Recent Advances in PDEs and Fluids

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A Quantitative Theory of Stochastic Homogenization

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References

Antoine Gloria, Stefan Neukamm, and Felix Otto. Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics - long version.

Antoine Gloria, Stefan Neukamm, and Felix Otto. An optimal quantitative two-scale expansion in stochastic homogenization of discrete elliptic equations.

Daniel Marahrens and Felix Otto, Annealed estimates on the Greens function.

[2] All available on http://www.mis.mpg.de/applan/members/felix-otto/publications/homogenization.html

1. Setting, motivation, and main result

Discrete calculus, Ensemble

Qualitative homogenization

Estimate of random homogenization error

Gradient, (negative) divergence, coefficients

Definition 1 For $u\colon \mathbb{Z}^d \to \mathbb{R}$ define $\nabla u\colon \mathbb{E}^d \to \mathbb{R}$ via

$$\nabla u(b) = u(x+e_i) - u(x) \quad \text{if } b = [x, x+e_i].$$

For $g\colon \mathbb{E}^d \to \mathbb{R}$ define $\nabla^* g\colon \mathbb{Z}^d \to \mathbb{R}$ via

$$\nabla^* g(x) = \sum_{i=1}^{d} (g(b_{i,-}) - g(b_{i,+})) \quad \text{if } b_{i,+} = [x, x+e_i],\ b_{i,-} = [x-e_i, x].$$

Definition 2

$$\Omega := \{a\colon \mathbb{E}^d \to [\lambda, 1]\}.$$

Ensemble of coefficients

Definition 3 Endow $\Omega = [\lambda, 1]^{\mathbb{E}^d}$ with product topology.
Consider probability measure $\langle\cdot\rangle$ on Ω.
Note: $\mathbb{Z}^d$ acts on Ω by shift:
For "shift vector" $z \in \mathbb{Z}^d$, $a(\cdot + z)$ denotes shifted $a \in \Omega$.

We call $\langle\cdot\rangle$ stationary if
$\forall\ z \in \mathbb{Z}^d$ a and $a(\cdot + z)$ have same distribution.

We call $\langle\cdot\rangle$ ergodic if
$\forall$ measurable $\zeta\colon \Omega \to \mathbb{R}$ it holds
ζ is shift invariant (i. e. $\forall\ z \in \mathbb{Z}^d \quad \zeta(a) = \zeta(a(\cdot + z))$)
$\Longrightarrow \zeta = const.$

Qualitative homogenization

Theorem 1[Kozlov, Papanicolaou & Varadhan '79]
Let $\langle\cdot\rangle$ be stationary & ergodic.
Then $\exists$ symmetric $d \times d$ matrix λid $\leq a_{hom} \leq$ id with following property:
For $\hat{f}(\hat{x})$, $\hat{x} \in \mathbb{R}^d$, bounded & compact support, and $a \in \Omega$ consider decaying solution $u_\epsilon(a;x)$, $x \in \mathbb{Z}^d$, of

$$\nabla^* a \nabla u_\epsilon = \epsilon^2 \hat{f}(\epsilon x).$$

Then

$$\lim_{\epsilon \downarrow 0} \left(\epsilon^d \sum_{x \in \mathbb{Z}^d} |u_\epsilon(a;x) - \hat{u}(\epsilon x)|^2 \right)^{\frac{1}{2}} = 0 \quad \text{for } \langle\cdot\rangle\text{-a. e. } a \in \Omega,$$

where $\hat{u}(\hat{x})$, $\hat{x} \in \mathbb{R}^d$, is decaying solution of

$$-\hat{\nabla} a_{hom} \hat{\nabla} \hat{u} = \hat{f}.$$

Estimate of random homogenization error

Theorem 2 [Marahrens & O. '12]
Let $\langle\cdot\rangle$ be stationary & satisfy LSI with constant $\rho > 0$.
For $\hat f(\hat x)$, $\hat x \in \mathbb{R}^d$, bounded & compact support, and $a \in \Omega$ consider decaying solution $u_\epsilon(a;x) \overset{\text{short}}{=} u_\epsilon(x)$, $x \in \mathbb{Z}^d$, of

