Adaptive evolution : a population approach

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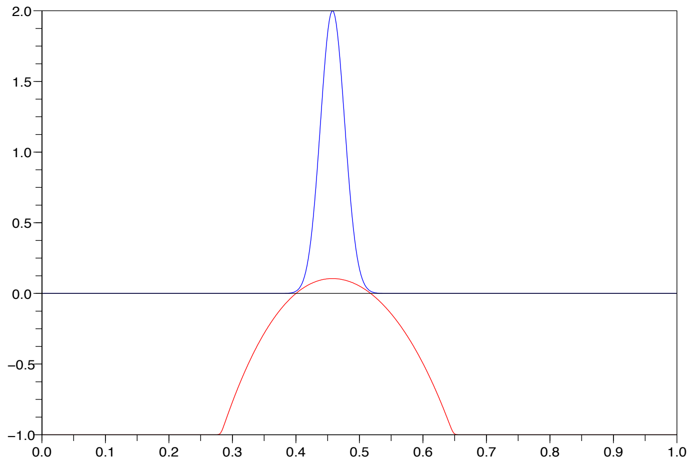
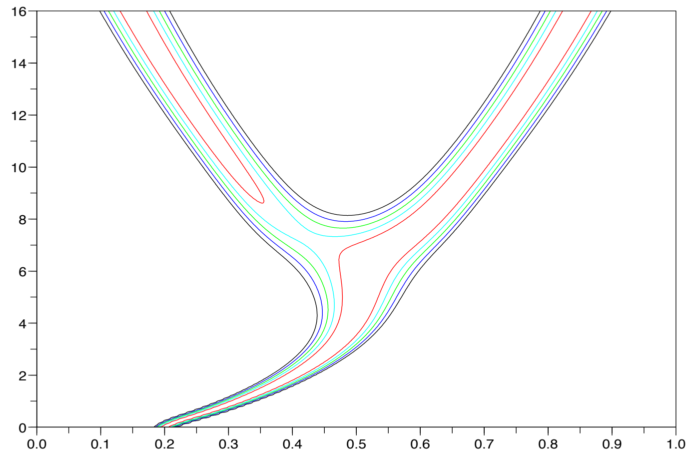




Asymptotic method

We have considered the asymptotic probem

$$\begin{cases} \varepsilon \frac{\partial}{\partial t} n_\varepsilon(x,t) - \varepsilon^2 \Delta n_\varepsilon = n_\varepsilon(x,t) R\Big(x, \varrho_\varepsilon(t)\Big), \\ \\ \varrho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(x,t) dx. \end{cases}$$

In the limit one can expect

$$0 = n(x,t) R\Big(x, \varrho(t)\Big),$$

$$n(x,t) = \varrho \delta_{\Gamma(t)}, \qquad \qquad \Gamma(t) \subset \Big\{ R(\cdot, \varrho(t)) = 0 \Big\}.$$

Asymptotic method

Theorem (Weak form) In $\mathbb{R}^d$, set

$$n_\varepsilon(x,t) = e^{\varphi_\varepsilon(x,t)/\varepsilon}.$$

- After extraction, $\varphi_\varepsilon \underset{\varepsilon_k \to 0}{\longrightarrow} \varphi$ (locally uniformly), $\varrho_\varepsilon(t) \underset{\varepsilon_k \to 0}{\longrightarrow} \bar{\varrho}(t)$

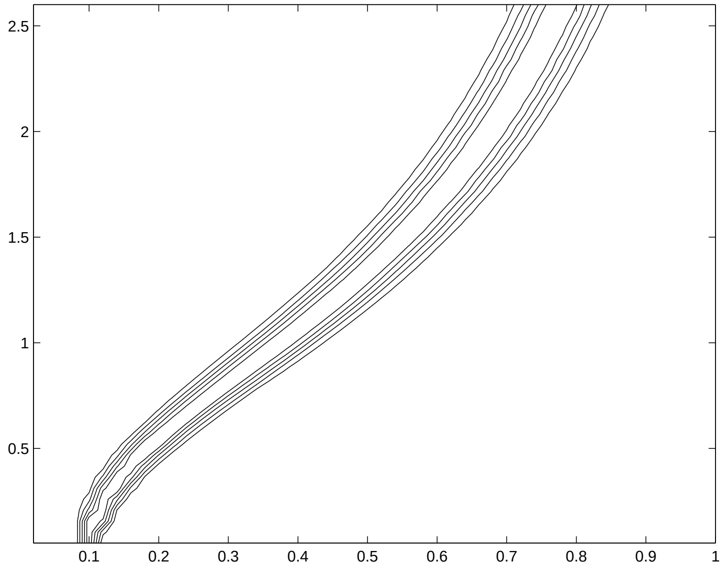
$$\begin{cases} \frac{\partial}{\partial t}\varphi(x,t) = R\big(x, \bar{\varrho}(t)\big) + |\nabla\varphi(x,t)\big)|^2 \\ \max_x \varphi(x,t) = 0 \qquad \Big(= \varphi(t, \bar{x}(t)) \Big). \end{cases}$$

