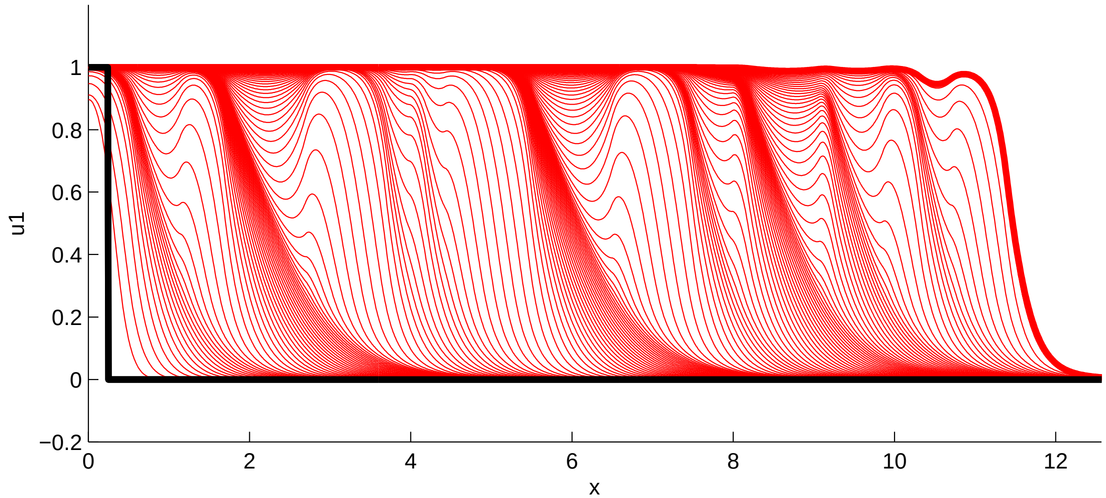
Traveling waves in an inhomogeneous medium

Jim Nolen (Duke University)



Including joint work with Lenya Ryzhik (Stanford),
Antoine Mellet (Univ. Maryland),
Jean-Michel Roquejoffre (Univ. Paul Sabatier, Toulouse)

The reaction-diffusion equation

$$u_t = u_{xx} + f(x, u), \quad x \in \mathbb{R},\ t > 0.$$

Solutions will behave like a traveling wave with moving interface. . . .

(i) How does the solution evolve at large times?

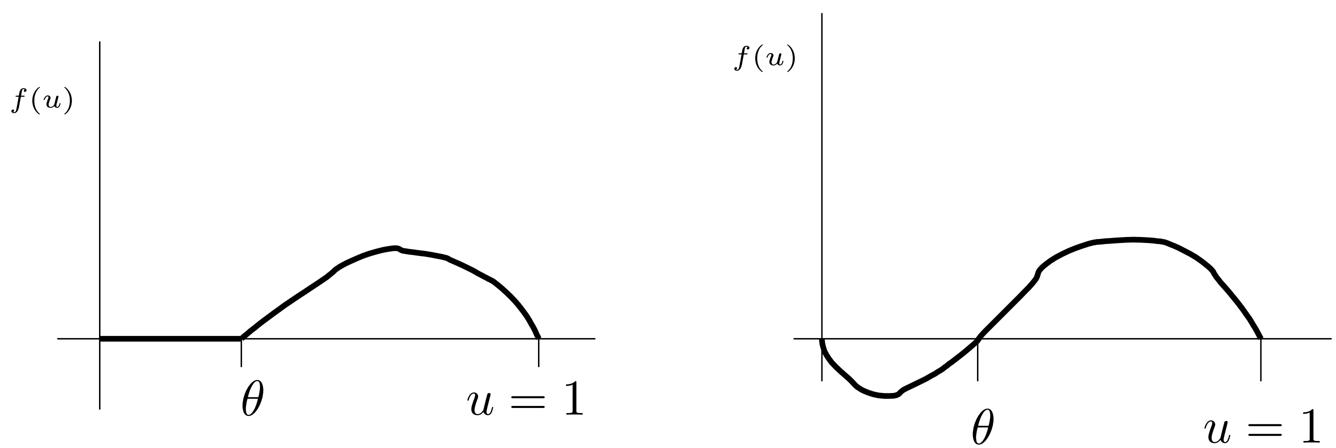
(ii) If $f(x, u)$ is random, what are the statistical properties of u?

