FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 37

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CONTENTS

Welcome back to the third quarter! The theme for this quarter, insofar as there is one, will be "useful ideas to know". We'll start with differentials for the first three lectures.

I prefer to start any topic with a number of examples, but in this case I'm going to spend a fair amount of time discussing technicalities, and then get to a number of examples. Here is the main message I want you to get. Differentials are an intuitive geometric notion, and we're going to figure out the right description of them algebraically. I find the algebraic manifestation a little non-intuitive, so I always like to tie it to the geometry. So please don't tune out of the statements. Also, I want you to notice that although the algebraic statements are odd, none of the proofs are hard or long.

This topic could have been done as soon as we knew about morphisms and quasicoherent sheaves.

1. MOTIVATION AND GAME PLAN

Suppose X is a "smooth" k-variety. We hope to define a tangent bundle. We'll see that the right way to do this will easily apply in much more general circumstances.

• We'll see that cotangent is more "natural" for schemes than tangent bundle. This is similar to the fact that the Zariski *cotangent* space is more natural than the *tangent space* (i.e. if A is a ring and $\mathfrak{m}$ is a maximal ideal, then $\mathfrak{m}/\mathfrak{m}^2$ is "more natural" than $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. So we'll define the cotangent sheaf first.

• Our construction will work for general X, even if X is not "smooth" (or even at all nice, e.g. finite type). The cotangent sheaf won't be locally free, but it will still be a quasicoherent sheaf.

• Better yet, this construction will work "relatively". For *any* $X \to Y$, we'll define $\Omega_{X/Y}$, a quasicoherent sheaf on X, the sheaf of *relative differentials*. This will specialize to the earlier

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case by taking $Y = \operatorname{Spec} k$. The idea is that this glues together the cotangent sheaves of the fibers of the family. (I drew an intuitive picture in the "smooth" case. I introduced the phrase "vertical (co)tangent vectors".)

2. The affine case: three definitions

We'll first study the affine case. Suppose A is a B-algebra, so we have a morphism of rings $\phi : B \to A$ and a morphism of schemes $\operatorname{Spec} A \to \operatorname{Spec} B$. I will define an A-module $\Omega_{A/B}$ in three ways. This is called the *module of relative differentials* or the *module of Kähler differentials*. The module of differentials will be defined to be this module, as well as a map $d : A \to \Omega_{A/B}$ satisfying three properties.
(i) additivity. $da + da' = d(a + a')$
(ii) Leibniz. $d(aa') = a\ da' + \ a'da$
(iii) triviality on pullbacks. $db = 0$ for $b \in \phi(B)$.

As motivation, think of the case $B = k$. So for example, $da^n = na^{n-1}da$, and more generally, if f is a polynomial in one variable, $df(a) = f'(a)\ da$ (where f' is defined formally: if $f = \sum c_i x^i$ then $f' = \sum c_i i x^{i-1}$).

I'll give you three definitions of this sheaf in the affine case (i.e. this module). The first is a concrete hands-on definition. The second is by universal property. And the third will globalize well, and will allow us to define $\Omega_{X/Y}$ conveniently in general.

The first two definitions are analogous to what we have seen for tensor product. Recall that there are two common definitions of $\otimes$. The first is in terms of formal symbols satisfying some rules. This is handy for showing certain things, e.g. if $M \to M'$ is surjective, then so is $M \otimes N \to M' \otimes N$. The second is by universal property.

2.1. First definition of differentials: explicit description. We define $\Omega_{A/B}$ to be finite A-linear combinations of symbols "da" for $a \in A$, subject to the three rules (i)–(iii) above. For example, take $A = k[x, y]$, $B = k$. Then a sample differential is $3x^2\ dy + 4dx \in \Omega_{A/B}$. We have identities such as $d(3xy^2) = 3y^2\ dx + 6xy\ dy$.

Key fact. Note that if A is generated over B (as an algebra) by $x_i \in A$ (where i lies in some index set, possibly infinite), subject to some relations r_j (where j lies in some index set, and each is a polynomial in some finite number of the x_i), then the A-module $\Omega_{A/B}$ is generated by the dx_i, subject to the relations (i)—(iii) and $dr_j = 0$. In short, we needn't take every single element of A; we can take a generating set. And we needn't take every single relation among these generating elements; we can take generators of the relations.

2.2. *Exercise.* Verify the above key fact.

In particular:

2.3. *Proposition. — If* A *is a finitely generated* B*-algebra, then* $\Omega_{A/B}$ *is a finite type (i.e. finitely generated)* A*-module. If* A *is a finitely presented* B*-algebra, then* $\Omega_{A/B}$ *is a finitely presented* A*-module.*

("Finitely presented" algebra means finite number of generators (=finite type) and finite number of relations. If A is Noetherian, then the two hypotheses are the same, so most of you will not care.)

Let's now see some examples. Among these examples are three particularly important kinds of ring maps that we often consider: adding free variables; localizing; and taking quotients. If we know how to deal with these, we know (at least in theory) how to deal with any ring map.

2.4. Example: taking a quotient. If $A = B/I$, then $\Omega_{A/B} = 0$ basically immediately: $da = 0$ for all $a \in A$, as each such a is the image of an element of B. This should be believable; in this case, there are no "vertical tangent vectors".

2.5. Example: adding variables. If $A = B[x_1, \dots, x_n]$, then $\Omega_{A/B} = A dx_1 \oplus \cdots \oplus A dx_n$. (Note that this argument applies even if we add an arbitrarily infinite number of indeterminates.) The intuitive geometry behind this makes the answer very reasonable. The cotangent bundle should indeed be trivial of rank n.

2.6. Example: two variables and one relation. If $B = \mathbb{C}$, and $A = \mathbb{C}[x,y]/(y^2 - x^3)$, then $\Omega_{A/B} = \mathbb{C}\ dx \oplus \mathbb{C}\ dy/(2y\ dy - 3x^2\ dx)$.

2.7. Example: localization. If S is a multiplicative set of B, and $A = S^{-1}B$, then $\Omega_{A/B} = 0$. Reason: Note that the quotient rule holds. (If $b = as$, then $db = a\ ds + s\ da$, which can be rearranged to give $da = (s\ db - b\ ds)/s^2$.) Thus if $a = b/s$, then $da = (s\ db - b\ ds)/s^2 = 0$. (If $A = B_f$ for example, this is intuitively believable; then $\operatorname{Spec} A$ is an open subset of $\operatorname{Spec} B$, so there should be no "vertical cotangent vectors".)

2.8. Exercise: localization (stronger form). If S is a multiplicative set of A, show that there is a natural isomorphism $\Omega_{S^{-1}A/B} \cong S^{-1}\Omega_{A/B}$. (Again, this should be believable from the intuitive picture of "vertical cotangent vectors".) If T is a multiplicative set of B, show that there is a natural isomorphism $\Omega_{S^{-1}A/T^{-1}B} \cong S^{-1}\Omega_{A/B}$ where S is the multiplicative set of A that is the image of the multiplicative set $T \subset B$.

2.9. *Exercise.* (a) *(pullback of differentials)* If

$$\begin{array}{ccc} A' & \longleftarrow & A \\ \uparrow & & \uparrow \\ B' & \longleftarrow & B \end{array}$$

is a commutative diagram, show that there is a natural homomorphism of A'-modules $A' \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$. An important special case is $B = B'$.

(b) *(differentials behave well with respect to base extension, affine case)* If furthermore the above diagram is a tensor diagram (i.e. $A' \cong B' \otimes_B A$) then show that $A' \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$ is an isomorphism.

2.10. *Exercise.* Suppose k is a field, and K is a separable algebraic extension of k. Show that $\Omega_{K/k} = 0$.

2.11. *Exercise (Jacobian description of $\Omega_{A/B}$).* — Suppose $A = B[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Then $\Omega_{A/B} = \{\oplus_i B dx_i\}/\{df_j = 0\}$ maybe interpreted as the cokernel of the Jacobian matrix $J : A^{\oplus r} \to A^{\oplus n}$.

I now want to tell you two handy (geometrically motivated) exact sequences. The arguments are a bit tricky. They are useful, but a little less useful than the foundation facts above.