$$\nabla^* a \nabla u_\epsilon = \epsilon^2 \hat f(\epsilon x).$$

i) Let $d > 2$. $\forall\ 2 \le p < \infty$, $r < \infty$:

$$\left\langle \left(\epsilon^d \sum_x |u_\epsilon(x) - \langle u_\epsilon(x)\rangle|^p \right)^{\frac{r}{p}} \right\rangle^{\frac{1}{r}} \le C(d,\lambda,\rho,r,p,\hat f)\epsilon.$$

ii) Let $d \ge 2$. $\forall\ \hat g(\hat x)$, $\hat x \in \mathbb{R}^d$, bounded & compact support, $r < \infty$:

$$\left\langle \left| \epsilon^d \sum_x (u_\epsilon(x) - \langle u_\epsilon(x)\rangle) \hat g(\epsilon x) \right|^r \right\rangle^{\frac{1}{r}} \le C(d,\lambda,\rho,r,\hat g,\hat f)\epsilon^{\frac{d}{2}}.$$

Interpolating between strong and weak error

Theorem 2'. For $u(a;x)$, $f(x)$ with $\nabla^* a \nabla u(a;\cdot) = f$, $g(x)$, $1 \le r < \infty$:

$$\Big\langle \Big| \sum_x (u - \langle u \rangle) g \Big|^r \Big\rangle^{\frac{1}{r}} \le C(d,\lambda,\rho,r,p) \Big(\sum_x |g|^p \Big)^{\frac{1}{p}} \Big(\sum_x |f|^q \Big)^{\frac{1}{q}},$$

where $1 < p, q \le d$ with $\frac{1}{p} + \frac{1}{q} = \frac{2}{d} + \frac{1}{2}$.

Specify to $d > 2$, $p = \frac{2d}{d+2}$ and thus $q = d$ and
$f(x) = \epsilon^2 \hat{f}(\epsilon x)$, $g(x) = \delta^d \hat{g}(\delta x)$

$$\begin{aligned} &\Big\langle \Big| \delta^d \sum_x (u_\epsilon(x) - \langle u_\epsilon(x) \rangle) \hat{g}(\delta x) \Big|^r \Big\rangle^{\frac{1}{r}} \\ &\le \; C(d,\lambda,\rho,r,p) \epsilon \delta^{\frac{d}{2}-1} \Big(\delta^d \sum_x |\hat{g}(\delta x)|^p \Big)^{\frac{1}{p}} \Big(\epsilon^d \sum_x |\hat{f}(\epsilon x)|^q \Big)^{\frac{1}{q}} \\ &\approx \; C(d,\lambda,\rho,r,p) \epsilon \delta^{\frac{d}{2}-1} \Big(\int_{\mathbb{R}^d} |\hat{g}(\hat{x})|^p d\hat{x} \Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}^d} |\hat{f}(\hat{x})|^q d\hat{x} \Big)^{\frac{1}{q}}. \end{aligned}$$

∞

Scaling is natural

$$\Big\langle\Big|\delta^d\sum_x(u_\epsilon(x)-\langle u_\epsilon(x)\rangle)\widehat{g}(\delta x)\Big|^r\Big\rangle^{\frac{1}{r}}$$
$$\leq\ C(d,\lambda,\rho,r,p)\epsilon\delta^{\frac{d}{2}-1}\Big(\int_{\mathbb{R}^d}|\widehat{g}(\widehat{x})|^p d\widehat{x}\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^d}|\widehat{f}(\widehat{x})|^q d\widehat{x}\Big)^{\frac{1}{q}}.$$

Consistent with (unproven but expected) covariance estimate

$$\Big|\Big\langle(u_\epsilon(x)-\langle u_\epsilon(x)\rangle)(u_\epsilon(x')-\langle u_\epsilon(x')\rangle)\Big\rangle\Big|\lesssim\epsilon^2(|x-x'|+1)^{2-d}.$$

