

The search for 3D Euler singularities: The interplay between analysis and numerics

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Abstract

Whether the 3D incompressible Euler or Navier-Stokes equations can develop a finite time singularity from smooth initial data with finite energy is one of the most challenging questions in applied mathematics and fluid dynamics. We review some recent theoretical and computational studies of the 3D Euler equations which show that there is a subtle dynamic depletion of nonlinear vortex stretching due to local geometric regularity of vortex filaments. We use both an Eulerian and a Lagrangian approach. Our study suggests that the convection term could have a nonlinear stabilizing effect for certain flow geometry. This is demonstrated through two reduced models of the 3D incompressible Navier-Stokes equations. The first model is a new exact 1D model for the axisymmetric Navier-Stokes equation along the symmetry axis. We show that local flattening of the vortex structure and the effect of convection could lead to dynamic depletion of the vortex stretching term. In the second model, we derive a 3D model of the axisymmetric Navier-Stokes equation by removing the convection term from the reformulated Navier-Stokes equations. We show both numerically and analytically that the solution of this 3D model develops a finite time singularity from smooth initial data with finite energy. Finally we present a new class of solutions to the 3D Euler equations which could lead to a strong nonlinear alignment in the vortex stretching term and have the potential to develop a finite time singularity. In the first example, we study a class of solutions whose velocity fields produce a “tornado” like structure. Near the center of the “tornado”, the angular velocity develops a very sharp gradient and becomes almost discontinuous. As a result, the solution approaches to a vortex sheet like structure as time evolves. Near the center of the tornado, there is a strong nonlinear alignment in the vortex stretching term, and the solution becomes increasingly singular with a scaling consistent with a finite time blow-up. However, as the thickness of the vortex sheet becomes smaller and smaller, the Kelvin-Helmholtz instability of the fluid flow eventually kicks in and destroys such nonlinear alignment, leading to the subsequent development of turbulent flow. In the second example, we consider the 3D Euler equation with initial boundary data of certain symmetry. By employing a highly adaptive mesh, we demonstrate that the solution of the 3D Euler equations seems to develop a stable and focusing finite time singularity. The scaling of the potential singular solution is not self-similar in the traditional sense. Instead the solution develops a highly anisotropic scaling in the singularity region. Such anisotropic scaling is not supported by the Navier-Stokes equations, thus is unstable under viscous perturbation.