2.12. *Theorem (the relative cotangent sequence, affine version). — Suppose* $C \to B \to A$ *are ring homomorphisms. Then there is a natural exact sequence of* A*-modules*

$$A \otimes_B \Omega_{B/C} \to \Omega_{A/C} \to \Omega_{A/B} \to 0.$$

Before proving this, I drew a picture motivating the statement. I drew pictures of two maps of schemes, $\operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} B \xrightarrow{g} \operatorname{Spec} C$, where $\operatorname{Spec} C$ was a point, $\operatorname{Spec} B$ was $\mathbb{A}^1$ (or a "smooth curve"), and $\operatorname{Spec} A$ was $\mathbb{A}^2$ (or a "smooth surface"). The tangent space to a point upstairs has a subspace that is the tangent space to the vertical fiber. The cokernel is the pullback of the tangent space to the image point in $\operatorname{Spec} B$. Thus we have an exact sequence $0 \to T_{\operatorname{Spec} A/\operatorname{Spec} B} \to T_{\operatorname{Spec} A/\operatorname{Spec} C} \to f^* T_{\operatorname{Spec} B/\operatorname{Spec} C} \to 0$. We want the corresponding sequence of cotangent vectors, so we dualize. We end up with precisely the statement of the Theorem, except we also have left-exactness. This discrepancy is because the statement of the theorem is more general; we'll see that in the "smooth" case, we'll indeed have left-exactness.

Proof. (Before we start, note that surjectivity is clear, from $da \mapsto da$. The composition over the middle term is clearly 0: $db \to db \to 0$.) We wish to identify $\Omega_{A/B}$ as the cokernel of $A \otimes_B \Omega_{B/C} \to \Omega_{A/C}$. Now $\Omega_{A/B}$ is exactly the same as $\Omega_{A/C}$, except we have extra relations: $db = 0$ for $b \in B$. These are precisely the images of $1 \otimes db$ on the left. □

2.13. *Theorem (Conormal exact sequence, affine version). — Suppose* B *is a* C*-algebra,* I *is an ideal of* B*, and* $A = B/I$*. Then there is a natural exact sequence of* A*-modules*

$$I/I^2 \xrightarrow{\delta : i \mapsto 1 \otimes di} A \otimes_B \Omega_{B/C} \xrightarrow{a \otimes db \mapsto a\, db} \Omega_{A/C} \longrightarrow 0.$$

Before getting to the proof, some discussion is necessary. (The discussion is trickier than the proof itself!)

The map δ is a bit subtle, so I'll get into its details before discussing the geometry. For any $i \in I$, $\delta i = 1 \otimes di$. Note first that this is well-defined: If $i, i' \in I$, $i \equiv i' \pmod{I^2}$, say $i - i' = i''i'''$ where $i'', i''' \in I$, then $\delta i - \delta i' = 1 \otimes (i''\, di''' + i'''\, di'') \in I\Omega_{B/C}$ is 0 in $A \otimes_B \Omega_{B/C} = (B/I) \otimes_B \Omega_{B/C}$. Next note that I/I^2 indeed is an $A = (B/I)$-module. Finally, note that the map $I/I^2 \to A \otimes_B \Omega_{B/C}$ is indeed a homomorphism of A-modules: If $a \in A$, $b \in I$, then $ab \mapsto 1 \otimes d(ab) = 1 \otimes (a\, db + b\, da) = 1 \otimes (a\, db) = a(1 \otimes db)$.

Having dispatched that formalism, let me get back to the geometry. I drew a picture where $\operatorname{Spec} C$ is a point, $\operatorname{Spec} B$ is a plane, and $\operatorname{Spec} A$ is something smooth in it. Let j be the inclusion. Then we have $0 \to T_{\operatorname{Spec} A/\operatorname{Spec} C} \to j^* T_{\operatorname{Spec} B/\operatorname{Spec} C} \to N_{\operatorname{Spec} B/\operatorname{Spec} C} \to 0$. Dualizing it, we get $0 \to N^\vee_{A/B} \to A \otimes \Omega_{B/C} \to \Omega_{A/C} \to 0$. This exact sequence reminds me of several things above and beyond the theorem. First of all, I/I^2 will later be the conormal bundle — hence the name of the theorem. Second, in good circumstances, the conormal exact sequence of Theorem 2.13 will be injective on the left.

2.14. *Aside: Why should I/I^2 be the conormal bundle?.* We'll *define* I/I^2 to be the conormal bundle later, so I'll try to give you an idea as to why this is reasonable. You believe now that $\mathfrak{m}/\mathfrak{m}^2$ should be the cotangent space to a point in $\mathbb{A}^n$. In other words, $(x_1, \dots, x_n)/(x_1, \dots, x_n)^2$ is the cotangent space to $\vec{0}$ in $\mathbb{A}^n$. Translation: it is the conormal space to the point $\vec{0} \in \mathbb{A}^n$. Then you might believe that in $\mathbb{A}^{n+m}$, $(x_1, \dots, x_n)/(x_1, \dots, x_n)^2$ is the conormal bundle to the coordinate n-plane $\mathbb{A}^m \subset \mathbb{A}^{n+m}$.

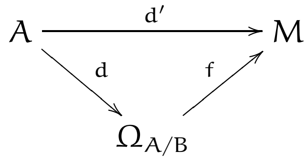
Let's finally prove the conormal exact sequence.

Proof of the conormal exact sequence (affine version) 2.13. We need to identify the cokernel of $\delta : I/I^2 \to A \otimes_B \Omega_{B/C}$ with $\Omega_{A/C}$. Consider $A \otimes_B \Omega_{B/C}$. As an A-module, it is generated by db ($b \in B$), subject to three relations: $dc = 0$ for $c \in \phi(C)$ (where $\phi : C \to B$ describes B as a C-algebra), additivity, and the Leibniz rule. Given any relation *in* B, d of that relation is 0.

Now $\Omega_{A/C}$ is defined similarly, except there are more relations *in* A; these are precisely the elements of $i \in B$. Thus we obtain $\Omega_{A/C}$ by starting out with $A \otimes_B \Omega_{B/C}$, and adding the additional relations di where $i \in I$. But this is precisely the image of δ! $\square$

2.15. Second definition: universal property. Here is a second definition that we'll use at least once, and is certainly important philosophically. Suppose A is a B-algebra, and M is a A-module. An *B-linear derivation of A into M* is a map $d : A \to M$ of B-modules (*not necessarily A-modules*) satisfying the Leibniz rule: $d(fg) = f\, dg + g\, df$. As an example, suppose $B = k$, and $A = k[x]$, and $M = A$. Then an example of a k-linear derivation is d/dx. As a second example, if $B = k$, $A = k[x]$, and $M = k$. Then an example of a k-linear derivation is $d/dx|_0$.

Then $d : A \to \Omega_{A/B}$ is defined by the following universal property: any other B-linear derivation $d' : A \to M$ factors uniquely through d:



Here f is a map of A-modules. (Note again that d and d' are not! They are only B-linear.) By universal property nonsense, if it exists, it is unique up to unique isomorphism. The candidate I described earlier clearly satisfies this universal property (in particular, it is a derivation!), hence this is it. [Thus Ω is the "unversal derivation". I should rewrite this paragraph at some point.]

The next result will give you more evidence that this deserves to be called the (relative) cotangent bundle.

2.16. *Proposition.* *Suppose* B *is a* k*-algebra, with residue field* k*. Then the natural map* $\delta : \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B k$ *is an isomorphism.*

I skipped this proof in class, but promised it in the notes.

Proof. By the conormal exact sequence 2.13 with $I = \mathfrak{m}$ and $A = C = k$, δ is a surjection (as $\Omega_{k/k} = 0$), so we need to show that it is injection, or equivalently that $\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is a surjection. But any element on the right is indeed a derivation from B to k (an earlier exercise from back in the dark ages on the Zariski tangent space), which is precisely an element of $\mathrm{Hom}_B(\Omega_{B/k}, k)$ (by the universal property of $\Omega_{B/k}$), which is canonically isomorphic to $\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k)$ as desired. □

Remark. As a corollary, this (in combination with the Jacobian exercise 2.11 above) gives a second proof of an exercise from the first quarter, showing the Jacobian criterion for nonsingular varieties over an algebraically closed field.

Aside. If you wish, you can use the universal property to show that $\Omega_{A/B}$ behaves well with respect to localization. For example, if S is a multiplicative set of A, then there is a natural isomorphism $\Omega_{S^{-1}A/B} \cong S^{-1}\Omega_{A/B}$. This can be used to give a different solution to Exercise 2.8. It can also be used to give a second definition of $\Omega_{X/Y}$ for a morphism of schemes $X \to Y$ (different from the one given below): we define it as a quasicoherent sheaf, by describing how it behaves on affine open sets, and showing that it behaves well with respect to distinguished localization.

Next day, I'll give a third definition which will globalize well, and we'll see that we already understand differentials for morphisms of schemes.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 38

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CONTENTS

Last day I introduced differentials on affine schemes, for a morphism $B \to A$. The differential was an A-module, as well as a homomorphism of B-modules, $d : A \to \Omega_{A/B}$. The A-module $\Omega_{A/B}$ is generated by da, and d satisfies 3 rules: additivity, Leibniz rule, and $db = 0$ (or $d1 = 0$). It satisfies a universal property: any derivation $A \to M$ uniquely factors through an A-module homomorphism $\Omega_{A/B} \to M$.