- And $n_\varepsilon(x,t) \underset{\varepsilon_k \to 0}{\rightharpoonup} n(x,t)$ weakly in measures,

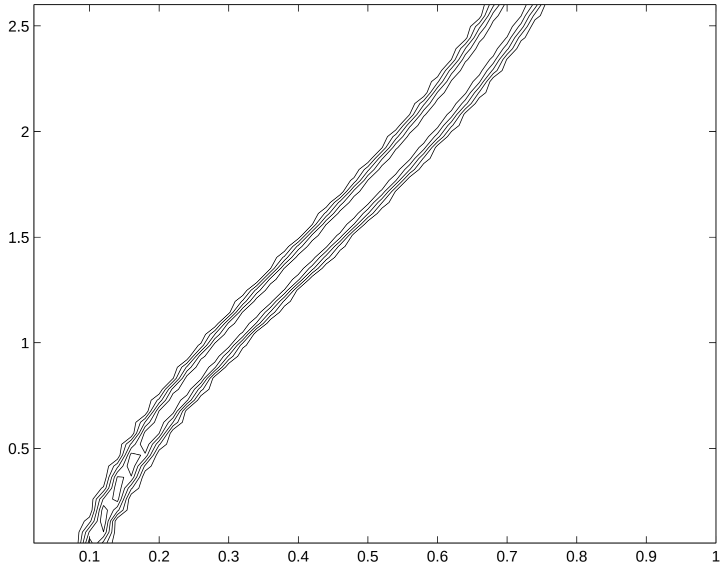
$$\text{supp}\big(n(t)\big) \subset \{\varphi(t) = 0\}$$

Asymptotic method

Numerical tests : $b(x) = .5 + x(2 - x)$



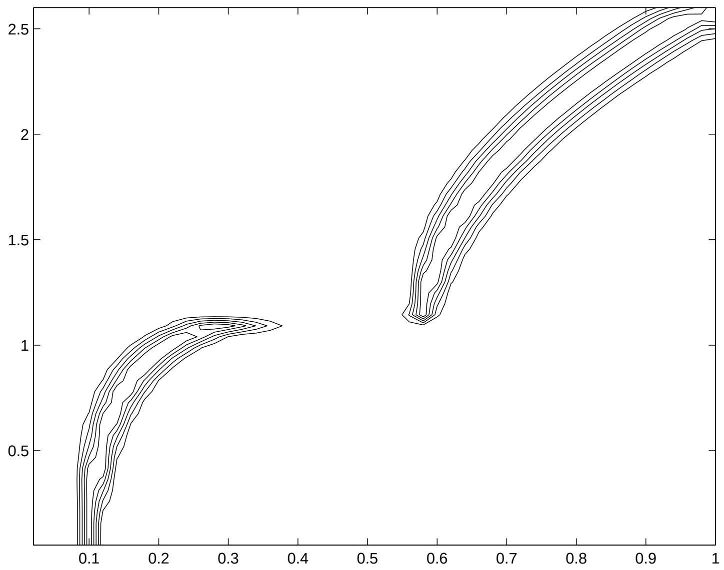
Direct simulation (1500 points)



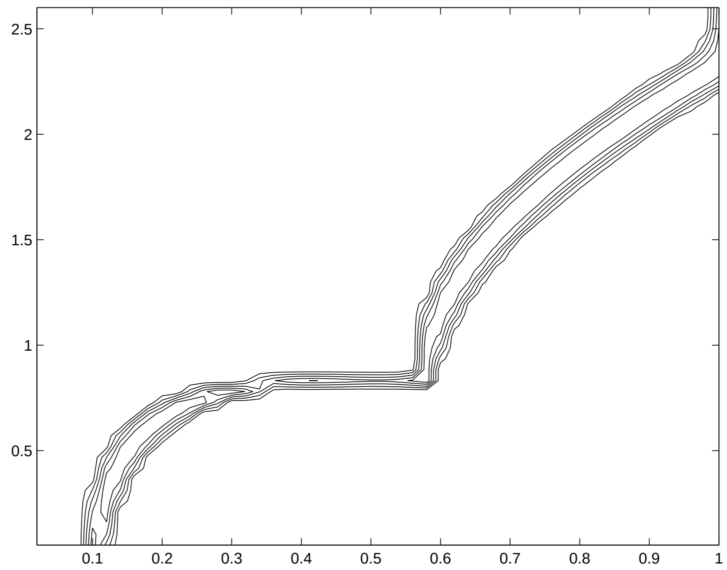
H.-J. solution (200 points)

