MODERN ALGEBRA MATH 210A MIDTERM

1. (a) Suppose H and K are two normal subgroups of a group G. Show that HK is a subgroup of G. Show that HK is normal.

(b) Prove that a group is abelian if and only if the function $f : G \to G$, given by $f(a) = a^{-1}$, is a group homomorphism.

2. Show that there is no simple group of order 30.

3. Suppose G is a finite group of order n. Let k be an integer relatively prime to n. Show that the map $x \mapsto x^k$ is surjective.

4. If G is a nonabelian group of order p^3, show that $|Z(G)| = p$. (Here p is a prime number.)

5. Recall that a solvable group is defined as one possessing a series:

$$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_s = G$$

where each quotient H_{i+1}/H_i is abelian. Recall that $[G, G]$ is the subgroup generated by terms of the form $[x, y] = xyx^{-1}y^{-1}$; $[G, G] \lhd G$; and $G/[G, G]$ is abelian. For any group define the following sequence of subgroups inductively by $G^{(0)} = G$, $G^{(1)} = [G, G]$, and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. This series of subgroups is called the *derived* or *commutator* series of G. Show that a group G is solvable if and only if $G^{(n)} = 1$ for some $n \geq 0$.

6. Let G be a finite group and p be the smallest prime dividing the order of G. Let H be a subgroup of index p. Show that H is normal.

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