An unfortunate misprint

Robert L. Jerrard

Department of Mathematics
University of Toronto

March 22, 2010, BIRS

Consider the equation[1]

$$\boxed{\varepsilon(u_{tt} - u_{xx}) + \frac{2}{\varepsilon}(u^2 - 1)(u - \varepsilon\alpha) = 0} \tag{1}$$

for $0 < \varepsilon \ll 1$.

Model problem related to

- mechanics: *undamped* dynamic phase transition
- cosmology: “decay of a false vacuum"

By analogy with well-studied parabolic equations, one expects (?) *relativistic accelerating* fronts.

Potential difficulty: the equation has a conserved energy, and accelerating fronts would release large amounts of it.

[1] The second time derivative may appear to be an unfortunate misprint, as suggested in the title of the talk. In fact it is not.

Theorem (J., 2010)

There exists initial data for which the corresponding solution u of (1) *satisfy*

$$\|u - U_\varepsilon\|_{L^2(K)} \le C\sqrt{\varepsilon}, \qquad K \text{ compact in } \mathbb{R}^2, \; C = C(K)$$

Here U_ε is an explicitly constructed function with a Lorenz-contracted interface that travels along a curve of constant Minkowskian *curvature* 2α *in* $\mathbb{R}^2$.

This result is stable with respect to suitable small perturbations in $H^1 \times L^2$ of the initial data.

The next few slides present most of the proof.

Step 1: change of variables.

Introduce polar coordinates $(r,\theta) \mapsto (r\cosh\theta, r\sinh\theta) = \psi(r,\theta)$.

Let $v = u \circ \psi$. Then

$$\boxed{\varepsilon(\frac{1}{r^2}v_{\theta\theta} - v_{rr} - \frac{1}{r}v_r) + \frac{2}{\varepsilon}(v^2-1)(v-\varepsilon\alpha) = 0.}$$

The θ variable is timelike and r is spacelike.

If we imagine $v_{\theta\theta} \approx 0$ and $\frac{1}{r} \approx c$, then the above looks like the equation for traveling waves for a reaction-diffusion equation:

$$\varepsilon(-q_{rr} - cq_r) + \frac{2}{\varepsilon}(q^2-1)(q-\varepsilon\alpha) = 0.$$

The heuristic $\frac{1}{r} \approx c$ may be reasonable for solutions with an interface concentrated at $r = \frac{1}{c}$.

The traveling wave equation has a unique solution

$$c = 2\alpha, \qquad q(r) = \tanh(\frac{r}{\varepsilon}).$$

Step 2: introduce pseudo-energy density

Define

$$e_\varepsilon(v) := \frac{\varepsilon}{2}(\frac{v_\theta^2}{r^2} + v_r^2) + \frac{1}{2\varepsilon^2}(v^2 - 1)^2.$$

Then

$$\begin{aligned} \frac{d}{d\theta} e_\varepsilon(v) &= \varepsilon(v_r v_\theta)_r + \varepsilon \frac{v_\theta}{r} v_r - 2\alpha v_\theta(1 - v^2) \\ &= \varepsilon(v_r v_\theta)_r + \varepsilon \frac{v_\theta}{r} v_r(1 - 2\alpha r) + 2\alpha\varepsilon v_\theta(v_r - \frac{1}{\varepsilon}(1 - v^2)) \\ &= \varepsilon(v_r v_\theta)_r + \text{Term 1} + \text{Term 2}. \end{aligned}$$

Note: the equation has a *conserved* energy that we are *not* using.

- Term 1 small if v_r concentrated near $r = \frac{1}{2\alpha}$
- Traveling wave $q = \tanh(r/\varepsilon)$ solves $q' - \frac{1}{\varepsilon}(1 - q^2) = 0$, so Term 2 small if $v \approx q$ in strong enough sense, up to translation.

Step 3: weighted energy and related functionals.
Write $r_0 = \frac{1}{2\alpha}$, and define

$$\eta_1(\theta) = \int_0^\infty w(r) e_\varepsilon(v) dr \ - \ \kappa_1$$

for $w(r) = \min\{1 + (r - r_0)^2, 2\}$ and $\kappa_1 =$ minimal interface energy.

Further define

$$\eta_2(\theta) = \int_0^\infty \varepsilon \frac{v_\theta^2}{r^2} + (r - r_0)^2 \left(\frac{\varepsilon}{2} v_r^2 + \frac{1}{2\varepsilon}(v^2 - 1)^2\right) dr$$

and

$$\eta_3(\theta) = \int_0^\infty |v - \mathrm{sign}(r - r_0)|^2 |r - r_0| \ dr.$$

Step 4: energy flux.