Asymptotic method

Numerical tests : $min(.45 + x.^2, .55 + .4 * x)$



Direct simulation (1500 points)



H.-J. solution (200 points)

Asymptotic method

Question for this course :

$$\begin{cases} \varepsilon \frac{\partial}{\partial t} n_\varepsilon(x,t) - \varepsilon^2 \Delta n_\varepsilon = n_\varepsilon(x,t) R\Big(x, \varrho_\varepsilon(t)\Big), \\ \varrho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(x,t) dx. \end{cases}$$

In the limit one can expect

$$0 = n(x,t) R\Big(x, \varrho(t)\Big),$$

$$n(x,t) = \varrho \delta_{\Gamma(t)}, \qquad \Gamma(t) \subset \Big\{ R(\cdot, \varrho(t)) = 0 \Big\}.$$

Are all the points equivalent on $\Gamma(t)$?

Are these pointwise Dirac, of distributed on the hypersurfaces ?

Can one describe better their dynamics ?

OUTLINE OF THE LECTURE

DYNAMICS OF THE FITTEST TRAIT

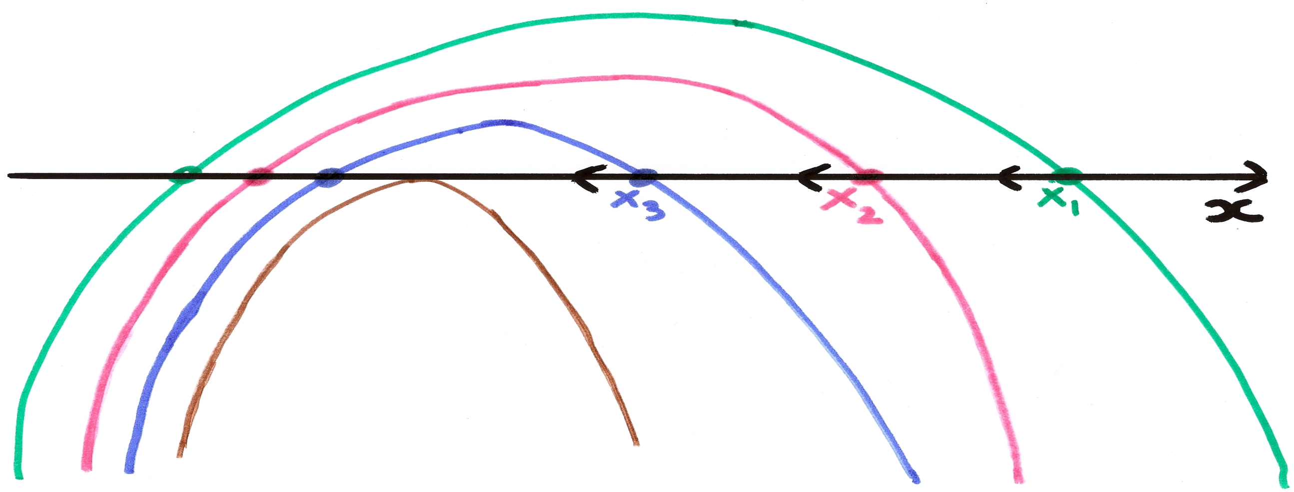
I. A simple case of canonical equation

II. Regularity for concave initial data

III. Canonical equation (general)

A simple case

Therefore in high dimension it makes sense to study the case when $R(x, \cdot)$ is CONCAVE



An not only the monotonic case in 1D

A simple case

$$\begin{cases} \frac{d}{dt} n(x,t) = n(x,t) R\Big(x, \varrho(t)\Big), \\ \varrho(t) = \int_{\mathbb{R}^d} n(x,t) dx. \end{cases}$$

- There are many steady states. For any $\bar{x}$

$$\bar{n}(x) = \bar{\varrho}\ \delta(x - \bar{x}).$$

choosing $\bar{\varrho}$ such that

$$R(\bar{x}, \bar{\varrho}) = 0.$$

A simple case

$$\begin{cases} \frac{d}{dt}n(x,t) = n(x,t)R\Big(x, \varrho(t)\Big), \\ \varrho(t) = \int_{\mathbb{R}^d} n(x,t)dx. \end{cases}$$