Pushed fronts in a homogeneous environment

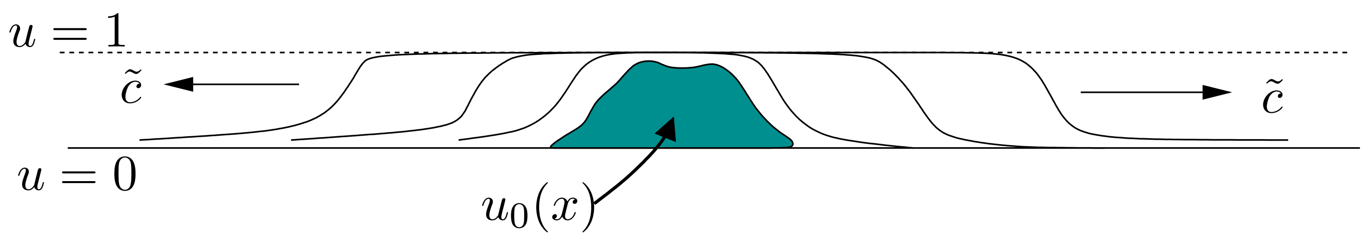
Suppose $u(t,x)$ satisfies

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, \ \ t > 0$$
$$u(0,x) = u_0(x) \in [0,1]$$

$f(u)$ is nonlinear and $\int_0^1 f(u)\,du > 0$:

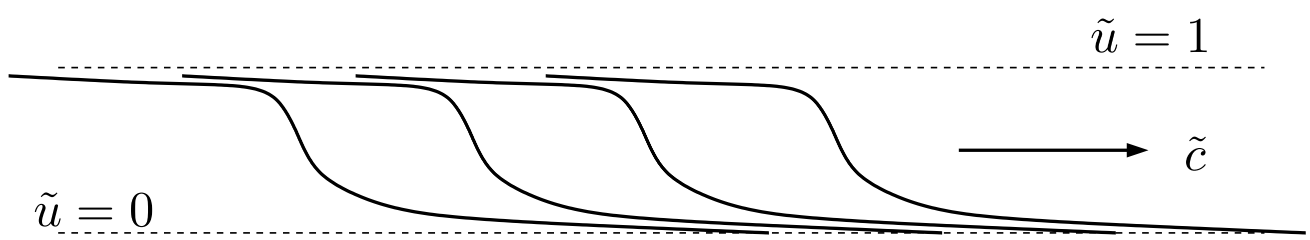


Diffusion + Reaction = front propagation



Traveling wave solutions:

$$\tilde{u}(t, x) = \tilde{u}(0, x - \tilde{c}t), \quad x \in \mathbb{R},\ t \in \mathbb{R}$$



Traveling wave solutions are attractors.

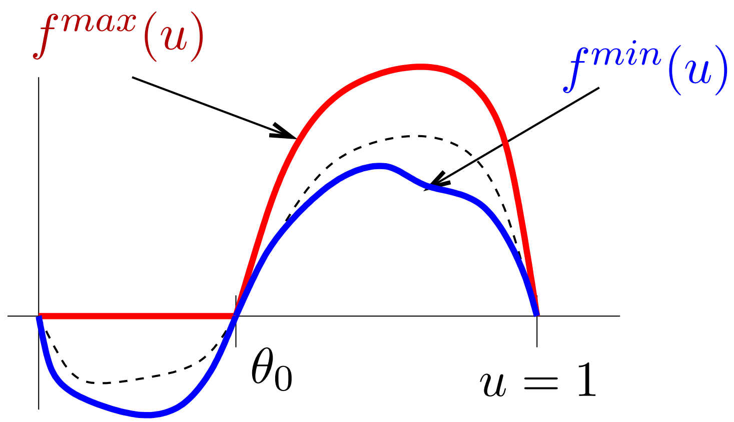
If $u(t,x)$ solves the initial value problem with appropriate "wave-like" initial data at $t = 0$, then for some $\tau \in \mathbb{R}$,

$$\sup_x |u(t,x) - \tilde{u}(t+\tau, x)| \leq Ce^{-rt}, \quad \forall\, t \geq 0$$

Kanel (1962), Aronson, Weinberger (1979), Fife, McLeod (1977).