1. A THIRD DEFINITION OF Ω, SUITABLE FOR EASY GLOBALIZATION

1.1. Third definition. We now want to globalize this definition for an arbitrary morphism of schemes $f : X \to Y$. We could do this "affine by affine"; we just need to make sure that the above notion behaves well with respect to "change of affine sets". Thus a relative differential on X would be the data of, for every affine $U \subset X$, a differential of the form $\sum a_i \, db_i$, and on the intersection of two affine open sets $U \cap U'$, with representatives $\sum a_i \, db_i$ on U and $\sum a'_i \, db'_i$ on the second, an equality on the overlap. Instead, we'll take a different tack. We'll get what intuitively seems to be a very weird definition! I'll give the definition, then give you some intuition, and then get back to the definition.

Suppose $f : X \to Y$ be any morphism of schemes. Recall that $\delta : X \to X \times_Y X$ is a locally closed immersion (Class 9, p. 5). Thus there is an open subscheme $U \subset X \times_Y X$ for which $\delta : X \to U$ is a closed immersion, cut out by a quasicoherent sheaf of ideals $\mathcal{I}$. Then $\mathcal{I}/\mathcal{I}^2$ is a quasicoherent sheaf naturally supported on X (affine-locally this is the statement that I/I^2 is naturally an A/I-module). We call this the *conormal sheaf to* X (or somewhat more precisely,to the locally closed immersion). (For the motivation for this name, see last day's notes.) We denote it by $\mathcal{N}^\vee_{X/X\times_Y X}$. Then we will *define* $\Omega_{X/Y}$ as this conormal sheaf.

(Small technical point for pedants: what does $\mathcal{I}^2$ mean? In general, if $\mathcal{I}$ and $\mathcal{J}$ are quasicoherent ideal sheaves on a scheme Z, what does $\mathcal{I}\mathcal{J}$ mean? Of course it means that on each affine, we take the product of the two corresponding ideals. To make sure this

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is well-defined, we need only check that if A is a ring, and $f \in A$, and $I, J \subset A$ are two ideals, then $(IJ)_f = I_f J_f$ in A_f.)

Brief aside on (co)normal sheaves to locally closed immersions. For any locally closed immersion $W \to Z$, we can define the *conormal sheaf* $\mathcal{N}^\vee_{W/Z}$, a quasicoherent sheaf on W, similarly, and the *normal sheaf* as its dual $\mathcal{N}_{W/Z} := \underline{\mathrm{Hom}}(\mathcal{N}^\vee, \mathcal{O}_W)$. This is somewhat imperfect notation, as it suggests that the dual of $\mathcal{N}$ is always $\mathcal{N}^\vee$. This is not always true, as for A-modules, the natural morphism from a module to its double-dual is not always an isomorphism. (Modules for which this is true are called *reflexive*, but we won't use this notion.)

1.2. *Exercise: normal bundles to effective Cartier divisors.* Suppose $D \subset X$ is an effective Cartier divisor. Show that the conormal sheaf $\mathcal{N}^\vee_{D/X}$ is $\mathcal{O}(-D)|_D$ (and in particular is an invertible sheaf), and hence that the normal sheaf is $\mathcal{O}(D)|_D$. It may be surprising that the normal sheaf should be locally free if $X \cong \mathbb{A}^2$ and D is the union of the two axes (and more generally if X is nonsingular but D is singular), because you may be used to thinking that the normal bundle is isomorphic to a "tubular neighborhood".

Let's get back to talking about differentials.

We now define the d operator $d : \mathcal{O}_X \to \Omega_{X/Y}$. Let $\pi_1, \pi_2 : X \times_Y X \to X$ be the two projections. Then define $d : \mathcal{O}_X \to \Omega_{X/Y}$ on the open set U as follows: $df = \pi_2^* f - \pi_1^* f$. (*Warning:* this is not a morphism of quasicoherent sheaves, although it *is* $\mathcal{O}_Y$-linear.) We'll soon see that this is indeed a derivation, and at the same time see that our new notion of differentials agrees with our old definition on affine open sets, and hence globalizes the definition.

Before we do, let me try to convince you that this is a reasonable definition to make. (This paragraph is informal, and is in no way mathematically rigorous.) Say for example that Y is a point, and X is something smooth. Then the tangent space to $X \times X$ is $T_X \oplus T_X$: $T_{X \times X} = T_X \oplus T_X$. Restrict this to the diagonal Δ, and look at the normal bundle exact sequence:

$$0 \to T_\Delta \to T_{X \times X}|_\Delta \to N_{\Delta, X} \to 0.$$

Now the left morphism sends v to (v, v), so the cokernel can be interpreted as $(v, -v)$. Thus $N_{\Delta, X}$ is isomorphic to T_X. Thus we can turn this on its head: we know how to find the normal bundle (or more precisely the conormal sheaf), and we can use this to define the tangent bundle (or more precisely the cotangent sheaf). (Experts may want to ponder the above paragraph when Y is more general, but where $X \to Y$ is "nice". You may wish to think in the category of manifolds, and let $X \to Y$ be a submersion.)

Let's now see how this works for the special case $\operatorname{Spec} A \to \operatorname{Spec} B$. Then the diagonal $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A \otimes_B A$ corresponds to the ideal I of $A \otimes_B A$ that is the cokernel of the ring map

$$\sum x_i \otimes y_i \to \sum x_i y_i.$$

The derivation is $d : A \to A \otimes_B A$, $a \mapsto da := 1 \otimes a - a \otimes 1$ (taken modulo I^2). (I shouldn't really call this "d" until I've verified that it agrees with our earlier definition, but bear with me.)

Let's check that this satisfies the 3 conditions, i.e. that it is a derivation. Two are immediate: it is linear, vanishes on elements of b. Let's check the Leibniz rule:

$$\begin{aligned} d(aa') - a\, da' - a'\, da &= 1 \otimes aa' - aa' \otimes 1 - a \otimes a' + aa' \otimes 1 - a' \otimes a + a'a \otimes 1 \\ &= -a \otimes a' - a' \otimes a + a'a \otimes 1 + 1 \otimes aa' \\ &= (1 \otimes a - a \otimes 1)(1 \otimes a' - a' \otimes 1) \\ &\in I^2. \end{aligned}$$

Thus by the universal property of $\Omega_{A/B}$, we get a natural morphism $\Omega_{A/B} \to I/I^2$ of A-modules.

1.3. *Theorem. — The natural morphism $f : \Omega_{A/B} \to I/I^2$ induced by the universal property of $\Omega_{A/B}$ is an isomorphism.*

Proof. We'll show this as follows. (i) We'll show that f is surjective, and (ii) we will describe $g : I/I^2 \to \Omega_{A/B}$ such that $g \circ f : \Omega_{A/B} \to \Omega_{A/B}$ is the identity. Both of these steps will be very short. Then we'll be done, as to show $f \circ g$ is the identity, we need only show (by surjectivity of g) that $(f \circ g)(f(a)) = f(a)$, which is true (by (ii) $g \circ f = id$).

(i) For surjectivity, we wish to show that I is generated (modulo I^2) by $a \otimes 1 - 1 \otimes a$ as a runs over the elements of A. This has a one sentence explanation: If $\sum x_i \otimes y_i \in I$, i.e. $\sum x_i y_i = 0$ in A, then $\sum_i x_i \otimes y_i = \sum_i x_i(1 \otimes y_i - y_i \otimes 1)$.

(ii) Define $g : I/I^2 \to \Omega_{A/B}$ by $x \otimes y \mapsto x\, dy$. We need to check that this is well-defined, i.e. that elements of I^2 are sent to 0, i.e. we need that

$$\left(\sum x_i \otimes y_i\right)\left(\sum x'_j \otimes y'_j\right) = \sum_{i,j} x_i x'_j \otimes y_i y'_j \mapsto 0$$

where $\sum_i x_i y_i = \sum x'_j y'_j = 0$. But by the Leibniz rule,

$$\begin{aligned} \sum_{i,j} x_i x'_j\, d(y_i y'_j) &= \sum_{i,j} x_i x'_j y_i\, dy'_j + \sum_{i,j} x_i x'_j y'_j\, dy_i \\ &= \left(\sum_i x_i y_i\right)\left(\sum_j x'_j\, dy'_j\right) + \left(\sum_i x_i\, dy_i\right)\left(\sum_j x'_j y'_j\right) \\ &= 0. \end{aligned}$$

Then $f \circ g$ is indeed the identity, as

$$da \xrightarrow{g} 1 \otimes a - a \otimes 1 \xrightarrow{f} 1\, da - a\, d1 = da$$

as desired. □

We can now use our understanding of how Ω works on affine open sets to state some global results.

1.4. *Exercise.* Suppose $f : X \to Y$ is locally of finite type, and X is locally Noetherian. Show that $\Omega_{X/Y}$ is a coherent sheaf on X.