Consistent with (unproven but expected) expansion

$$u_\epsilon(\tfrac{\widehat{x}}{\epsilon})\ \approx\ \widehat{u}(\widehat{x})+\epsilon\sum_{k=1}^d\phi_k(\tfrac{\widehat{x}}{\epsilon})\widehat{\partial}_k\widehat{u}(\widehat{x}),$$
$$|\langle\phi_k(x)\phi_k(x')\rangle|\ \lesssim\ (|x-x'|+1)^{2-d}.$$

Unpublished work by Naddaf & Spencer

Theorem 3 (Naddaf & Spencer).
$\langle\cdot\rangle$ stationary, Spectral Gap. Suppose $1-\lambda \le c(d)$.
Then for $u(a;x)$, $f(x)$ with $\nabla^* a \nabla u(a;\cdot) = f$:

$$\left\langle \left| \sum_x (u - \langle u\rangle) f \right|^2 \right\rangle^{\frac{1}{2}} \le C(d,\rho) \Big(\sum_x |f|^q \Big)^{\frac{2}{q}} \quad \text{where } q = \frac{4d}{d+4}.$$

Spectral Gap: l. h. s. $\le \frac{1}{\rho} \langle \sum_b \Big(\frac{\partial}{\partial a(b)} \sum_x uf \Big)^2 \rangle$

Formula: $\displaystyle \frac{\partial}{\partial a(b)} \sum_x uf = -(\nabla u(b))^2$

Meyer's estimate: $\sum_b (\nabla u(b))^4 \le C(d) \Big(\sum_x |f|^q \Big)^{\frac{4}{q}}$

Our contribution: Remove the condition $1-\lambda \le c(d)$.

2. The Logarithmic Sobolev Inequality

Why is it a natural assumption?

How do we use it?

Logarithmic Sobolev Inequality and Spectral Gap

Definition 4 $\langle\cdot\rangle$ satisfies LSI with constant $\rho > 0$:

$$\forall \text{ measurable } \zeta(a) > 0 \quad \langle \zeta \ln \frac{\zeta}{\langle\zeta\rangle} \rangle \;\le\; \frac{1}{2\rho}\langle \frac{1}{\zeta}|\partial\zeta|^2_{\ell^2}\rangle.$$

Here $|\partial\zeta|^2_{\ell^2} := \sum_b (\frac{\partial\zeta}{\partial a(b)})^2$.

Lemma 1 Let $\langle\cdot\rangle$ satisfies LSI with constant $\rho > 0$. Then $\langle\cdot\rangle$ has a Spectral Gap (SG) with constant ρ:

$$\forall \text{ measurable } \zeta(a) \quad \rho\langle(\zeta - \langle\zeta\rangle)^2\rangle \;\le\; \langle|\partial\zeta|^2_{\ell^2}\rangle.$$

Spectral Gap encodes Central Limit Theorem scaling

Spectral Gap with constant $\rho > 0$ means

$$\forall\, \zeta(a) \quad \rho\langle(\zeta - \langle\zeta\rangle)^2\rangle \;\le\; \langle|\partial\zeta|_{\ell^2}^2\rangle.$$

Remark 1 Let $\langle\cdot\rangle$ be stationary and satisfy SG with constant $\rho > 0$. Let $S \subset \mathbb{E}^d$ be finite; consider

$$\zeta(a) = |S|^{-\frac{1}{2}} \sum_{b\in S} (a(b) - \langle a\rangle).$$

Then

$$\langle\zeta^2\rangle \le \frac{1}{\rho}.$$

Tensorization principle

Lemma 2[Gross '75] Let $\langle\cdot\rangle_0$ be probability measure on $[\lambda, 1]$ that satisfies LSI with constant ρ, i. e,

$$\forall\, \zeta(a) > 0 \quad \langle \zeta \ln \frac{\zeta}{\langle\zeta\rangle_0} \rangle_0 \le \frac{1}{2\rho} \langle \frac{1}{\zeta} | \frac{d\zeta}{da} |^2 \rangle_0 .$$

Let $\langle\cdot\rangle$ denote the corresponding product measure on Ω. Then $\langle\cdot\rangle$ satisfies LSI with constant ρ.