The inhomogeneous environment

$$u_t = u_{xx} + f(x,u), \quad x \in \mathbb{R}, \quad t > 0; \qquad u(0,x) = u_0(x).$$



- $f^{min}(u) \le f(x,u) \le f^{max}(u)$
- $\int_0^1 f^{min}(u)\,du > 0$
- For example: $f(x,u) = g(x) f_0(u), \qquad g(x) > 0.$

If $f(x,u)$ is periodic in x there are **pulsed traveling waves**

$$\tilde{u}(t + \frac{L}{\tilde{c}}, x) = \tilde{u}(t, x - L)$$

For example, see Berestycki, Hamel (2002), Xin (1992, 1993).

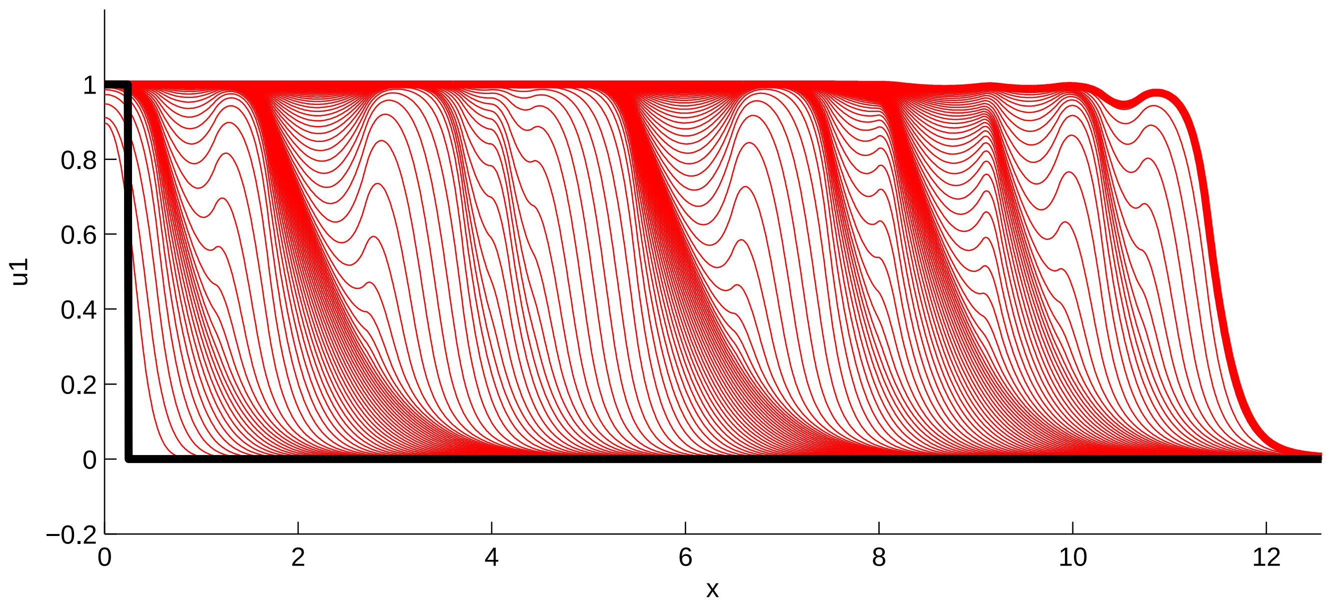
What if we do not impose a periodic structure on f?

What does the solution look like?

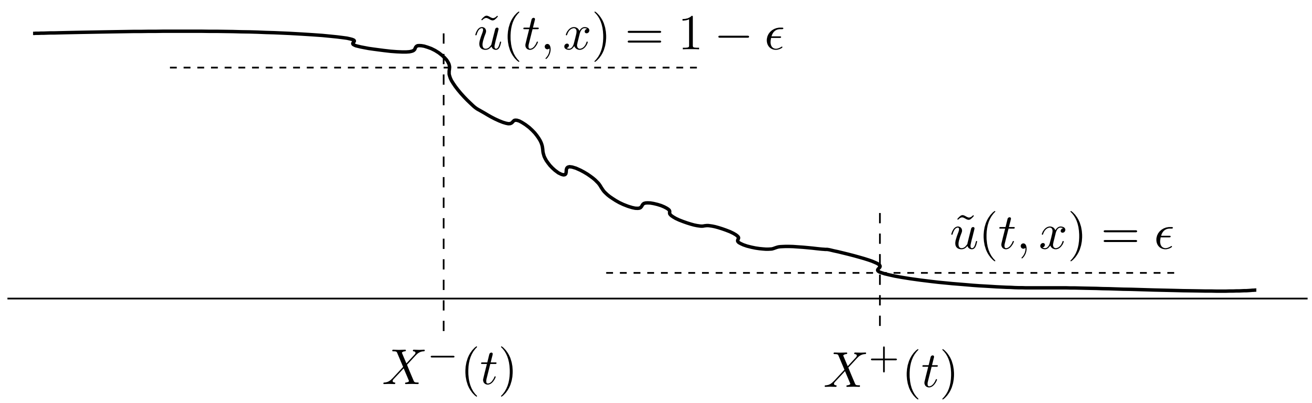
The initial data is a step function (in black).

