18.034 PROBLEM SET 4

Hand in seven of the following problems. Due March 3 in class. No lates will be accepted. Discussion is encouraged, with two caveats: (a) write up your solutions by yourself, and (b) give credit when others came up with ideas (you won't be penalized for this). Give explanations, not just answers. References are to Boyce and DiPrima; the answers to problems from the book are in the back of the book.

1. Find the eight complex solutions of $r^8 - 1 = 0$. In other words, find all the eighth-roots of 1. (Hint: one is $(1+i)/\sqrt{2}$.)
2. Which of the following operators are linear? (a) $L(f(t)) = 1$. (b) $L(f(t)) = 0$. (c) $L(f(t)) = f(t)^2$. (d) $L(f(t)) = \int_0^t f(s)ds$. (e) $L(f(t)) = \int_0^t f(s)s^3 ds$. (f) $L(f(t)) = e^t f(t)$.
3. Using de Moivre's formula, show that $\cos(x) = (e^{ix} + e^{-ix})/2$ and $\sin(x) = (e^{ix} - e^{-ix})/(2i)$. Using the second formula, compute
$$\int e^{rt} \sin(bt) dt$$
without using anything painful like integration by parts.
4. Find the general solution to the following differential equations. Use the exponential shift law. (a) $y'' + 2y' + y = e^{-t}$. (b) $y'' + 2y' + y = \cos t$. (c) $y'' + 4y = \cos 2t + e^t$.
5. Find the general solution to $(D-1)^{100} y = e^t$.
6. Problems 3.8.14 and 3.8.15, p. 191.
7. Problems 3.9.1 and 3.9.3, p. 198.
8. (a) Using the first formula in Problem 3 and the binomial theorem, prove that (for all non-negative integers n)
$$(\cos x)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \cos((n-2k)x).$$
(b) Use (b) to show that (for all non-negative integers n)
$$\frac{1}{\pi} \int_0^{\pi} (\cos x)^{2n} dx = \frac{(2n)!}{2^{2n} n! n!}.$$
(c) If n is an integer, give a one-sentence explanation why $\int_0^{\pi} (\cos x)^{2n+1} dx = 0$ (Hint: Look first at the graph of $y = \cos x$ for $0 \le x \le \pi$, and then describe something about the graph of $y = (\cos x)^{2n+1}$ that forces the integral to be zero on this region.)

Date: February 25, 2000.