Compute:

$$\boxed{\eta_1'(\theta) \le \eta_2(\theta) + \int_0^\infty \left(\sqrt{\varepsilon} v_r - \frac{1}{\sqrt{\varepsilon}}(1 - v^2) \right)^2 dr =: \eta_2(\theta) + \eta_4(\theta).}$$

The new term η_4 is small if v is approximately solves the first-order equation characterizing traveling waves.

Step 5: some stability estimates. First, straightforward estimates show that

$$\boxed{\eta_3(\theta) \le \eta_3(0) + \int_0^\theta \eta_2(\phi) d\phi.}$$

This uses the pointwise inequality

$$\frac{v_\theta^2}{r^2} + \frac{1}{4\varepsilon}(r - r_0)^2(v^2 - 1)^2 \ge |r - r_0||(v - \frac{v^3}{3})_\theta|.$$

Further straightforward estimates show that

$$\boxed{\eta_2(\theta) + \eta_4(\theta) \le C(\eta_1(\theta) + \eta_3(\theta))}$$

Heuristically,

η_3 small $\Rightarrow$interface present $\Rightarrow \eta_1 \ge \eta_2$, and

surplus energy dominates $\sqrt{\varepsilon} v_r - \frac{1}{\sqrt{\varepsilon}}(1 - v^2)$

Step 6: Main conclusion:

$$\boxed{(\eta_2+\eta_4)(\theta) \le C\int_0^\theta (\eta_1+\eta_3)d\phi \le C\int_0^\theta (\eta_2+\eta_4)d\phi.}$$

Thus

$$\eta_i(\theta) \le Ce^{C\theta} \sup_i \eta_i(0) \quad \approx \quad Ce^{C\theta}\varepsilon^2 \quad \text{for good data.}$$

This forces $v(0,r) \approx q(\frac{r-r_0}{\varepsilon})$ and $v_\theta(0,r) \approx 0$, a nearly stationary interface near $r = r_0$.

In particular

$$\int \frac{1}{r^2} {v_\theta}^2 dr \quad \le \quad C(k)\varepsilon$$

for $|\theta| \le k$ so that

$$\boxed{\int \frac{1}{r^2}|v(r,\theta) - v(r,0)|^2 dr \le C(k)\varepsilon.}$$

Thus the interface remains concentrated near $\{(\theta, r) : r = \frac{1}{2\alpha}\}$, which is a curve of constant curvature 2α.

Essentially the same argument yields much more general results.

Consider the PDE for $u : \mathbb{R}_t \times \mathbb{R}^N \to \mathbb{R}$

$$\boxed{\varepsilon \Box u + \frac{2}{\varepsilon}(u^2 - 1)u = \alpha(1 - u^2)} \tag{2}$$

where α is a fixed smooth function; or qualitatively similar equations with more general nonlinearities.

Suppose that Γ is

- a timelike hypersurface, smooth in $(-T, T) \times \mathbb{R}^N$,
- with prescribed *Minkowski* mean curvature α,
- and with velocity 0 at $t = 0$.

Theorem (J., 2010)

For initial data exhibiting an interface near $\{x : (x,0) \in \Gamma\}$ *and with surplus energy* $O(\varepsilon^2)$*, the solution* u *of* (2) *satisfies*

$$\|u - U_\varepsilon\|_{L^2(K)} \le C\sqrt{\varepsilon}, \qquad K \text{ compact in } (-T,T)\times R^N, \; C = C(K,\Gamma)$$

where

$$U_\varepsilon(t,x) = q\left(\frac{d(t,x)}{\varepsilon}\right)$$

for a specific profile q*, and* d *is the signed Minkowski distance to* Γ*, so that*

$$-d_t^2 + |\nabla d|^2 = 1 \text{ near } \Gamma, \qquad d = 0 \text{ on } \Gamma.$$

The first theorem can surely be proved by other techniques, eg splitting the equation.

I do not know of any viable alternative approach to the second theorem.

Some related work

- Decay of a false vacuum:
  - formal results on quantum tunneling from higher-energy stable state
  - exact radial outward accelerating solutions in $\mathbb{R}^{1+3}$ (?)

  Coleman, Callan-Coleman, Coleman-Glaser-Martin, late 70s.

- scalar elliptic PDE and minimal/prescribed curvature surfaces: Modica-Mortola, Mortola, Hutchinson-Tonegawa, Pacard-Ritoré, del Pino-Kowalczyk-Wei....
- scalar parabolic PDE and mean curvature flow: Bronsard-Kohn, de Mottoni-Schatzmann, X. Chen, Evans-Soner-Souganidis, Ilmanen, Soner
- scalar parabolic PDE and front propagation: very long history

On wave equations:

- Many related results in (J 2009) concerning $\alpha = 0$.
- prior to that, all work addressed dynamics of *point* defects, eg: J, Lin, Gustafson-Sigal, Stuart