- There are many steady states. For any $\bar{x}$

$$\bar{n}(x) = \bar{\varrho}\ \delta(x - \bar{x}), \qquad R(\bar{x}, \bar{\varrho}) = 0.$$

- They are stable by perturbation of the weight $\bar{\varrho}$ (strong topology)

$$\frac{d}{dt}\varrho(t) = \varrho(t)R(\bar{x}, \varrho(t)).$$

A simple case

$$\begin{cases} \frac{d}{dt}n(x,t) = n(x,t)R\Big(x, \varrho(t)\Big), \\ \varrho(t) = \int_{\mathbb{R}^d} n(x,t)dx. \end{cases}$$

• There are many steady states. For any $\bar{x}$

$$\bar{n}(x) = \bar{\varrho}\ \delta(x - \bar{x}), \qquad R(\bar{x}, \bar{\varrho}) = 0.$$

• They are stable by perturbation of the weight $\bar{\varrho}$ (strong topology)

$$\frac{d}{dt}\varrho(t) = \varrho(t)R(\bar{x}, \varrho(t)).$$

• But they are unstable by approximation in measures (weak topology)... a direct way to see this

A simple case

Replace $\bar{n}^0(x) = \bar{\varrho}^0 \ \delta(x - \bar{x}^0)$ by a concentrated gaussian

$$n_\varepsilon^0(x) = e^{\varphi_\varepsilon^0(x)/\varepsilon} \approx \bar{\varrho}^0 \ \delta(x - \bar{x}^0), \qquad \max \varphi_\varepsilon^0(x) = \varphi_\varepsilon^0(\bar{x}^0) \approx 0$$

We expect

- fast dynamic on $\bar{\varrho}(t)$
- a slow dynamics on $n_\varepsilon(x,t)$

Therefore we rescale as

$$\begin{cases} \varepsilon \frac{d}{dt} n_\varepsilon(x,t) = n_\varepsilon(x,t) R\Big(x, \varrho_\varepsilon(t)\Big), \\ \varrho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(x,t) dx. \end{cases}$$

A simple case

$$\begin{cases} \varepsilon \frac{d}{dt} n_\varepsilon(x,t) = n_\varepsilon(x,t) R\Big(x, \varrho_\varepsilon(t)\Big), \\ \varrho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(x,t) dx. \end{cases}$$

Then, set

$$n_\varepsilon(x,t) = e^{\varphi_\varepsilon(x,t)/\varepsilon}$$

$$\frac{d}{dt}\varphi_\varepsilon(x,t) = R\Big(x, \varrho_\varepsilon(t)\Big), \qquad \max_{x\in\mathbb{R}} \varphi_\varepsilon(x,t) = o(1).$$

Since φ_ε is obviously smooth. In the limit

$$\frac{d}{dt}\varphi(x,t) = R\Big(x, \bar{\varrho}(t)\Big), \qquad \max_{x\in\mathbb{R}} \varphi(x,t) = 0.$$

A simple case

$$\frac{\partial}{\partial t}\varphi(x,t) = R\Big(x, \varrho(t)\Big), \qquad \max_{x\in\mathbb{R}} \varphi(x,t) = 0.$$

Assume

$$\varphi^0(x), \qquad R(x,\cdot) \qquad \text{are CONCAVE and smooth}$$

Then $\varphi_\varepsilon(x,t)$, $\varphi(x,t)$ are also concave and smooth in x.

Can we go further ? Is $\varrho_\varepsilon(t)$ smooth ?

Define $\bar{x}_\varepsilon(t)$ as the maximum point of $\varphi_\varepsilon(t)$

$$\nabla\varphi_\varepsilon(\bar{x}_\varepsilon(t), t) = 0,$$

A simple case

Claim

$$\frac{d}{dt}\bar{x}_\varepsilon(t) = \Big(- D^2\varphi_\varepsilon(\bar{x}_\varepsilon(t), t)\Big)^{-1} \cdot \nabla R\Big(\bar{x}_\varepsilon(t), \bar{\varrho}_\varepsilon(t)\Big).$$