The relative cotangent exact sequence and the conormal exact sequence for schemes now directly follow.

1.5. *Theorem. — (Relative cotangent exact sequence) Suppose* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be morphisms of schemes. Then there is an exact sequence of quasicoherent sheaves on* X

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

(Conormal exact sequence) Suppose $f : X \to Y$ *morphism of schemes,* $Z \hookrightarrow X$ *closed subscheme of* X*, with ideal sheaf* $\mathcal{I}$*. Then there is an exact sequence of sheaves on* Z*:*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0.$$

Similarly, the sheaf of relative differentials pull back, and behave well under base change.

1.6. *Theorem (pullback of differentials). — (a) If*

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

is a commutative diagram of schemes, there is a natural homomorphism of quasicoherent sheaves on X' $g^*\Omega_{X/Y} \to \Omega_{X'/Y'}$*. An important special case is* $Y = Y'$*.*
(b) (Ω behaves well under base change) If furthermore the above diagram is a tensor diagram (i.e. $X' \cong X \otimes_Y Y'$*) then* $g^*\Omega_{X/Y} \to \Omega_{X'/Y'}$ *is an isomorphism.*

This follows immediately from an Exercise in last day's notes. Part (a) implicitly came up in our earlier discussion of the Riemann-Hurwitz formula.

As a particular case of part (b), the fiber of the sheaf of relative differentials is indeed the sheaf of differentials of the fiber. *Thus this notion indeed glues together the differentials on each fiber.*

2. EXAMPLES

2.1. The projective line. As an important first example, let's consider $\mathbb{P}^1_k$, with the usual projective coordinates x_0 and x_1. As usual, the first patch corresponds to $x_0 \neq 0$, and is of the form $\operatorname{Spec} k[x_{1/0}]$ where $x_{1/0} = x_1/x_0$. The second patch corresponds to $x_1 \neq 0$, and is of the form $\operatorname{Spec} k[x_{0/1}]$ where $x_{0/1} = x_0/x_1$.

Both patches are isomorphic to $\mathbb{A}^1_k$, and $\Omega_{\mathbb{A}^1_k} = \mathcal{O}_{\mathbb{A}^1_k}$. (More precisely, $\Omega_{k[x]/k} = k[x]\ dx$.) Thus $\Omega_{\mathbb{P}^1_k}$ is an invertible sheaf (a line bundle). Now we have classified the invertible sheaves on $\mathbb{P}^1_k$ — they are each of the form $\mathcal{O}(m)$. So which invertible sheaf is $\Omega_{\mathbb{P}^1/k}$?

Let's take a section, $dx_{1/0}$ on the first patch. It has no zeros or poles there, so let's check what happens on the other patch. As $x_{1/0} = 1/x_{0/1}$, we have $dx_{1/0} = -(1/x_{0/1}^2)\ dx_{0/1}$. Thus this section has a double pole where $x_{0/1} = 0$. Hence $\Omega_{\mathbb{P}^1_k/k} \cong \mathcal{O}(-2)$.

Note that the above argument did not depend on k being a field, and indeed we could replace k with any ring A (or indeed with any base scheme).

2.2. A plane curve. Consider next the plane curve $y^2 = x^3 - x$ in $\mathbb{A}^2_k$, where the characteristic of k is not 2. Then the differentials are generated by dx and dy, subject to the constraint that

$$2y\ dy = (3x^2 - 1)\ dx.$$

Thus in the locus where $y \neq 0$, dx is a generator (as dy can be expressed in terms of dx). Similarly, in the locus where $3x^2 - 1 \neq 0$, dy is a generator. These two loci cover the entire curve, as solving $y = 0$ gives $x^3 - x = 0$, i.e. $x = 0$ or ± 1, and in each of these cases $3x^2 - 1 \neq 0$.

Now consider the differential dx. Where does it vanish? Answer: precisely where $y = 0$. You should find this believable from the picture (which I gave in class).

2.3. Exercise: differentials on hyperelliptic curves. Consider the double cover $f : C \to \mathbb{P}^1_k$ branched over $2g+2$ distinct points. (We saw earlier that this curve has genus g.) Then $\Omega_{C/k}$ is again an invertible sheaf. What is its degree? (Hint: let x be a coordinate on one of the coordinate patches of $\mathbb{P}^1_k$. Consider f^*dx on C, and count poles and zeros.) In class I gave a sketch showing that you should expect the answer to be $2g - 2$.

2.4. Exercise: differentials on nonsingular plane curves. Suppose C is a nonsingular plane curve of degree d in $\mathbb{P}^2_k$, where k is algebraically closed. By considering coordinate patches, find the degree of $\Omega_{C/k}$. Make any reasonable simplifying assumption (so that you believe that your result still holds for "most" curves).

Because Ω behaves well under pullback, note that the assumption that k is algebraically closed may be quickly excised:

2.5. *Exercise.* Suppose that C is a nonsingular projective curve over k such that $\Omega_{C/k}$ is an invertible sheaf. (We'll see that for nonsingular curves, the sheaf of differentials is always locally free. But we don't yet know that.) Let $C_{\overline{k}} = C \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. Show that $\Omega_{C_{\overline{k}}/\overline{k}}$ is locally free, and that

$$\deg \Omega_{C_{\overline{k}}/\overline{k}} = \deg \Omega_{C/k}.$$

2.6. Projective space. We next examine the differentials of projective space $\mathbb{P}^n_k$. As projective space is covered by affine open sets of the form $\mathbb{A}^n$, on which the differential form a rank n locally free sheaf, $\Omega_{\mathbb{P}^n_k/k}$ is also a rank n locally free sheaf.

2.7. *Theorem (the Euler exact sequence). — The sheaf of differentials $\Omega_{\mathbb{P}^n_k/k}$ satisfies the following exact sequence*

$$0 \to \Omega_{\mathbb{P}^n_A} \to \mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n_A} \to 0.$$

This is handy, because you can get a hold of Ω in a concrete way. Next day I will give an explicit example, to give you some practice.

I discussed some philosophy behind this theorem. Next day, I'll give a proof, and repeat the philosophy.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 39 AND 40

RAVI VAKIL

CONTENTS

These are notes from both class 39 and class 40.

Today: the Euler exact sequence. Discussion of nonsingular varieties over algebraically closed fields: Bertini's theorem, the Riemann-Hurwitz formula, and the (co)normal exact sequence for nonsingular subvarieties of nonsingular varieties.

We have now established the general theory of differentials, and we are now going to apply it.

1. PROJECTIVE SPACE AND THE EULER EXACT SEQUENCE

We next examine the differentials of projective space $\mathbb{P}^n_k$, or more generally $\mathbb{P}^n_A$ where A is an arbitrary ring. As projective space is covered by affine open sets, on which the differentials form a rank n locally free sheaf, $\Omega_{\mathbb{P}^n_A/A}$ is also a rank n locally free sheaf.

1.1. *Important Theorem (the Euler exact sequence). — The sheaf of differentials $\Omega_{\mathbb{P}^n_A/A}$ satisfies the following exact sequence*

$$0 \to \Omega_{\mathbb{P}^n_A} \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n_A} \to 0.$$

This is handy, because you can get a hold of Ω in a concrete way. Here is an explicit example, to give you practice.

1.2. *Exercise.* Show that $H^1(\mathbb{P}^n_A, T_{\mathbb{P}^n_A}) = 0$. (This later turns out to be an important calculation for the following reason. If X is a nonsingular variety, $H^1(X, T_X)$ parametrizes deformations of the variety. Thus projective space can't deform, and is "rigid".)

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Let's prove the Euler exact sequence. I find this an amazing fact, and while I can prove it, I don't understand in my bones why this is true. Maybe someone can give me some enlightenment.

Proof. (What's really going on in this proof is that we consider those differentials on $\mathbb{A}_A^{n+1} \setminus \{0\}$ that are pullbacks of differentials on $\mathbb{P}_A^n$.)

I'll describe a map $\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}$, and later identify the kernel with $\Omega_{X/Y}$. The map is given by

$$(s_0, s_1, \ldots, s_n) \mapsto x_0 s_0 + x_1 s_1 + \cdots + x_n s_n.$$

Note that this is a degree 1 map.

