"YOU CAN'T GET THERE FROM HERE": WHY YOU CAN'T TRISECT AN ANGLE, DOUBLE THE CUBE, OR SQUARE THE CIRCLE (WITH STRAIGHTEDGE AND COMPASS)

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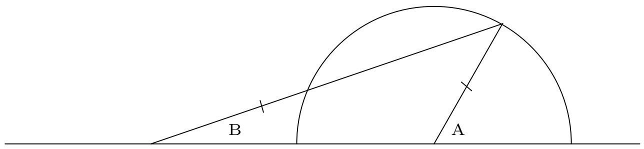
The three impossibilities:

(i) You can't trisect an angle with straightedge and compass.
(ii) You can't double the cube with straightedge and compass.
(iii) You can't square the circle with straightedge and compass.

1. (Archimedes' trick) Show that you can trisect an angle using a "marked straightedge" (such as a ruler). Hint: First show that in the following figure, angle B is one third angle A.



2. Show that $\cos 20$ is a root of

$$8x^3 - 6x - 1 = 0. \tag{1}$$

(Hint: Work out the formula for $\cos 3\theta$ in terms of $\cos\theta$, and use $\cos 60 = 1/2$.) Show that (1) has no rational roots (so $\cos 20$ is irrational).

Constructible numbers are the real numbers that you can write down using rational numbers, the four arithmetic operations, and square roots. (All intermediate calculations must be real as well.)

3. Let S be the set of points you could then construct using straightedge and compass, starting with the points $(0,0)$ and $(1,0)$. Show that S is precisely the set of points (x,y) where x and y are constructible. (This has many parts, but none of them are technically difficult. For example, show that if you have two line segments of length a, and $b \neq 0$, and a rational number r, then you can construct line segments of lengths $a \pm b$, ab, a/b, $\sqrt{a}$, and ra.)

A *real field* F is a subset of $\mathbb{R}$ containing 0 and 1, closed under the four operations. In other words, if you add, subtract, multiply, or divide two elements of F, you'll get another element of F. Suppose $z > 0$ is an element of F such that $\sqrt{z}$ isn't

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in F. Then the elements of the form $a + b\sqrt{z}$ (where a, $b \in F$) also form a real field, and this is called a *quadratic extension* of F, and is denoted $F(\sqrt{z})$. Define the *conjugate* of $a + b\sqrt{z}$ to be $a - b\sqrt{z}$. Hence constructable numbers are those numbers that are elements of *towers* of quadratic extensions over $\mathbb{Q}$.

4. If a, b, c, $d \in F$, and

$$a + b\sqrt{z} = c + d\sqrt{z},$$

show that $a = b = c = d = 0$.

To prove the three impossibilities, it suffices to show that $\cos 20$, $\sqrt[3]{2}$, and π aren't constructible.

5. Prove the *Sneaky lemma*: Suppose F is a real field. Suppose you have a cubic with coefficients in F, and no root in F. Then it has no roots in a quadratic extension of F.

6. Use the sneaky lemma to show the first two impossibilities.

7. To prove the third impossibility, we need the (hard!) fact that π is not algebraic.

(a) Show that any element of a quadratic extension of a real field F satisfies a polynomial equation (of degree at most 2 in fact) whose coefficients are in F.
(b) Suppose x is a real number that satisfies a polynomial equation $P(t) = 0$ whose coefficients are in $F(\sqrt{z})$. Show that x also satisfies a polynomial equation whose coefficients are in F. Hint: write $P(t) = Q(t) + R(t)\sqrt{z}$, where Q and R have coefficients in F. Then x is a solution of

$$(Q(t) + R(t)\sqrt{z})(Q(t) - R(t)\sqrt{z}) = 0.$$

(c) Hence show that any constructible number is algebraic, i.e. satisfies some polynomial equation with rational coefficients.
(d) Finally, prove the third impossibility.

8. (a) Show that a regular pentagon is constructible. (Hint: first show that $\sin 18 = (\sqrt{5} - 1)/4$.) (b) Show that a regular nonagon is *not* constructible.

In fact, the regular 17-gon is constructible!

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