Indeed, differentiate in time $\nabla\varphi_\varepsilon(\bar{x}_\varepsilon(t), t) = 0$

$$\frac{d}{dt}\bar{x}_\varepsilon(t) \cdot D^2\varphi_\varepsilon(\bar{x}_\varepsilon(t), t) + \nabla\frac{\partial}{\partial t}\varphi_\varepsilon(\bar{x}_\varepsilon(t), t) = 0,$$

and using

$$\frac{\partial}{\partial t}\varphi_\varepsilon(x, t) = R\Big(x, \varrho_\varepsilon(t)\Big),$$

we find

$$\frac{d}{dt}\bar{x}_\varepsilon(t) \cdot D^2\varphi_\varepsilon(\bar{x}_\varepsilon(t), t) = -\nabla R\Big(\bar{x}_\varepsilon(t), \bar{\varrho}_\varepsilon(t)\Big).$$

A simple case

Therefore $x_\varepsilon(t)$ is at least uniformly Lipschitz continuous,

$$x_\varepsilon(t) \underset{\varepsilon\to 0}{\longrightarrow} \bar{x}(t) \text{ uniformly (for some subsequence)}$$

Furthermore

$$\max_x \varphi(x,t) = 0 = \varphi(\bar{x}(t),t), \quad \nabla\varphi(\bar{x}(t),t) = 0, \quad \frac{\partial}{\partial t}\varphi(\bar{x}(t),t) = 0$$

$$\frac{\partial}{\partial t}\varphi(x,t) = R\Big(x, \bar{\varrho}(t)\Big),$$

and thus

$$R\Big(\bar{x}(t), \bar{\varrho}(t)\Big) = 0$$

and $\bar{\varrho}(t)$ is Lipschitz and

$$\frac{d}{dt}\bar{x}(t) = \Big(-D^2\varphi(\bar{x}(t),t)\Big)^{-1} \cdot \nabla R\Big(\bar{x}(t), \bar{\varrho}(t)\Big).$$

A simple case

Conclusions

1. $\bar{x}(t)$ moves toward increasing values of $R(x, \bar{\varrho}(t))$

2. Not a usual WKB expansion.

$$\varrho_\varepsilon(t) = \int n_\varepsilon(x,t)dx = \int e^{\frac{\varphi_\varepsilon(x,t)-\varphi_\varepsilon(t,\bar{x}_\varepsilon)}{\varepsilon}} dx \; e^{\frac{\varphi_\varepsilon(t,\bar{x}_\varepsilon)}{\varepsilon}}$$

$$\approx \int e^{-C\frac{|x-x_\varepsilon|^2}{\varepsilon}} dx \; e^{\frac{\varphi_\varepsilon(t,\bar{x}_\varepsilon)}{\varepsilon}} \qquad \approx C\sqrt{\varepsilon}^d \; e^{\frac{\varphi_\varepsilon(t,\bar{x}_\varepsilon)}{\varepsilon}}$$

Gaussian type : $\varphi_\varepsilon\Big(t, \bar{x}_\varepsilon(t)\Big) = O(\varepsilon \ln(\varepsilon))$.

Strong theory

$$\begin{cases} \varepsilon \frac{\partial}{\partial t} n_\varepsilon(x,t) - \varepsilon^2 \Delta n_\varepsilon = n_\varepsilon(x,t) R\Big(x, \varrho_\varepsilon(t)\Big), \\ \\ \varrho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(x,t) dx. \end{cases}$$

A smoothness regime exists with the assumptions

$$-K_1 I \le D^2 R\Big(x, \varrho\Big) \le -K_2 I \qquad \text{(idendity matrix)},$$

$$-L_1 I \le D^2 \varphi^0 \le -L_2 I, \quad L_1 \; L_2 large.$$

Theorem With these assumptions, the solution to the Hamilton-Jacobi equation

$$\frac{\partial}{\partial t} \varphi_\varepsilon(x,t) = R\Big(x, \varrho_\varepsilon(t)\Big) + |\nabla \varphi_\varepsilon(x,t)|^2 + \varepsilon \Delta \varphi_\varepsilon$$

satisfies $\qquad -L_1 I \le D^2 \varphi_\varepsilon(x,t) \le -L_2 I.$

Canonical equation

Proof (1D)

$$\frac{\partial}{\partial t}\varphi''(t,x) = R''\Big(x, \varrho(t)\Big) + 2|\varphi''(t,x)|^2 + 2\nabla\varphi.\nabla\varphi''$$

$$M(t) = \max_x \varphi''(t,x)$$

$$\frac{d}{dt}M(t) \leq -K_2 + 2M(t)^2$$

therefore $M(t) \leq -\sqrt{K2/2}$ (if initially true). Similarly

$$\frac{d}{dt}\min_x \varphi''(t,x) \geq -K_1 + 2[\min_x \varphi''(t,x)]^2$$

and this controls from below.