Now I have to identify the kernel of this map with differentials, and I can do this on each open set (so long as I do it in a way that works simultaneously for each open set). So let's consider the open set U_0, where $x_0 \neq 0$, and we have coordinates $x_{j/0} = x_j/x_0$ ($1 \leq j \leq n$). Given a differential

$$f_1(x_{1/0}, \ldots, x_{n/0})\ dx_{1/0} + \cdots + f_n(x_{1/0}, \ldots, x_{n/0})\ dx_{n/0}$$

we must produce $n+1$ sections of $\mathcal{O}(-1)$. As motivation, let me just look at the first term, and pretend that the projective coordinates are actual coordinates.

$$\begin{aligned} f_1\ dx_{1/0} &= f_1\ d(x_1/x_0) \\ &= f_1 \frac{x_0\ dx_1 - x_1\ dx_0}{x_0^2} \\ &= -\frac{x_1}{x_0^2} f_1\ dx_0 + \frac{f_1}{x_0}\ dx_1 \end{aligned}$$

Note that x_0 times the "coefficient of dx_0" plus x_1 times the "coefficient of dx_1" is 0, and also both coefficients are of homogeneous degree -1. Motivated by this, we take:

$$f_1\ dx_{1/0} + \cdots + f_n\ dx_{n/0} \mapsto \left(-\frac{x_1}{x_0^2} f_1 - \cdots - \frac{x_n}{x_0^2} f_n, \frac{f_1}{x_0}, \frac{f_2}{x_0}, \cdots, \frac{f_n}{x_0} \right) \tag{1}$$

Note that over U_0, this indeed gives an injection of $\Omega_{\mathbb{P}_A^n}$ to $\mathcal{O}(-1)^{\oplus(n+1)}$ that surjects onto the kernel of $\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_X$ (if $(g_0, \ldots, g_n)$ is in the kernel, take $f_i = x_0 g_i$ for $i > 0$).

Let's make sure this construction, applied to two different coordinate patches (say U_0 and U_1) gives the same answer. (This verification is best ignored on a first reading.) Note that

$$\begin{aligned} f_1\ dx_{1/0} + f_2\ dx_{2/0} + \cdots &= f_1\ d\frac{1}{x_{0/1}} + f_2\ d\frac{x_{2/1}}{x_{0/1}} + \cdots \\ &= -\frac{f_1}{x_{0/1}^2} dx_{0/1} + \frac{f_2}{x_{0/1}} dx_{2/1} - \frac{f_2 x_{2/1}}{x_{0/1}^2} dx_{0/1} + \cdots \\ &= -\frac{f_1 + f_2 x_{2/1} + \cdots}{x_{0/1}^2} dx_{0/1} + \frac{f_2 x_1}{x_0} dx_{2/1} + \cdots . \end{aligned}$$

Under this map, the $dx_{2/1}$ term goes to the second factor (where the factors are indexed 0 through n) in $\mathcal{O}(-1)^{\oplus(n+1)}$, and yields f_2/x_0 as desired (and similarly for $dx_{j/1}$ for $j > 2$).

Also, the $dx_{0/1}$ term goes to the "zero" factor, and yields

$$(\sum_{j=1}^{n} f_i(x_i/x_1)/(x_0/x_1)^2)/x_1 = f_i x_i/x_0^2$$

as desired. Finally, the "first" factor must be correct because the sum over i of x_i times the ith factor is 0. □

Generalizations of the Euler exact sequence are quite useful. We won't use them later this year, so I'll state them without proof. Note that the argument applies without change if $\operatorname{Spec} A$ is replaced by an arbitrary base scheme. The Euler exact sequence further generalizes in a number of ways. As a first step, suppose V is a rank $n+1$ locally free sheaf (or vector bundle) on a scheme X. Then $\Omega_{\mathbb{P}V/X}$ sits in an Euler exact sequence:

$$0 \to \Omega_{\mathbb{P}V/X} \to \mathcal{O}(-1) \otimes V^\vee \to \mathcal{O}_X \to 0$$

If $\pi : \mathbb{P}V \to X$, the map $\mathcal{O}(-1) \otimes V^\vee \to \mathcal{O}_X$ is induced by $V^\vee \otimes \pi_* \mathcal{O}(1) \cong (V^\vee \otimes V) \otimes \mathcal{O}_X \to \mathcal{O}_X$, where $V^\vee \otimes V \to A$ is the trace map.

For another generalization, fix a base field, and let $G(m, n+1)$ be the space of vector spaces of dimension m in an $(n+1)$-dimensional vector space V. (This is called the *Grassmannian*. We have not shown that this is actually a variety in any natural way, but it is. The case $m = 1$ is $\mathbb{P}^n$.) Then over $G(m, n+1)$ we have a short exact sequence of locally free sheaves

$$0 \to S \to V \otimes \mathcal{O}_{G(m,n+1)} \to Q \to 0$$

where $V \otimes \mathcal{O}_{G(m,n+1)}$ is a trivial bundle, and S is the "universal subbundle" (such that over a point $[V' \subset V]$ of the Grassmannian $G(m, n+1)$, $S|_{[V' \subset V]}$ is V if you can see what that means). Then

$$\Omega_{G(m,n+1)/k} \cong \underline{\operatorname{Hom}}(Q, S). \tag{2}$$

1.3. *Exercise.* In the case of projective space, $m = 1$, $S = \mathcal{O}(-1)$. Verify (2) in this case.

This Grassmannian fact generalizes further to Grassmannian bundles.

2. Varieties over algebraically closed fields

We'll now discuss differentials in the case of interest to most people: varieties over algebraically closed fields. I'd like to begin with a couple of remarks.

2.1. *Remark: nonsingularity may be checked at closed points.* Recall from the first quarter a deep fact about regular local rings that we haven't proved: Any localization of a regular local ring at a prime is again regular local ring. (For a reference, see Matsumura's *Commutative Algebra*, p. 139.) I'm going to continue to use this without proof. It is possible I'll write up a proof later. But in any case, if this bothers you, you could re-define nonsingularity of locally finite type schemes over fields to be what other people call "nonsingularity at closed points", and the results of this section will hold.

2.2. *Remark for non-algebraically closed people.* Even if you are interested in non-algebraically closed fields, this section should still be of interest to you. In particular, if X is a variety over a field k, and $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, then $X_{\overline{k}}$ nonsingular implies that X is nonsingular. (You may wish to prove this yourself. By Remark 2.1, it suffices to check at closed points.) *Possible exercise.* In fact if k is separably closed, then $X_{\overline{k}}$ is nonsingular if and only if X is nonsingular, but this is a little bit harder.

Suppose for the rest of this section that X is a pure n-dimensional locally finite type scheme over an algebraically closed field k (e.g. a k-variety).

2.3. *Proposition.* — *$\Omega_{X/k}$ is locally free of rank n if and only if X is nonsingular.*

Proof. By Remark 2.1, it suffices to prove that $\Omega_{X/k}$ is locally free of rank n if and only if the closed points of X is nonsingular. Now $\Omega_{X/k}$ is locally free of rank n if and only if its fibers at all the closed points are rank n (recall that fibers jump in closed subsets). As the fiber of the cotangent sheaf is canonically isomorphic to the Zariski tangent space at closed points (done earlier), the Zariski tangent space at every closed point must have dimension n, i.e. the closed points are all nonsingular. □

Using this Proposition, we can get a new result using a neat trick.

2.4. *Theorem.* — *If X is integral, there is an dense open subset U of X which is nonsingular.*

Proof. The $n = 0$ case is immediate, so we assume $n > 0$.

We will show that the rank at the generic point is n. Then by uppersemicontinuity of the rank of a coherent sheaf (done earlier), it must be n in an open neighborhood of the generic point, and we are done by Proposition 2.3.

We thus have to check that if K is the fraction field of a dimension n integral finite-type k-scheme, i.e. if K is a transcendence degree n extension of k, then $\Omega_{K/k}$ is an n-dimensional vector space. But any transcendence degree $n > 1$ extension is separably generated: we can find n algebraically independent elements of K over k, say $x_1, \dots, x_n$, such that $K/k(x_1, \dots, x_n)$ is separable. (This is a fact about transcendence theory.) Then $\Omega_{K/k}$ is generated by $dx_1, \dots, dx_n$ (as $dx_1, \dots, dx_n$ generate $\Omega_{k(x_1,\dots,x_n)/k}$, and any element of K is separable over $k(x_1, \dots, x_n)$ — this is summarized most compactly using the affine form of the relative cotangent sequence). □

2.5. *Bertini's Theorem.* — *Suppose X is a nonsingular closed subvariety of $\mathbb{P}^n_k$ (where the standing hypothesis for this section, that k is algebraically closed, holds). Then there is an open subset of hyperplanes H of $\mathbb{P}^n_k$ such that H doesn't contain any component of X, and the scheme $H \cap X$ is a nonsingular variety. More precisely, this is an open subset of the dual projective space $\mathbb{P}^{n\vee}_k$. In particular, there exists a hyperplane H in $\mathbb{P}^n_k$ not containing any component of X such that the scheme $H \cap X$ is also a nonsingular variety.*

(We've already shown in our section on cohomology that if X is connected, then $H \cap X$ is connected.)

We may have used this before to show the existence of nonsingular curves of any genus, for example, although I don't think we did. (We discussed Bertini in class 35, p. 4.)

