MODERN ALGEBRA (MATH 210) PROBLEM SET 5

1. Suppose G is a finite group. Let S be the subset of G consisting of all elements whose order is a power of p. If S is a subgroup, show that S is a p-Sylow subgroup, and that it is normal. Conversely, show that if there is a single p-Sylow subgroup, then it is S.

2. Show that the group of rotations of a tetrahedron is isomorphic to A_4.

3. Let $\phi(n)$ be the number of integers smaller than n that are relatively prime to n. If p is a prime, and m is positive integer, show that $m \mid \phi(p^m - 1)$.

4. Suppose G is the quotient product of n finite cyclic groups by some subgroup. Show that G can be written as the product of m cyclic groups, where $m \leq n$.

5. Show that $(\mathbb{Z}/2^n)^*$ is not cyclic for $n \geq 3$. (One possible hint: show the result for $n = 3$ and work by induction. A hint for a different proof: find 2 distinct subgroups of order 2.)

6. Show that $(\mathbb{Z}/3^n)^*$ *is* cyclic for all positive integers n.

7. Let P be a 2-Sylow subgroup of S_8. Show that P can be written as a semidirect product $D_8 \times D_8 \rtimes \mathbb{Z}/2$. Show that P can be written as a semidirect product $(\mathbb{Z}/2)^4 \rtimes D_8$.

8. Let F be the free group on two generators x and y (whose elements can be written as words with two letters x and y). Let $\phi : F \to D_{2n}$ (where D_{2n} is written interpreted as the symmetries of a regular n-gone), where x maps to a rotation by $2\pi/n$ and y maps to a reflection. Show that the kernel of π is generated by x^n, y^2, $xyxy$. (Hence D_{2n} has a *presentation* $\langle x, y : x^n = y^2 = xyxy = e\rangle$. Hint: remember the related problem about S_6.)

This set is due Friday, Nov. 12 at noon at Jarod Alper's door, 380–J.

Date: Wednesday, November 3, 2004.