Remark 2 The uniform distribution $\langle\cdot\rangle_0$ on $[\lambda, 1]$ satisfies LSI with constant $\rho = \frac{1}{2(1-\lambda)^2}$.

Ensemble with integrable correlations ok

LSI with constant $\rho > 0$ means

$$\forall\, \zeta(a) > 0 \quad \langle \zeta \ln \frac{\zeta}{\langle \zeta \rangle} \rangle \;\le\; \frac{1}{2\rho} \langle \frac{1}{\zeta} |\partial \zeta|^2_{\ell^2} \rangle .$$

Remark 3 Let $\langle \cdot \rangle$ satisfy LSI with constant ρ.
For $\phi \in \ell^1_x$ let $\langle \cdot \rangle'$ be the push forward of $\langle \cdot \rangle$ under $a \mapsto a' := \phi * a$, where $a'(b) = \sum_{x \in \mathbb{Z}^d} \phi(x) a(x+b)$.
Then $\langle \cdot \rangle$ satisfy LSI with constant $\frac{\rho}{|\phi|_{\ell^1}}$.

Our use of LSI: Reverse Jensen

LSI with constant $\rho > 0$ means

$$\forall\, \zeta(a) > 0 \quad \langle \zeta \ln \frac{\zeta}{\langle \zeta \rangle} \rangle \;\leq\; \frac{1}{2\rho} \langle \frac{1}{\zeta} |\partial \zeta|^2_{\ell^2} \rangle.$$

Lemma 3 Let $\langle \cdot \rangle$ satisfy LSI with constant ρ.
Then $\forall\ \zeta(a)$, $p < \infty$, $\delta > 0$:

$$\langle |\zeta|^{2p} \rangle^{\frac{1}{2p}} \leq C(\rho, p, \delta) \langle |\zeta| \rangle + \delta \langle |\partial \zeta|^{2p}_{\ell^2} \rangle^{\frac{1}{2p}}.$$

3. Quenched estimates on Green's function

optimal spatially weighted estimates
on first and second mixed derivative of Green's function

Green's function and its derivatives

Definition 5
Green's function $G(a;x,y) \overset{\text{short}}{=} G(x,y)$:

$$\nabla^* a \nabla G(\cdot,y) = \delta(\cdot - y).$$

First derivative $\nabla G(b,y)$, second mixed derivative $\nabla\nabla G(b,e)$

Why Green's function and what estimates do we need?

Goal $\langle|\sum(u-\langle u\rangle)g|^2\rangle^{\frac{1}{2}} \lesssim (\sum|g|^p)^{\frac{1}{p}}(\sum|f|^q)^{\frac{1}{q}}$

Spectral Gap l. h. s. $\le \frac{1}{\rho}\Big\langle \sum_b \Big(\sum_x \frac{\partial u(x)}{\partial a(b)} g(x)\Big)^2\Big\rangle$

Formula $\frac{\partial u(x)}{\partial a(b)} = -\nabla G(x,b)\nabla u(b) = -\sum_y \nabla G(x,b)\nabla G(b,y) f(y)$

For HLS need $\langle|\nabla G(b,y)|^4\rangle^{\frac{1}{4}} \lesssim (|b-y|+1)^{1-d}$

Not true $|\nabla G(b,y)| \le C(d,\lambda)(|b-y|+1)^{1-d}$

True $\Big(R^{-d}\sum_{R\le|b-y|\le 2R}|\nabla G(b,y)|^2\Big)^{\frac{1}{2}} \le C(d,\lambda)(R+1)^{1-d}$

Stationarity yields $\langle|\nabla G(b,y)|^2\rangle^{\frac{1}{2}} \le C(d,\lambda)(|b-y|+1)^{1-d}$

Overcoming Gap: Use reverse Hölder given by LSI

Have $\langle|\nabla G(b,y)|^2\rangle^{\frac{1}{2}} \lesssim (|b-y|+1)^{1-d}$.
Want $\langle|\nabla G(b,y)|^4\rangle^{\frac{1}{4}} \lesssim (|b-y|+1)^{1-d}$.