Note that this implies that a general degree $d > 0$ hypersurface in $\mathbb{P}^n_k$ also intersects X in a nonsingular subvariety of codimension 1 in X: replace $X \hookrightarrow \mathbb{P}^n$ with the composition $X \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ where the latter morphism is the dth Veronese map.

Proof. In order to keep the language of the proof as clean as possible, I'll assume X is irreducible, but essentially the same proof applies in general.

The central idea of the proof is quite naive and straightforward. We'll describe the hyperplanes that are "bad", and show that they form a closed subset of dimension at most $n-1$ of $\mathbb{P}^{n\vee}_k$, and hence that the complement is a dense open subset. More precisely, we will define a projective variety $Y \subset X \times \mathbb{P}^{n\vee}_k$ that will be:

$$Y = \{(p \in X, H \subset \mathbb{P}^n_k) : p \in H, p \text{ is a singular point of } H \cap X, \text{ or } X \subset H\}$$

We will see that $\dim Y \leq n - 1$. Thus the image of Y in $\mathbb{P}^{n\vee}_k$ will be a closed subset (the image of a closed subset by a projective hence closed morphism!), of dimension of $n - 1$, and its complement is open.

We'll show that Y has dimension $n - 1$ as follows. Consider the map $Y \to X$, sending (p, H) to p. Then a little thought will convince you that there is a $(n - \dim X - 1)$-dimensional family of hyperplanes through $p \in X$ such that $X \cap H$ is singular at p, or such that X is contained in H. (Those two conditions can be summarized quickly as: H contains the "first-order formal neighborhood of p in X", $\operatorname{Spec} \mathcal{O}_{X,p}/\mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$.) Hence we expect Y to be a projective bundle, whose fibers are dimension $n - \dim X - 1$, and hence that Y has dimension at most $\dim X + (n - \dim X - 1) = n - 1$. In fact this is the case, but we'll show a little less (e.g. we won't show that $Y \to X$ is a projective bundle) because we don't need to prove this full statement to complete our proof of Bertini's theorem.

Let's put this strategy into action. We first define Y more precisely, in terms of equations on $\mathbb{P}^n \times \mathbb{P}^{n\vee}$, where the coordinates on $\mathbb{P}^n$ are $x_0, \ldots x_n$, and the dual coordinates on $\mathbb{P}^{n\vee}$ are $a_0, \ldots, a_n$. Suppose X is cut out by $f_1, \ldots, f_r$. (We will soon verify that this definition of Y is independent of these equations.) Then we take these equations as some of the defining equations of Y. (So far we have defined the subscheme $X \times \mathbb{P}^{n\vee}$.) We also add the equation $a_0 x_0 + \cdots + a_n x_n = 0$. (So far we have described the subscheme of $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ corresponding to points (p, H) where $p \in X$ and $p \in H$.) Note that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}$$

has corank equal to $\dim X$ at all closed points of X — this is precisely the Jacobian condition for nonsingularity (class 12, p. 3, 1.6). (Although we won't use this fact, in fact it has that corank $\dim X$ everywhere on X. Reason: the locus where the corank jumps is a closed locus, as this is described by equations, namely determinants of minors. Thus as the corank is constant at all closed points, it is constant everywhere.) We then require that the Jacobian matrix with a new row $(a_0, \cdots, a_n)$ has corank $\geq \dim X$ (hence $= \dim X$). This is cut out by equations (determinants of minors). By the Jacobian description of the Zariski tangent space, this condition encodes the requirement that the Zariski tangent space of $H \cap X$ at p has dimension precisely $\dim X$, which is $\dim H \cap X + 1$ (i.e. $H \cap X$ is singular at p) if H does not contain X, or if H contains X. This is precisely the notion that we hoped to capture.

Before getting on with our proof, let's do an example to convince ourselves that this algebra is describing the geometry we desire. Consider the plane conic $x_0^2 - x_1^2 - x_2^2 = 0$ over a field of characteristic not 2, which I picture as the circle $x^2 + y^2 = 1$ from the real picture in the chart U_0. (At this point I drew a picture.) Consider the point $(1, 1, 0)$, corresponding to $(1, 0)$ on the circle. We expect the tangent line in the affine plane to be $x = 1$, which should correspond to $x_0 - x_1 = 0$. Let's see what the algebra gives us. The Jacobian matrix is $\begin{pmatrix} 2x_0 & -2x_1 & -2x_2 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 0 \end{pmatrix}$, which indeed has rank 1 as expected. Our recipe asks that the matrix $\begin{pmatrix} 2 & -2 & 0 \\ a_0 & a_1 & a_2 \end{pmatrix}$ have rank 1, which means that $(a_0, a_1, a_2) = (a_0, -a_0, 0)$, and also that $a_0 x_0 + a_1 x_1 + a_2 x_2 = 0$, which is precisely what we wanted!

Returning to our construction, we can see that the Y just described is independent of the choice of $f_1, \ldots, f_r$ (although we won't need this fact).

Here's why. It suffices to show that if we add in a redundant equation (some homogeneous f_0 that is a $k[x_0, \ldots, x_n]$-linear combination of the f_i), we get the same Y (as then if we had a completely different set of f's, we could add them in one at a time, and then remove the old f's one at a time). If we add in a redundant equation, then that row in the Jacobian matrix will be a $k[x_0, \ldots, x_n]$-linear combination of other rows, and thus the rank remains unchanged. (There is a slight issue I am glossing over here — f_0 may vanish on Y despite not being a linear combination of $f_1, \ldots, f_n$.)

We'll next show that $\dim Y = n - 1$. For each $p \in X$, let Z_p be the locus of hyperplanes containing p, such that $H \cap X$ is singular at p, or else contains all of X; what is the dimension of Z_p? (For those who have heard of these words: what is the dimension of the locus of hyperplanes containing a first-order formal neighborhood of p in X?) Suppose $\dim X = d$. Then this should impose $d + 1$ conditions on hyperplanes. This means that it is a codimension $d + 1$, or dimension $n - d - 1$, projective space. Thus we should expect $Y \to X$ to be a projective bundle of relative dimension $n - d - 1$ over a variety of dimension d, and hence that $\dim Y = n - 1$. For convenience, I'll verify a little less: that $\dim Y \leq n - 1$.

Suppose Y has dimension N. Let $H_1, \ldots, H_d$ be general hyperplanes such that $H_1 \cap \cdots \cap H_d \cap X$ is a finite set of points (this was an exercise from long ago, class 31, ex. 1.5, p. 4). Then if $\pi : Y \to X$ is the projection to X, then (using Krull's Principal Ideal Theorem)

$$n - d - 1 = \dim Y \cap \pi^* H_1 \cap \cdots \cap \pi^* H_d \geq \dim Y - d$$

from which $\dim Y \leq n - 1$. $\square$

2.6. *Exercise-.* Show that Bertini's theorem still holds even if X is singular in dimension 0. (This isn't that important.)

2.7. *Remark..* The image in $\mathbb{P}^n$ tends to be a divisor. This is classically called the *dual variety*. The following exercise will give you some sense of it.

2.8. *Exercise.* Suppose $C \subset \mathbb{P}^2$ is a nonsingular conic over a field of characteristic not 2. Show that the dual variety is also a conic. (More precisely, suppose C is cut out by $f(x_0, x_1, x_2) = 0$. Show that $\{(a_0, a_1, a_2) : a_0 x_0 + a_1 x_1 + a_2 x_2 = 0\}$ is cut out by a quadratic equation.) Thus for example, through a general point in the plane, there are two tangents to C. (The points on a line in the dual plane corresponds to those lines through a point of the original plane.)

We'll soon find the degree of the dual to a degree d curve (after we discuss the Riemann-Hurwitz formula), at least modulo some assumptions.

2.9. The Riemann-Hurwitz formula.

We're now ready to discuss and prove the Riemann-Hurwitz formula. We continue to work over an algebraically closed field k. Everything below can be mildly modified to work for a perfect field, e.g. any field of characteristic 0, and I'll describe this at the end of the discussion (Remark 2.17).

Definition (separable morphisms). A finite morphism between integral schemes $f : X \to Y$ is said to be *separable* if it is dominant, and the induced extension of function fields $FF(X)/FF(Y)$ is a separable extension. (Similarly, a generically finite morphism is *generically separable* if it is dominant, and the induced extension of function fields is a separable extension. We may not use this notion.) Note that this comes for free in characteristic 0.