The plot shows $u(t, x)$ at regularly-spaced points in time.

$g(x)$ was randomly generated.



The interface width does **not** spread out as $t \to \infty$.

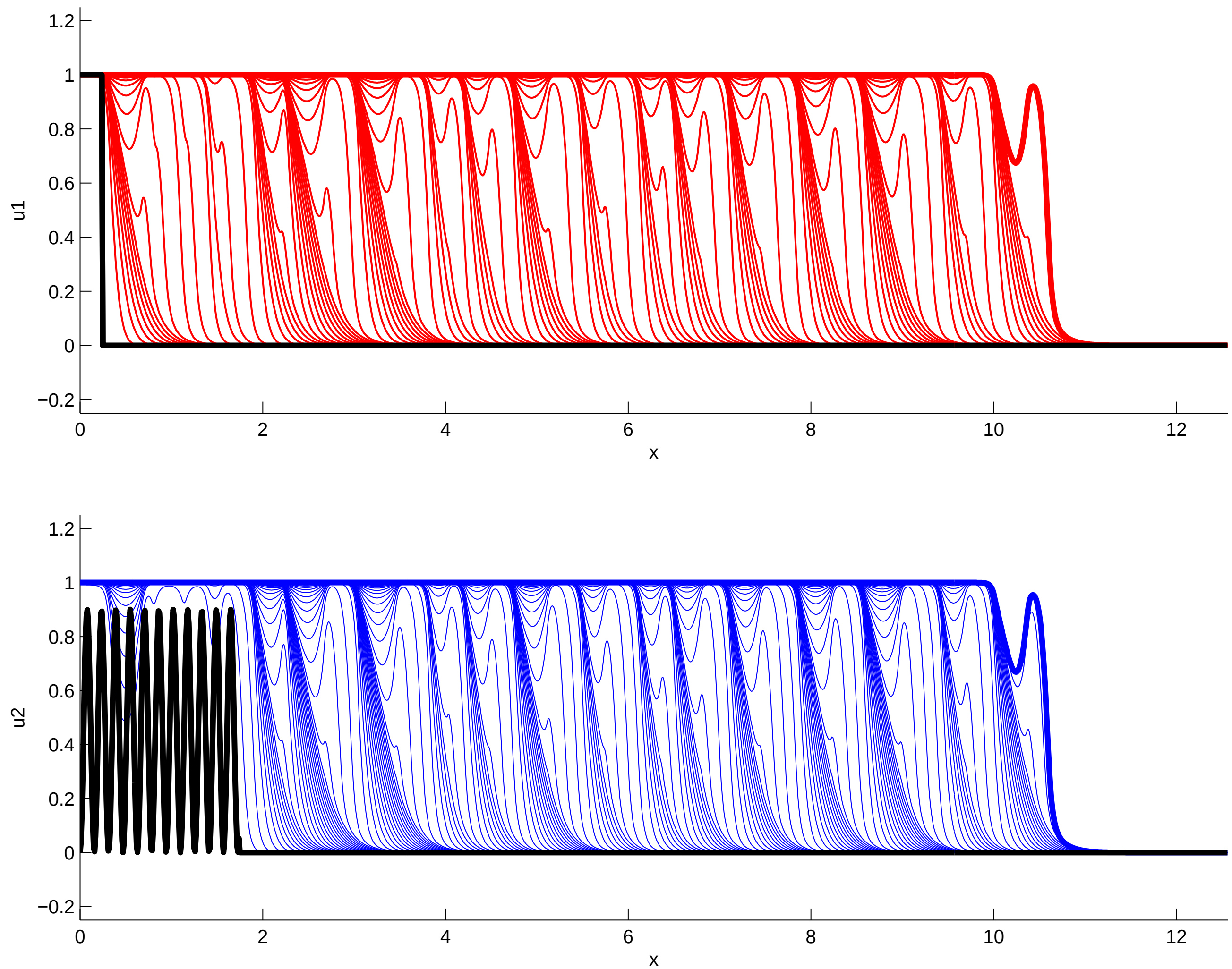


For some universal constant C,

$$|X^+(t) - X^-(t)| \leq C$$

holds for all t sufficiently large.

Two solutions with different initial data (in black).



A Generalized Traveling Wave:

There **exists** a right-moving transition-front solution $\tilde{u}(t,x)$ of

$$\tilde{u}_t = \tilde{u}_{xx} + g(x) f_0(\tilde{u}), \quad x \in \mathbb{R}, \;\; t \in \mathbb{R}.$$

It is **unique** up to a time shift. Also, $\tilde{u}_t > 0$, for all $x \in \mathbb{R}$, $t \in \mathbb{R}$.

This solution is an **attractor**: if $u_0(x)$ is wave-like, then there is a time shift τ and constants $C, r > 0$ such that

$$\sup_{x \in \mathbb{R}} |u(x,t) - \tilde{u}(t+\tau, x)| \le C e^{-rt}$$

holds for all $t \ge 0$.

Mellet, Roquejoffre, Sire (2009),

N., Ryzhik (2009),

Mellet, N., Ryzhik, Roquejoffre (2009)

What if f is random?

Suppose that

$$f = g(x,\omega) f_0(u)$$

where $g(x,\omega) : \mathbb{R} \times \Omega \to (0,\infty)$ is a stationary random field, with suitable bounds and regularity.