Strong theory

As in the simple case, one can build the maximum point $\bar{x}_\varepsilon(t)$ of $\varphi_\varepsilon(t)$ and

$$\nabla \varphi_\varepsilon(\bar{x}_\varepsilon(t), t) = 0,$$

and the equation

$$\frac{d}{dt}\bar{x}_\varepsilon(t) \cdot D^2\varphi_\varepsilon(\bar{x}_\varepsilon(t), t) + \nabla \frac{\partial}{\partial t}\varphi_\varepsilon(\bar{x}_\varepsilon(t), t) = 0,$$

and using the H.-J. equation

$$\frac{\partial}{\partial t}\nabla\varphi_\varepsilon(\bar{x}_\varepsilon(t), t) = \nabla R\Big(x, \varrho_\varepsilon(t)\Big) + 2D^2\varphi_\varepsilon(\bar{x}_\varepsilon(t), t).\nabla\varphi_\varepsilon(\bar{x}_\varepsilon(t), t) + O(\varepsilon).$$

We still find

$$\frac{d}{dt}\bar{x}_\varepsilon(t) \cdot D^2\varphi_\varepsilon(\bar{x}_\varepsilon(t), t) = -\nabla R\Big(\bar{x}_\varepsilon(t), \bar{\varrho}_\varepsilon(t)\Big).$$

Strong theory

Therefore, for some subsequence,

$$x_\varepsilon(t) \underset{\varepsilon\to 0}{\longrightarrow} \bar{x}(t) \qquad (\text{ uniformly})$$

$$\varphi_\varepsilon(x,t) \underset{\varepsilon\to 0}{\longrightarrow} \varphi(x,t) \in W_x^{3,\infty}$$

$$\begin{cases} \frac{\partial}{\partial t}\varphi(x,t) = R\Big(x, \varrho(t)\Big) + |\nabla\varphi(x,t)|^2 \\ \max_x \varphi(x,t) = 0 = \varphi(\bar{x}(t),t), \end{cases}$$

and

$$n_\varepsilon(x,t) \rightharpoonup \bar{\varrho}(t)\delta(x-\bar{x}(t)),$$

Canonical equation

Theorem (A. Lorz, S. Mirrahimi, BP) With the concavity and smoothness assumptions

(i) $n_\varepsilon(x,t) \rightharpoonup \bar{\varrho}(t)\delta(x-\bar{x}(t)),$

(ii) $\bar{x}(t),\ \bar{\varrho}(t)$ are 'smooth'

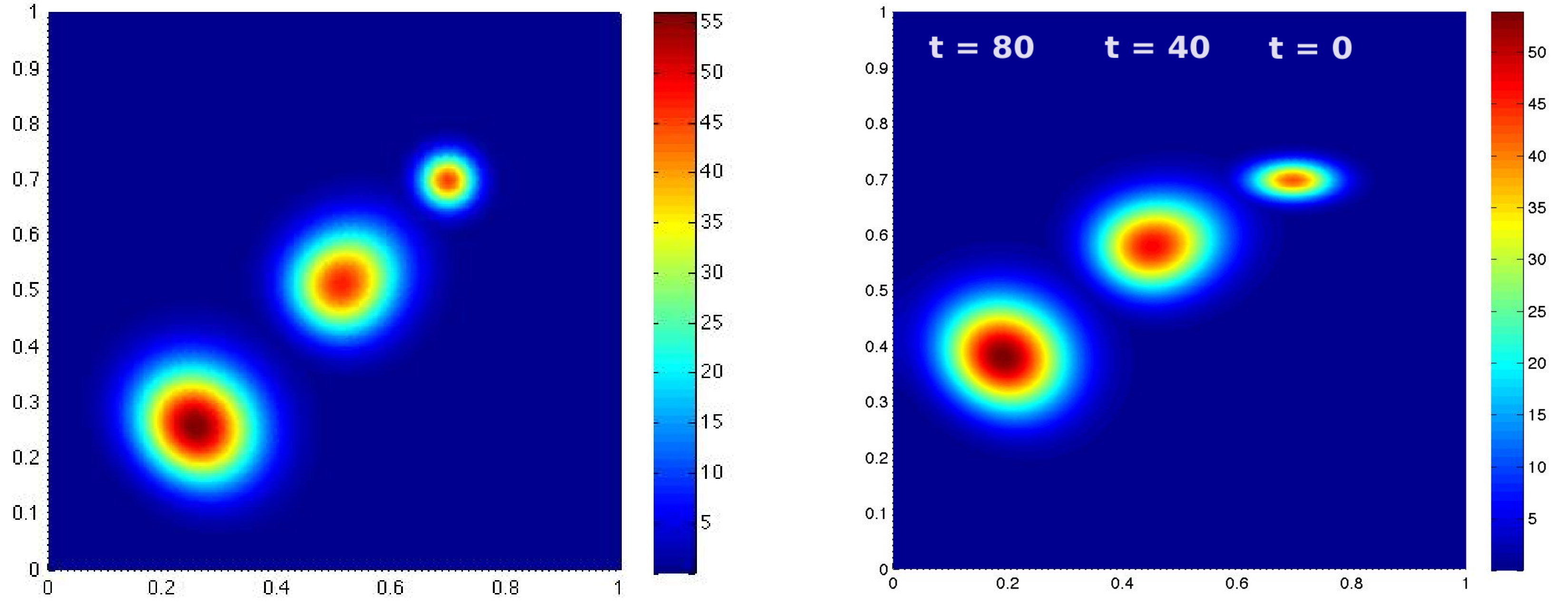
(iii) $R\Big(\bar{x}(t),\bar{\varrho}(t)\Big) = 0$