2.10. *Proposition. — If $f : X \to Y$ is a finite separable morphism of nonsingular integral curves, then we have an exact sequence*

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0.$$

Proof. We have right-exactness by the relative cotangent sequence, so we need to check only that $\phi : f^*\Omega_{Y/k} \to \Omega_{X/k}$ is injective. Now $\Omega_{Y/k}$ is an invertible sheaf on Y, so $f^*\Omega_{Y/k}$ is an invertible sheaf on X. Thus it has no torsion subsheaf, so we need only check that ϕ is an inclusion at the generic point. We thus tensor with $\mathcal{O}_\eta$ where η is the generic point of X. This is an exact functor (it is localization), and $\mathcal{O}_\eta \otimes \Omega_{X/Y} = 0$ (as $FF(X)/FF(Y)$ is a separable by hypothesis, and Ω for separable field extensions is 0 by Ex. 2.10, class 37, which was also Ex. 4, problem set 17). Also, $\mathcal{O}_\eta \otimes f^*\Omega_{Y/k}$ and $\mathcal{O}_\eta \otimes \Omega_{X/k}$ are both one-dimensional $\mathcal{O}_\eta$-vector spaces (they are the stalks of invertible sheaves at the generic point). Thus by considering

$$\mathcal{O}_\eta \otimes f^*\Omega_{Y/k} \to \mathcal{O}_\eta \otimes \Omega_{X/k} \to \mathcal{O}_\eta \otimes \Omega_{X/Y} \to 0$$

(which is

$$\mathcal{O}_\eta \to \mathcal{O}_\eta \to 0 \to 0)$$

we see that $\mathcal{O}_\eta \otimes f^*\Omega_{Y/k} \to \mathcal{O}_\eta \otimes \Omega_{X/k}$ is injective, and thus that $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is injective. □

2.11. It is worth noting what goes wrong for non-separable morphisms. For example, suppose k is a field of characteristic p, consider the map $f : \mathbb{A}^1_k = \operatorname{Spec} k[t] \to \mathbb{A}^1_k = \operatorname{Spec} k[u]$ given by $u = t^p$. Then $\Omega_{\mathbb{A}^1_k/k}$ is the trivial invertible sheaf generated by dt. As another (similar but different) example, if $K = k(x)$ and $K' = K(x^p)$, then the inclusion $K' \hookrightarrow K$ induces $f : \operatorname{Spec} K[t] \to \operatorname{Spec} K'[t]$. Once again, Ω_f is an invertible sheaf, generated by dx (which in this case is pulled back from $\Omega_{K/K'}$ on $\operatorname{Spec} K$). In both of these cases, we have maps from one affine line to another, and there are vertical tangent vectors.

2.12. The sheaf $\Omega_{X/Y}$ on the right side of Proposition 2.10 is a coherent sheaf not supported at the generic point. Hence it is supported at a finite number of points. These are called the *ramification points* (and the images downstairs are called the *branch points*). I drew a picture here.

Let's check out what happens at closed points. We have two discrete valuation rings, say $\operatorname{Spec} A \to \operatorname{Spec} B$. I've assumed that we are working over an algebraically closed field k, so this morphism $B \to A$ induces an isomorphism of residue fields (with k). Suppose their uniformizers are s and t respectively, with $t \mapsto us^n$ where u is a unit of A. Then

$$dt = d(us^n) = uns^{n-1}\, ds + s^n\, du.$$

This vanishes to order at least $n-1$, and precisely $n-1$ if n doesn't divide the characteristic. The former case is called *tame* ramification, and the latter is called *wild* ramification. We call this order the *ramification order* at this point of X.

Define the *ramification divisor* on X as the sum of all points with their corresponding ramifications (only finitely many of which are non-zero). The image of this divisor on Y is called the *branch divisor*.

2.13. *Straightforward exercise: interpreting the ramification divisor in terms of number of preimages.* Suppose all the ramification above $y \in Y$ is tame. Show that the degree of the branch divisor at y is $\deg(f : X \to Y) - \#f^{-1}(y)$. Thus the multiplicity of the branch divisor counts the extent to which the number of preimages is less than the degree.

2.14. *Proposition. — Suppose R is the ramification divisor of $f : X \to Y$. Then $\Omega_X(-R) \cong f^*\Omega_Y$.*

Note that we are making no assumption that X or Y is projective.

Proof. This says that we can interpret the invertible sheaf $f^*\Omega_Y$ over an open set of X as those differentials on X vanishing along the ramification divisor. But that is the content of Proposition 2.10. □

Then the Riemann-Hurwitz formula follows!

2.15. *Theorem (Riemann-Hurwitz). — Suppose* $f : X \to Y$ *is a finite separable morphism of curves. Let* $n = \deg f$. *Then* $2g(X) - 2 = n(2g(Y) - 2) + \deg R$.

Note that we now need the projective hypotheses in order to take degrees of invertible sheaves.

Proof. This follows by taking the degree of both sides of Proposition 2.14 (and using the fact that the pullback of a degree d line bundle by a finite degree n morphism is dn, which was an earlier exercise, Ex. 3.1, class 29, p. 3, or Ex. 2, problem set 13). $\square$

2.16. *Exercise: degree of dual curves.* Describe the degree of the dual to a nonsingular degree d plane curve C as follows. Pick a general point $p \in \mathbb{P}^2$. Find the number of tangents to C through p, by noting that projection from p gives a degree d map to $\mathbb{P}^1$ (why?) by a curve of known genus (you've calculated this before), and that ramification of this cover of $\mathbb{P}^1$ corresponds to a tangents through p. (Feel free to make assumptions, e.g. that for a general p this branched cover has the simplest possible branching — this should be a back-of-an-envelope calculation.)

2.17. *Remark: Riemann-Hurwitz over perfect fields.* This discussion can be extended to work when the base field is not algebraically closed; perfect will suffice. The place we assumed that the base field was algebraically closed was after we reduced to understanding the ramification of the morphism of the spectrum of one discrete valuation ring over our base field k to the spectrum of another, and we assumed that this map induced an isomorphism of residue fields. In general, it can be a finite extension. Let's analyze this case explicitly. Consider a map $\operatorname{Spec} A \to \operatorname{Spec} B$ of spectra of discrete valuation rings, corresponding to a ring extension $B \to A$. Let s be the uniformizer of A, and t the uniformizer of B. Let $\mathfrak{m}$ be the maximal ideal of A, and $\mathfrak{n}$ the maximal ideal of A. Then as $A/\mathfrak{m}$ is a finite extension of $B/\mathfrak{n}$, it is generated over $B/\mathfrak{n}$ by a single element (we're invoking here the theorem of the primitive element, and we use the "perfect" assumption here). Let s' be any lift of this element of $A/\mathfrak{m}$ to A. Then A is generated over B by s and s', so $\Omega_{A/B}$ is generated by ds and ds'. The contribution of ds is as described above. You can show that $ds' = 0$. Thus all calculations above carry without change, except for the following.

(i) We have to compute the degree of the ramification divisor appropriately: we need to include as a factor the degree of the field extension of the residue field of the point on the *source* (over k).

(ii) Exercise 2.13 doesn't work, but can be patched by replacing $\#f^{-1}(y)$ with the number of *geometric* preimages.

As an example of what happens differently in (ii), consider the degree 2 finite morphism $X = \operatorname{Spec} \mathbb{Z}[i] \to Y = \operatorname{Spec} \mathbb{Z}$. We can compute $\Omega_{\mathbb{Z}[i]/\mathbb{Z}}$ directly, as $\mathbb{Z}[i] \cong \mathbb{Z}[x]/(x^2+1)$: $\Omega_{\mathbb{Z}[i]/\mathbb{Z}} \cong \mathbb{Z}[i]dx/(2dx)$. In other words, it is supported at the prime $(1+i)$ (the unique prime above $[(2)] \in \operatorname{Spec} \mathbb{Z}$). However, the number of preimages of points in $\operatorname{Spec} \mathbb{Z}$ is not always 2 away from the point $[(2)]$; half the time (including, for example, over $[(3)]$) there is one point, but the field extension is separable.

2.18. *Exercise (aside): Artin-Schreier covers.* In characteristic 0, the only connected unbranched cover of $\mathbb{A}^1$ is the isomorphism $\mathbb{A}^1 \xrightarrow{\sim} \mathbb{A}^1$; that was an earlier example/exercise, when we discussed Riemann-Hurwitz the first time. In positive characteristic, this needn't be true, because of wild ramification. Show that the morphism corresponding to $k[x] \to k[x,y]/(y^p - x^p - y)$ is such a map. (Once the theory of the algebraic fundamental group is developed, this translates to: "$\mathbb{A}^1$ is not simply connected in characteristic p.")

2.19. The conormal exact sequence for nonsingular varieties.

Recall the conormal exact sequence. Suppose $f : X \to Y$ morphism of schemes, $Z \hookrightarrow X$ closed subscheme of X, with ideal sheaf $\mathcal{I}$. Then there is an exact sequence of sheaves on Z:

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0.$$

I promised you that in good situations this is exact on the left as well, as our geometric intuition predicts. Now let $Z = \operatorname{Spec} k$ (where $k = \overline{k}$), and Y a nonsingular k-variety, and $X \subset Y$ an irreducible closed subscheme cut out by the quasicoherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$.