Let $\{\pi_x\}_{x\in\mathbb{R}}$ be a group of measure-preserving transformations which act ergodically on $(\Omega, \mathcal{F}, \mathbb{P})$ so that $g(x+h,\omega) = g(x, \pi_h\omega)$.

In this case, the preceding results hold with probability one.

A Law of Large Numbers for the interface

Let $X(t,\omega)$ be the random interface position:

$$X(t,\omega) = \sup\{x \in \mathbb{R} \,|u(t,x,\omega) = \frac{1}{2}\}.$$

Then $X(t,\omega)$ satisfies

$$\lim_{t\to\infty} \frac{X(t,\omega)}{t} = \tilde{c}, \quad \text{almost surely, and in } L^1(\Omega).$$

The constant $\tilde{c} > 0$ is independent of the initial data.

N., Ryzhik (2009)

See Freidlin-Gärtner (1979) for a related result with K.P.P.-type nonlinearity.

A Central Limit Theorem

If the environment is sufficiently mixing, then

(i) There is $\kappa^2 \geq 0$ such that

$$\frac{X(t,\omega) - t\tilde{c}}{\sqrt{t}} \to N(0,\kappa^2), \qquad \text{as } t \to \infty.$$

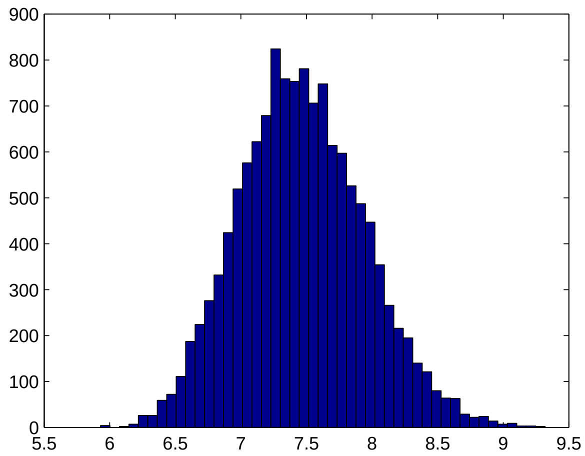
(ii) If $\kappa^2 > 0$, the family of continuous process $\{Y_n(t)\}_{n=1}^{\infty}$ defined by