(iv) $\frac{d}{dt}\bar{x}(t) = \Big(-D^2\varphi(\bar{x}(t),t)\Big)^{-1}.\nabla R\Big(\bar{x}(t),\bar{\varrho}(t)\Big)$

Remark One can extract $\bar{\varrho}(t)$ from (iii) and (iv) is an ODE with a unique solution once $\bar{\varphi}$ is known.

Canonical equation

Consequence 1 : Through the matrix $\left(-D^2\varphi(\bar{x}(t),t)\right)^{-1}$, the microscopic shape of the Dirac plays a role



Canonical equation

Consequence 2 : Long time behavior

$$\frac{d}{dt}\bar{x}(t) = \Big(- D^2\varphi(\bar{x}(t), t)\Big)^{-1}.\nabla R\Big(\bar{x}(t), \bar{\varrho}(t)\Big)$$

$$\frac{d}{dt}R\Big(\bar{x}(t), \bar{\varrho}(t)\Big) = \nabla R\Big(\bar{x}(t), \bar{\varrho}(t)\Big)\Big(- D^2\varphi(\bar{x}(t), t)\Big)^{-1}.\nabla R\Big(\bar{x}(t), \bar{\varrho}(t)\Big)$$

$$+ R_\varrho\Big(\bar{x}(t), \bar{\varrho}(t)\Big)\frac{d}{dt}\bar{\varrho}(t)$$

$$= 0$$

Therefore $\frac{d}{dt}\bar{\varrho}(t) \geq 0$,

$$\bar{\varrho}(t) \underset{t\to\infty}{\longrightarrow} \bar{\varrho}_\infty$$

Canonical equation

Consequence 2 : Long time behavior (cont'd)

$$\nabla R\Big(\bar{x}_\infty, \bar{\varrho}_\infty\Big)\Big(-D^2\varphi\Big)^{-1}.\nabla R\Big(\bar{x}_\infty, \bar{\varrho}_\infty\Big) \to 0, \qquad \nabla R\Big(\bar{x}_\infty, \bar{\varrho}_\infty\Big) = 0.$$

