MODERN ALGEBRA (MATH 210) PROBLEM SET 3

This set is due Friday, Oct. 22 at noon at Jarod Alper's door, 380–J.

1. Show that there is no simple group of order 56. Show that there is no simple group of order 351.

2. If G is a nonabelian group, show that $G/Z(G)$ is not cyclic.

3. Suppose that H is a normal subgroup of G. Show that G is solvable if and only if both H and G/H are solvable.

4. Show that if G is a finite group and $H < G$ is a proper subgroup, then there exist elements of G not conjugate to any element of H. (In other words, the union of all conjugate subgroups of H cannot be all of G.)

5. Show that if p is the smallest prime dividing $|G|$ then any subgroup of G of index p is a normal subgroup of G.

6. Prove that if H has finite index n then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n!$.

7. The set of n-cycles in S_n form a conjugacy class in S_n. If n is odd, how many conjugacy classes does this set form in A_n?

8. (a) Let Ω be an infinite set. Let D the subgroup of S_Ω consisting of permutations which move only a finite number of elements of Ω and let A be the set of all elements $\sigma \in D$ such that σ acts as an even permutation on the (finite) set of points it moves. Prove that A is an infinite simple group.

(b) Prove that if $H \neq \{e\}$ is a normal subgroup of S_Ω, then H contains A, i.e. A is the unique nontrivial minimal normal subgroup of S_Ω.

9. This exercise shows that for $n \neq 6$, every automorphism of S_n is inner. Fix an integer $n \geq 2$ with $n \neq 6$.

(a) Prove that the automorphism group of a group G permutes the conjugacy classes of G, i.e. for each $\sigma \in \mathrm{Aut}(G)$ and each conjugacy class $\mathcal{K}$ of G the set $\sigma(\mathcal{K})$ is also a conjugacy class of G.

(b) Let $\mathcal{K}$ be the conjugacy class of transpositions in S_n and let $\mathcal{K}'$ be the conjugacy class of any element of order 2 in S_n that is not a transposition. Prove that $|\mathcal{K}| \neq |\mathcal{K}'|$. Deduce that any automorphism of S_n sends transpositions to transpositions.

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(c) Prove that for each $\sigma \in \operatorname{Aut}(S_n)$

$$\sigma : (12) \mapsto (ab_2), \qquad \sigma : (13) \mapsto (ab_3), \ldots, \sigma : (1n) \mapsto (ab_n)$$

for some distinct integers $a, b_2, b_3, \ldots, b_n \in \{1, 2, \ldots, n\}$. As (12),(13), ..., $(1n)$ generate S_n, deduce that any automorphism of S_n is uniquely determined by its action on these elements. Hence show that S_n has at most $n!$ automorphisms and conclude that $\operatorname{Aut}(S_n) = \operatorname{Inn}(S_n)$ for $n \neq 6$.

10. We now show that $\operatorname{Inn}(S_6)$ is of index at most 2 in $\operatorname{Aut}(S_6)$. Let $\mathcal{K}$ be the conjugacy class of transpositions in S_6 and let $\mathcal{K}'$ be the conjugacy class of any element of order 2 in S_6 that is not a transposition. Prove that $|\mathcal{K}| \neq |\mathcal{K}'|$ unless $\mathcal{K}'$ is the conjugacy class of products of three disjoint transpositions. Deduce that $\operatorname{Aut}(S_6)$ has a subgroup of index at most 2 which sends transpositions to transpositions. Then prove that $|\operatorname{Aut}(S_6) : \operatorname{Inn}(S_6)| \leq 2$.

11. Finally, we exhibit an outer automorphism of S_6. (There are other, more beautiful, descriptions.) Let $t'_1 = (12)(34)(56)$, $t'_2 = (14)(25)(36)$, $t'_3 = (13)(24)(56)$, $t'_4 = (12)(36)(45)$, $t'_5 = (14)(23)(56)$. Show that $t'_1, \ldots, t'_5$ satisfy the following relations:

- $(t'_i)^2 = e$ for all i;
- $(t'_i t'_j)^2 = e$ for all i and j with $|i-j| \geq 2$;
- $(t'_i t'_j)^3 = e$ for all i and j with $|i-j| = 1$.

Use this to show that the map $(i(i+1)) \mapsto t'_i$ gives an automorphism of S_6. (In the process, you will likely have to show that the relations above define S_6. Your argument will also presumably prove the obvious generalization to S_n.)

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