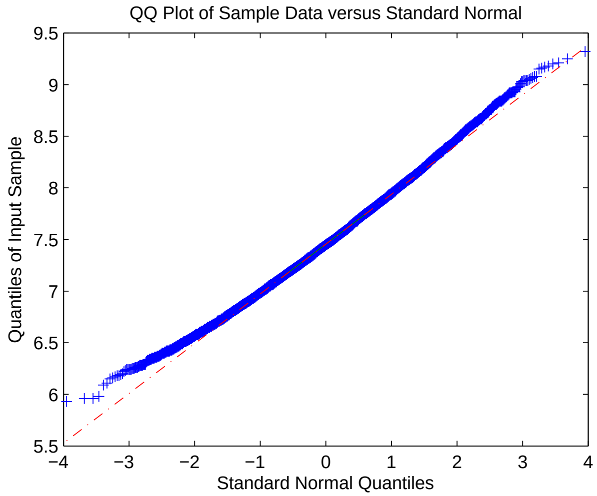
$$Y_n(t,\omega) = \frac{X(nt,\omega) - nt\tilde{c}}{\kappa\sqrt{n}}, \quad t \in [0,1],$$

converges weakly (as $n \to \infty$) to a standard Brownian motion on $[0,1]$, in the sense of weak convergence of measures on $C([0,1])$ with the topology of uniform convergence.

N. (2009)

Numerical observation of Gaussian fluctuations in interface position:





Left: Histogram for the random variable $X(t, \omega)$, 13,000 samples.

Right: Quantile-quantile plot vs. normal distribution.

Bounds on the variance κ^2

One can construct random media for which $\kappa^2 > 0$. Under the scaling

$$f(x,u) \to f(\frac{x}{L},u), \quad L > 0$$

the variance is bounded by

$$C_1 L \leq \kappa^2(L) \leq C_2 L$$

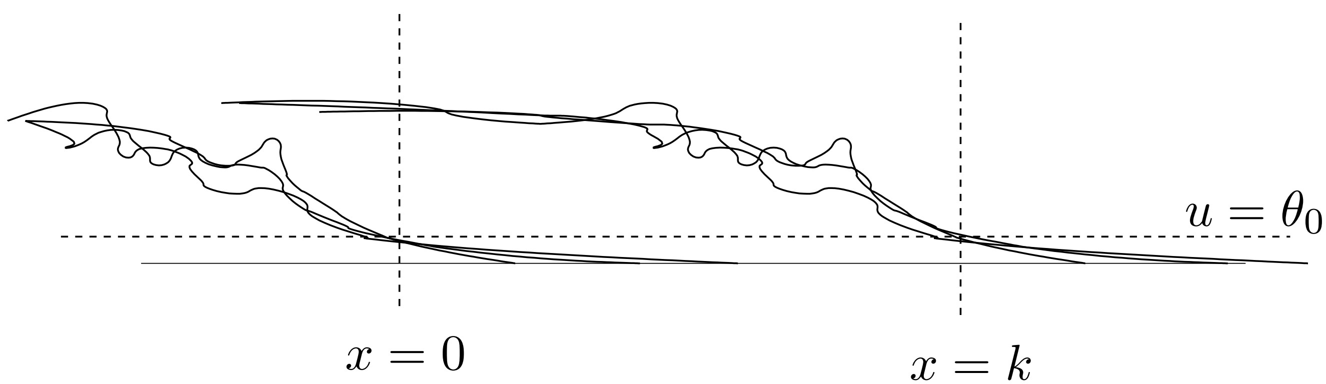
for L sufficiently large, while $0 < C_3 < \tilde{c}(L) \leq C_4$.

Statistical invariance of the generalized traveling wave:

We may normalize $\tilde{X}(0,\omega) = 0$, so that

$$\tilde{u}(T_k(\omega), x+k, \omega) = \tilde{u}(0, x, \pi_k\omega), \qquad \forall\, k \in \mathbb{R}$$

$T_k = T_k(\omega)$ is the hitting time to $x = k$: $\tilde{X}(T_k, \omega) = k$.
Increments $\Delta T_k = T_{k+1} - T_k$ are stationary with respect to k.



In this sense, the profile is **statistically invariant** with respect to reference point $x = k$.

How do we obtain a CLT for $X(t,\omega)$?