Reverse Hölder $\langle|\nabla G(b,y)|^4\rangle^{\frac{1}{4}} \lesssim \langle|\nabla G(b,y)|^2\rangle^{\frac{1}{2}} + \delta\langle|\partial\nabla G(b,y)|_{\ell^2}{}^4\rangle^{\frac{1}{4}}$

Formula $\frac{\partial}{\partial a(e)}\nabla G(b,y) = -\nabla\nabla G(b,e)\nabla G(e,y)$

Need $\sum_e((|b-e|+1)^{d-1}\nabla\nabla G(b,e))^4 \leq C(d,\lambda)$

Have $\sum_e((|b-e|+1)^{\alpha}\nabla\nabla G(b,e))^2 \leq C(d,\lambda)$ for some $\alpha > \frac{d}{2}-1$

Overcoming gap: Work with $\nabla\nabla G$ instead on ∇G

Would need $\sum_e ((|b-e|+1)^{d-1}\nabla\nabla G(b,e))^4 \leq C(d,\lambda)$.
Have only $\sum_e ((|b-e|+1)^{\alpha}\nabla\nabla G(b,e))^2 \leq C(d,\lambda)$
for some $\alpha > \frac{d}{2} - 1$

Way out: apply reverse Hölder to $\nabla\nabla G(b,b')$

Have Formula $\frac{\partial}{\partial a(e)}\nabla\nabla G(b,b') = -\nabla\nabla G(b,e)\nabla\nabla G(e,b')$

Need $\langle|\nabla\nabla G(b,e)|\rangle \leq C(d,\lambda)(|b-e|+1)^{-d}$

Get via stationarity from

$$\left(R^{-d}\sum_{b:8R\leq|b-y|\leq 16R} R^{-d}\sum_{e:|e-y|\leq R}|\nabla\nabla G(b,e)|^2\right)^{\frac{1}{2}} \leq C(d,\lambda)(R+1)^{-d}$$

Optimal spatially weighted estimate on first and second mixed derivative

Lemma 4[De Giorgi '57] $\forall\ a \in \Omega$, $y \in \mathbb{Z}^d$, radius R

$$\Big(R^{-d}\sum_{b:R\le|b-y|\le 2R} (\nabla G)^2(b,y)\Big)^{\frac{1}{2}} \ \le\ C(d,\lambda)R^{1-d},$$

$$\Big(R^{-d}\sum_{b:8R\le|b-y|\le 16R} R^{-d}\sum_{e:|e-y|\le R} (\nabla\nabla G)^2(b,e)\Big)^{\frac{1}{2}} \ \le\ C(d,\lambda)R^{-d},$$

where $|b-y|$ denotes the distance between the midpoint of edge b and site y.

Suboptimal only partially spatially averaged estimate on second mixed derivative of Green's function

Lemma 5 $\exists\, \alpha = \alpha(d,\lambda) > 0\ \forall\ a \in \Omega,\ e \in \mathbb{E}^d$

$$\sum_b (|b-e|+1)^{2\alpha} (\nabla\nabla G)^2(b,e) \ \le\ C(d,\lambda),$$

4. Annealed estimates on Green's function

optimal annealed estimates

on derivatives of Green's function

from low to high stochastic moments

estimate of random homogenization error

Elliptic regularity + Stationarity (Delmotte & Deuschel)

