MODERN ALGEBRA (MATH 210) PROBLEM SET 4

1. Let G be a group of order 7. Show that $\mathrm{Aut}(H)$ is abelian of order 6.

2. Prove that $\mathbb{Q}$ is neither a finitely generated group nor a free abelian group.

3. Suppose $B_1 + \cdots + B_n = A$ and $B_{i+1} \cap (B_1 + \cdots B_i) = 0$ for all i. Show that $B_1 \oplus \cdots \oplus B_n \to A$ is an isomorphism.

4. Describe a 2-Sylow subgroup of S_8. How many are there?

5. Describe all 2-Sylow subgroups of S_4. Show that this group is isomorphic to the symmetry group of a square.

6. Let P be a normal Sylow p-subgroup of G and let H be any subgroup of G. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

7. (Dummit and Foote p. 168) Let n and m be positive integers with $d = \gcd(n, m)$. Let $Z_n = \langle x \rangle$ and $Z_m = \langle y \rangle$. Let A be the central product of $\langle x \rangle$ and $\langle y \rangle$ with an element of order d identified:

$$\langle x, y : x^n = y^m = 1, xy = yx, x^{n/d} = y^{m/d} \rangle.$$

Describe A as the direct product of two cyclic groups.

8. (Dummit and Foote p. 169) For any group G define the *dual group* of G (denoted $\hat{G}$) to be the set of all homomorphisms from G into the multiplicative group of roots of unity in $\mathbb{C}$. Define a group operation in $\hat{G}$ by pointwise multiplication of functions: if χ, ψ are homomorphisms from G into the group of roots of unity then $\chi\psi$ is the homomorphism given by $(\chi\psi)(g) = \chi(g)\psi(g)$ for all $g \in G$, where the latter multiplication takes place in $\mathbb{C}$.

(a) Show that this operation on $\hat{G}$ makes $\hat{G}$ into an abelian group. [Show that the identity is the map $g \mapsto 1$ for all $g \in G$ and the inverse of $\chi \in \hat{G}$ is the map $g \mapsto \chi(g)^{-1}$.]

(b) If G is a finite abelian group, prove that $\hat{G} \cong G$. [Write G as $\langle x_1 \rangle \times \cdots \times \langle x_r \rangle$ and if n_i is the order of x_i define χ_i to be the homomorphism which sends x_i to $e^{2\pi i/n_i}$ and sends x_j to 1, for all $j \neq i$. Prove χ_i has order n_i in $\hat{G}$ and $\hat{G} = \langle \chi_1 \rangle \times \cdots \times \langle \chi_r \rangle$.]

This result is often phrased: a finite abelian group is self-dual. It implies that the lattice diagram (of subgroups) of a finite abelian group is the same when it is turned upside down. Note however that there is no *natural* isomorphism between G and its dual; the isomorphism depends on a choice of a set of generators for G. This is frequently stated in the form: a finite abelian group is *noncanonically* isomorphic to its dual.

This set is due Friday, Oct. 29 at noon at Jarod Alper's door, 380–J.

Date: Wednesday, October 20, 2004.