$$\nabla R\Big(\bar{x}(t), \bar{\varrho}_\infty\Big) \approx 0.$$

For a concave function this implies

$$\bar{x}(t) \longrightarrow \bar{x}_\infty$$

with the characterization

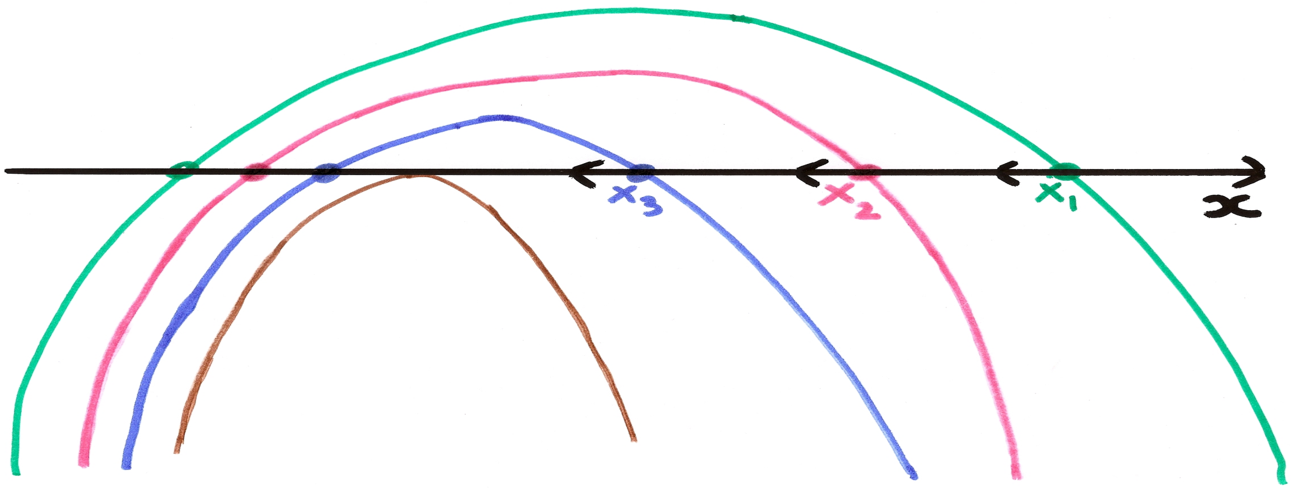
$$\max_x R(x, \bar{\varrho}_\infty) = 0 = R(\bar{x}_\infty, \bar{\varrho}_\infty)$$

$$= \min_{\rho \le \bar{\varrho}_\infty} \max R(x, \bar{\varrho}_\infty).$$

Canonical equation

Consequence 2 : Long time behavior (cont'd)

$$\nabla R\big(\bar{x}_\infty, \bar{\varrho}_\infty\big) = 0, \qquad R\big(\bar{x}_\infty, \bar{\varrho}_\infty\big) = 0$$



The limits $\varepsilon \to 0$, $t \to \infty$ is the same as the direct limit $t \to \infty$!

Canonical equation

Consequence 3 : What happens for several Dirac masses ?

For 2 Dirac masses

$$n(x,t) = \varrho_1(t)\delta(x-\bar{x}_1(t)) + \varrho_2(t)\delta(x-\bar{x}_2(t))$$

then

$$R\Big(\bar{x}_1(t), \bar{\varrho}(t)\Big) = 0, \qquad R\Big(\bar{x}_2(t), \bar{\varrho}(t)\Big) = 0,$$

$$\frac{d}{dt}\bar{x}_i(t) = \Big(-D^2\varphi(\bar{x}_i(t),t)\Big)^{-1}.\nabla R\Big(\bar{x}_i(t), \bar{\varrho}(t)\Big), \qquad i = 1,\ 2$$

These are 4 equations for 3 unknowns...

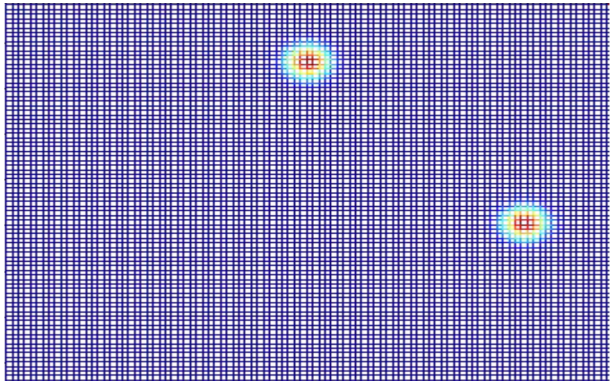
Canonical equation

Incompatible with the concavity assumption because $\varphi_\varepsilon(x,t) = \varepsilon \ln n_\varepsilon(x,t)$ should have two maxima.

One can go around and use the ansatz

$$n_\varepsilon = n_\varepsilon^1 + n_\varepsilon^2 = e^{\varphi_\varepsilon^1/\varepsilon} + e^{\varphi_\varepsilon^2/\varepsilon}$$

One indeed observes a single Dirac mass :



Open question

• Is there a broader smothness regime to derive the canonical equation ?

• Is there another rescaling ?

• Banching conditions : φ vanishes at fourth order, is it possible to use another transform ?

$$\begin{cases} \frac{\partial}{\partial t}\varphi(x,t) = R\Big(x, I_1(t), I_2(t)\Big) + |\nabla\varphi(x,t)\Big)|^2 \\ \max_x \varphi(x,t) = 0 \qquad \Big(= \varphi(t, \bar{x}_1(t)) = \varphi(t, \bar{x}_2(t)) \Big). \end{cases}$$

Two Lagrange multipliers, one constraint.