From pointwise in a but averaged in x

$$\Big(R^{-d}\sum_{R\le|b-y|\le 2R}|\nabla G(b,y)|^2\Big)^{\frac{1}{2}} \le C(d,\lambda)(R+1)^{1-d}$$

$$\Big(R^{-d}\sum_{b:8R\le|b-y|\le 16R}R^{-d}\sum_{e:|e-y|\le R}|\nabla\nabla G(b,e)|^2\Big)^{\frac{1}{2}} \le C(d,\lambda)(R+1)^{-d}$$

to pointwise in x but averaged in a

$$\langle|\nabla G(b,y)|^2\rangle^{\frac{1}{2}} \le C(d,\lambda)(|b-y|+1)^{1-d}$$

$$\langle|\nabla\nabla G(b,e)|\rangle \le C(d,\lambda)(|b-e|+1)^{-d}$$

Elliptic regularity + Good ergodicity via LSI (M.&O.)

Ingredients (reverse Hölder, formula, elliptic regularity)

$\langle|\zeta|^{2p}\rangle^{\frac{1}{2p}} \leq C(\rho,p,\delta)\langle|\zeta|\rangle + \delta\langle|\partial\zeta|_{\ell^2}^{2p}\rangle^{\frac{1}{2p}}$ for all $p<\infty, \delta>0$

$\frac{\partial}{\partial a(e)}\nabla\nabla G(b,b') = -\nabla\nabla G(b,e)\nabla\nabla G(e,b')$

$\sum_e((|b-e|+1)^{\alpha_0}\nabla\nabla G(b,e))^2 \leq C(d,\lambda)$ for some $\alpha_0(d,\lambda)>0$

Conclusion

$$\sup_{b,b'}(|b-b'|+1)^d\langle|\nabla\nabla G(b,b')|^{2p}\rangle^{\frac{1}{2p}}$$

$$\leq \; C(d,\lambda,\rho,p)\sup_{b,b'}(|b-b'|+1)^d\langle|\nabla\nabla G(b,b')|\rangle \quad \text{for all } \infty>p\geq p_0(d,\lambda).$$

Quick conclusion

$$\sup_{b,b'}(|b-b'|+1)^d\langle|\nabla\nabla G(b,b')|^{2p}\rangle^{\frac{1}{2p}} \lesssim \sup_{b,b'}(|b-b'|+1)^d\langle|\nabla\nabla G(b,b')|\rangle \lesssim 1$$

From there $\langle|\nabla G(b,y)|^4\rangle^{\frac{1}{4}} \lesssim |b-y|^{1-d}$

Spectral Gap $\langle(\zeta-\langle\zeta\rangle)^2\rangle^{\frac{1}{2}} \lesssim \langle|\partial\zeta|^2_{\ell^2}\rangle^{\frac{1}{2}}$

Formula $\frac{\partial u(x)}{\partial a(b)} = -\sum_y \nabla G(x,b)\nabla G(b,y)f(y)$

HLS $\left|x \mapsto \sum_y |x-y|^{1-d} f(y)\right|_{\ell^{\tilde{q}}} \lesssim |f|_{\ell^q}$ for $\frac{1}{\tilde{q}} + \frac{1}{d} = \frac{1}{q}$

Get $\langle|\sum(u-\langle u\rangle)g|^2\rangle^{\frac{1}{2}} \lesssim (\sum|g|^p)^{\frac{1}{p}}(\sum|f|^q)^{\frac{1}{q}}$ for $\frac{1}{p}+\frac{1}{q} = \frac{2}{d}+\frac{1}{2}$

For $f(x) = \epsilon^2\hat{f}(\epsilon x)$, $g(x) = \delta^d\hat{g}(\delta x)$ by rescaling

$\langle|\sum(u-\langle u\rangle)g|^2\rangle^{\frac{1}{2}} \lesssim \epsilon\delta^{\frac{d}{2}-1}$

A new twist on elliptic regularity theory

De Giorgi: $\exists\, \alpha_0(d,\lambda) > 0$

$\forall\, a \in \Omega,\ R < \infty,\ u\ a$-harmonic in B_{2R}

$$\sup_{x,x' \in B_R} \frac{|u(x)-u(x')|}{|x-x'|^{\alpha_0}} \leq C(d,\lambda) R^{-\alpha_0} \sup_{x \in B_{2R}} |u(x)|$$

