_\Omega \frac{1}{(1-u)^p}dx.$$

Since $p > 1$, we have that $\frac{3p+1}{2} < 2p$, and we are done!!!!