Consider the hitting times

$$T_k(\omega) = \inf\{t \geq 0 | \tilde{X}(t,\omega) = k\}.$$

Then

$$\frac{T_n - \tilde{\tau} n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(\Delta T_k - \mathbb{E}[\Delta T_k]\right),$$

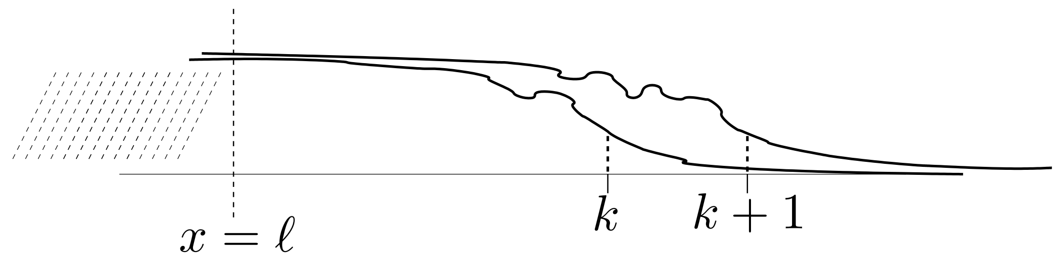
where $\Delta T_k = T_{k+1} - T_k$.

For the traveling wave, the increments ΔT_k are identically distributed, but **not** independent.

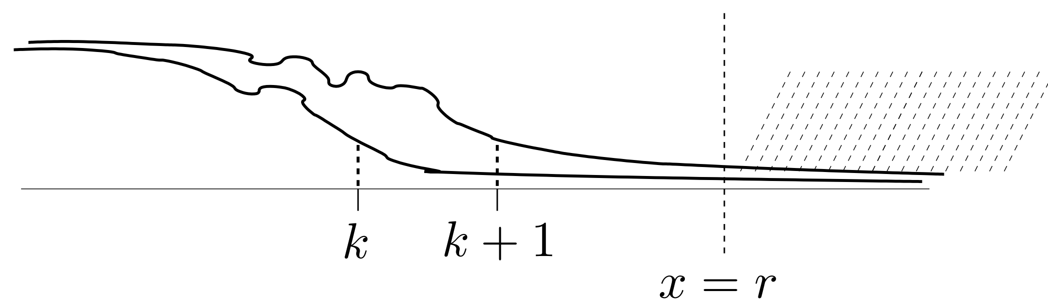
Stability of the wave under perturbations of the environment enables us to show that

$$\Delta T_k = T_{k+1} - T_k$$

does not depend strongly on the **distant past**:



or **distant future**:



ΔT_k depends primarily on the local environment near $x = k$.

Many interesting problems to consider:

- Propagation in multiple dimensions
- Systems of equations, propagating pulses

Thank you for your attention!

References:

Traveling waves: Nolen, Ryzhik, AIHP-Analyse Nonlineaire, **26**, 2009.

Stability: Mellet, Nolen, Roquejoffre, Ryzhik, Comm. PDE, **34**, 2009.

Central Limit Theorem: Nolen, preprint (2009).

The mixing condition

Define the family of σ-algebras

$$\mathcal{F}_k^- = \sigma\left(g(x,\omega)|\, x \le k\right)$$
$$\mathcal{F}_k^+ = \sigma\left(g(x,\omega)|\, x \ge k\right)$$

$$\mathcal{F}_k^- \subset \mathcal{F}_{k+1}^- \subset \mathcal{F}, \qquad \text{and} \qquad \mathcal{F} \supset \mathcal{F}_k^+ \supset \mathcal{F}_{k+1}^+$$

We say the environment is ϕ-mixing if for all $j \ge k$ and any $\xi \in L^2(\Omega, \mathcal{F}_k^-, \mathbb{P})$ and $\eta \in L^2(\Omega, \mathcal{F}_j^+, \mathbb{P})$,

$$|\mathbb{E}\left[\xi\eta\right] - \mathbb{E}[\xi]\mathbb{E}[\eta]| \le \sqrt{\phi(j-k)}\left(\mathbb{E}[\xi^2]\mathbb{E}[\eta^2]\right)^{1/2}$$

for $\phi(n) : \mathbb{Z}^+ \to [0,\infty)$ is nonincreasing. If $\sum_{n\ge1}\sqrt{\phi(n)} < \infty$, then the invariance principle holds.