MODERN ALGEBRA (MATH 210) PRACTICE FINAL EXAM

My office hours will be Sunday 2–4 pm, Monday 8–10 pm, Tuesday 8–10 pm.

1. Suppose P and Q are two normal subgroups of a group G such that $P \cap Q = \{e\}$. Show that any two elements $p \in P$, $q \in Q$ commute.

2. Suppose G is a finite subgroup, with only one Sylow subgroup for each prime. Show that G is isomorphic to the product of its Sylow subgroups.

3. *(Burnside's Lemma)* Let G act on a finite set X. If N is the number of orbits, then

$$N = \frac{1}{|G|} \sum_{\tau \in G} \mathrm{Fix}(\tau)$$

where $\mathrm{Fix}(\tau)$ is the number of $x \in X$ fixed by τ. (Hint: $\sum_{\tau \in G} \mathrm{Fix}(\tau) = \#\{(\tau \in G, x \in X) : \tau x = x\}$.)

4. Let $\phi : R \to S$ be a homomorphism of commutative rings.
(a) Prove that if $\mathfrak{p}$ is a prime ideal of S then either $\phi^{-1}(\mathfrak{p}) = R$ or $\phi^{-1}(\mathfrak{p})$ is a prime ideal of R.
(b) Prove that if $\mathfrak{m}$ is a maximal ideal of S and ϕ is surjective then $\phi^{-1}(\mathfrak{m})$ is a maximal ideal of R. Give an example to show that this need not be the case if ϕ is not surjective.

5. Let R be a commutative ring. We say that an element $x \in R$ is *nilpotent* if $x^n = 0$ for some $n \in \mathbb{Z}^+$. Prove that the set of nilpotent elements form an ideal — called the *nilradical* of R and denoted by $\mathfrak{N}(R)$.

6. Let τ be the golden mean $\frac{1+\sqrt{5}}{2}$. Show that $\mathbb{Z}[\tau]$ has infinitely many units.

7. Show that if an irreducible cubic in $\mathbb{Q}[x]$ has two complex roots and one real root, then its splitting field is a degree 6 extension of $\mathbb{Q}$.

8. Suppose p is an odd prime. Let ζ be a primitive root of unity. Show that the minimal polynomial for ζ over $\mathbb{Q}$ is $t^{p-1} + t^{p-2} + \cdots + 1$. Show that $\mathbb{Q}(\zeta)$ is Galois over $\mathbb{Q}$, of degree p, and describe the Galois group.

9. Suppose q is a prime power. Show that there are $(q^3 - q)/3$ irreducible monic degree 3 polynomials in $\mathbb{F}_q[x]$. (*Hint:* Consider the elements of $\mathbb{F}_{q^3}$ and their minimal polynomials over q.)

Good luck!

Date: Friday, December 2, 2004.