M. & O.: $\forall\ \alpha < 1,\ R < \infty,\ a \in \Omega$

$\exists C(a;\alpha,R)$ such that

1) $\forall u\ a$-harmonic in B_{2R}

$$\sup_{x,x' \in B_R} \frac{|u(x)-u(x')|}{|x-x'|^{\alpha}} \leq C(\alpha,R) R^{-\alpha} \sup_{x \in B_{2R}} |u(x)|$$

2) $\forall p < \infty \quad \langle C^p(\alpha,R) \rangle \leq C(d,\lambda,\rho,p)$

Optimal low moment annealed estimate, estimate of vertical derivative

Lemma 6[Delmotte & Deuschel '05] Let $\langle\cdot\rangle$ be stationary. Then $\forall\ y \in \mathbb{Z}^d,\ b, e \in \mathbb{E}^d$:

$$\begin{aligned} \langle|\nabla G(b,y)|^2\rangle^{\frac{1}{2}} &\le C(d,\lambda)(|b-y|+1)^{1-d}, \\ \langle|\nabla\nabla G(b,e)|\rangle &\le C(d,\lambda)(|b-e|+1)^{-d}. \end{aligned}$$

Lemma 7 $\exists\ p_0 = p_0(d,\lambda) < \infty\ \forall\ p \ge p_0$

$$\begin{aligned} &\sup_{b,e}(|b-e|+1)^d \langle|\partial\nabla\nabla G(b,e)|_{\ell^2}{}^{2p}\rangle^{\frac{1}{2p}} \\ &\le\ C(d,\lambda,p)\sup_{b,e}(|b-e|+1)^d\langle|\nabla\nabla G(b,e)|^{2p}\rangle^{\frac{1}{2p}}. \end{aligned}$$

Optimal high moment annealed estimates

Proposition Let $\langle\cdot\rangle$ be stationary and satisfy LSI with constant $\rho > 0$. Then $\forall\ p < \infty,\ y \in \mathbb{Z}^d,\ b, e \in \mathbb{E}^d$:

$$\begin{aligned}\langle|\nabla G(b,y)|^{2p}\rangle^{\frac{1}{2p}} &\leq C(d,\lambda,\rho,p)(|b-y|+1)^{1-d},\\ \langle|\nabla\nabla G(b,e)|^{2p}\rangle^{\frac{1}{2p}} &\leq C(d,\lambda,\rho,p)(|b-e|+1)^{-d}.\end{aligned}$$

Estimate of random homogenization error

Theorem 2' Let $\langle\cdot\rangle$ be stationary and satisfy LSI with constant $\rho > 0$. $\forall\ f(x)$, $x \in \mathbb{Z}^d$, compactly supported, $a \in \Omega$ consider decaying solution $u(a;x) \overset{\text{short}}{=} u(x)$ of

$$\nabla^* a \nabla u = f.$$

i) Let $d > 2$. Then $\forall\ 2 \le p < \infty$, $r < \infty$:

$$\Big\langle \Big(\sum_x |u - \langle u\rangle|^p \Big)^{\frac{r}{p}} \Big\rangle^{\frac{1}{r}} \le C(d,\lambda,\rho,r,p) \Big(\sum_x |f|^q \Big)^{\frac{1}{q}},$$

where $\frac{1}{q} = \frac{1}{p} + \frac{1}{d}$.
ii) Let $d \ge 2$. Then $\forall\ g(x)$, $x \in \mathbb{Z}^d$, $r < \infty$:

$$\Big\langle \Big(\sum_x (u - \langle u\rangle) g \Big)^{r} \Big\rangle^{\frac{1}{r}} \le C(d,\lambda,\rho,r,p) \Big(\sum_x |g|^p \Big)^{\frac{1}{p}} \Big(\sum_x |f|^q \Big)^{\frac{1}{q}},$$

where $1 < p, q \le d$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{2}{d} + \frac{1}{2}$.