2.20. *Theorem (conormal exact sequence for nonsingular varieties). — X is nonsingular if and only if (i) $\Omega_{X/k}$ is locally free, and (ii) the conormal exact sequence is exact on the left also:*

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0.$$

Moreover, if Y is nonsingular, then $\mathcal{I}$ is locally generated by $\operatorname{codim}(X,Y)$ elements, and $\mathcal{I}/\mathcal{I}^2$ is a locally free of rank $\operatorname{codim}(X,Y)$.

This latter condition is the definition of something being a *local complete intersection* in a nonsingular scheme.

You can read a proof of this in Hartshorne II.8.17. I'm not going to present it in class, as we'll never use it. The only case I've ever seen used is the implication that if X is nonsingular, then (i) and (ii) hold; and we've already checked (i). This implication (that in the case of a nonsingular subvariety of a nonsingular variety, the conormal and hence normal exact sequence is exact) is very useful for relating the differentials on a nonsingular subvariety to the normal bundle.

The real content is that in the case of a nonsingular subvariety of a nonsingular variety, the conormal exact sequence is exact on the left as well, and in this nice case we have a short exact sequence of locally free sheaves (vector bundles). By dualizing, i.e. applying $\underline{\operatorname{Hom}}(\cdot, \mathcal{O}_X)$, we obtain the *normal exact sequence*

$$0 \to T_{X/k} \to T_{Y/k} \to \mathcal{N}_{X/Y} \to 0$$

which is very handy. Note that dualizing an exact sequence will give you a left-exact sequence in general, but dualizing an exact sequence of locally free sheaves will always be locally free. (In fact, all you need is that the third term is locally free. I could make this an exercise; it may also follow if I define Ext soon after defining Tor, as an exercise.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 41 AND 42

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CONTENTS

Today: Flatness; Tor**; ideal-theoretic characterization of flatness; for coherent modules over a Noetherian local ring flat = free, flatness over a nonsingular curve.**

1. INTRODUCTION TO FLATNESS

We come next to the important concept of flatness. This topic is also not a hard topic, and we could have dealt with it as soon as we had discussed quasicoherent sheaves and morphisms. But it is an intuitively unexpected one, and the algebra and geometry are not obviously connected, so we've left it for relatively late. It is answer to many of your geometric prayers, but you just haven't realized it yet.

The notion of flatness apparently was first defined in Serre's landmark "GAGA" paper.

Here are some of the reasons it is a good concept. We would like to make sense of the notion of "fibration" in the algebraic category (i.e. in algebraic geometry, as opposed to differential geometry), and it turns out that flatness is essential to this definition. It turns out that flat is the right algebraic version of a "nice" or "continuous" family, and this notion is more general than you might think. For example, the double cover $\mathbb{A}^1 \to \mathbb{A}^1$ over an algebraically closed field given by $y \mapsto x^2$ is a flat family, which we interpret as two points coming together to a fat point. The fact that the degree of this map always is 2 is a symptom of how this family is well-behaved. Another key example is that of a family of smooth curves degenerating to a nodal curve, that I sketched on the board in class. One can prove things about smooth curves by first proving them about the nodal curve, and then showing that the result behaves well in flat families. In general, we'll see that certain things behave well in nice families, such as cohomology groups (and even better Euler characteristics) of fibers.

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There is a second flavor of prayer that is answered. It would be very nice if tensor product (of quasicoherent sheaves, or of modules over a ring) were an exact functor, and certain statements of results and proofs we have seen would be cleaner if this were true. Those modules for which tensoring is always exact are flat (this will be the definition!), and hence for flat modules (or quasicoherent sheaves, or soon, morphisms) we'll be able to get some very useful statements. A flip side of that is that exact sequences of *flat* modules remain exact when tensored with *any* other module.

In this section, we'll discuss flat morphisms. When introducing a new notion, I prefer to start with a number of geometric examples, and figure out the algebra on the fly. In this case, because there is enough algebra, I'll instead discuss the algebra at some length and then later explain why you care geometrically. This will require more patience than usual on your part.

2. Algebraic definition and easy facts

Many facts about flatness are easy or immediate, and a few are tricky. I'm going to try to make clear which is which, to help you remember the easy facts and the idea of proof for the harder facts.

The definition of a *flat A-module* is very simple. Recall that if

$$0 \to N' \to N \to N'' \to 0 \tag{1}$$

is a short exact sequence of A-modules, and M is another A-module, then

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact. In other words, $M \otimes_A$ is a right-exact functor. We say that M *is a flat A-module* if $M \otimes_A$ is an exact functor, i.e. if for all exact sequences (1),

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact as well.

Exercise. If $N' \to N \to N''$ is exact and M is a flat A-module, show that $M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$ is exact. Hence *any* exact sequence of A-modules remains exact upon tensoring with M. (We've seen things like this before, so this should be fairly straightforward.)

We say that *a ring homomorphism* $B \to A$ *is flat* if A is flat as a B-module. (We don't care about the algebra structure of A.)

Here are two key examples of flat ring homomorphisms:

(i) free modules A-modules are clearly flat.
(ii) Localizations are flat: Suppose S is a multiplicative subset of B. Then $B \to S^{-1}B$ is a flat ring morphism.

Exercise. Verify (ii). We have used this before: localization is an exact functor.

Here is a useful way of recognizing when a module is *not* flat. Flat modules are torsion-free. More precisely, if x is a non-zero-divisor of A, and M is a flat A-module, then $M \xrightarrow{\times x} M$ is injective. Reason: apply the exact functor $M \otimes_A$ to the exact sequence $0 \longrightarrow A \xrightarrow{\times x} A$.

We make some quick but important observations:

2.1. *Proposition (flatness is a stalk/prime-local property). — An A-module M is flat if and only if $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module for all primes $\mathfrak{p}$.*

Proof. Suppose first that M is flat. Given any exact sequence of $A_\mathfrak{p}$-modules (1),

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact too. But $M \otimes_A N$ is canonically isomorphic to $M \otimes_{A_\mathfrak{p}} N_\mathfrak{p}$ (exercise: show this!), so $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module.

Suppose next that M is *not* flat. Then there is some short exact sequence (1) that upon tensoring with M becomes

$$0 \to K \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0 \tag{2}$$

where $K \neq 0$ is the kernel of $M \otimes_A N' \to M \otimes_A N$. Then as $K \neq 0$, K has non-empty support, so there is some prime $\mathfrak{p}$ such that $K_\mathfrak{p} \neq 0$. Then

$$0 \to N'_\mathfrak{p} \to N_\mathfrak{p} \to N''_\mathfrak{p} \to 0 \tag{3}$$

is a short exact sequence of $A_\mathfrak{p}$-modules (recall that localization is exact — see (ii) before the statement of the Proposition), but is no longer exact upon tensoring (over $A_\mathfrak{p}$) with $M_\mathfrak{p}$ (as

$$0 \to K_\mathfrak{p} \to M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N'_\mathfrak{p} \to M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} \to M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N''_\mathfrak{p} \to 0 \tag{4}$$

is exact). (Here we use that localization commutes with tensor product.) □

2.2. *Proposition (flatness is preserved by change of base ring). — If M flat A-module, $A \to B$ is a homomorphism, then $M \otimes_A B$ is a flat B-module.*

Proof. Exercise. □

2.3. *Proposition (transitivity of flatness). — If B is a flat A-algebra, and M is B-flat, then it is also A-flat.*

Proof. Exercise. (Hint: consider the natural isomorphism $(M \otimes_A B) \otimes_B \cdot \cong M \otimes_B (B \otimes_A \cdot)$.) □

The extension of this notion to schemes is straightforward.

2.4. *Definition: flat quasicoherent sheaf.* We say that a quasicoherent sheaf $\mathcal{F}$ on a scheme X is flat (over X) if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module. In light of Proposition 2.1, we can check this notion on affine open cover of X.

2.5. *Definition: flat morphism.* Similarly, we say that a morphism of schemes $\pi : X \to Y$ is flat if for all $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,\pi(x)}$-module. Again, we can check locally, on maps of affine schemes.

We can combine these two definitions into a single definition.

2.6. *Definition: flat quasicoherent sheaf over some base.* Suppose $\pi : X \to Y$ is a morphism of schemes, and $\mathcal{F}$ is a quasicoherent sheaf on X. We say that $\mathcal{F}$ *is flat over* Y if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,\pi(x)}$-module.

Definitions 2.4 and 2.5 correspond to the cases $X = Y$ and $\mathcal{F} = \mathcal{O}_X$ respectively.

This definition can be extended without change to the category of ringed spaces, but we won't need this.

All of the Propositions above carry over naturally. For example, flatness is preserved by base change. (More explicitly: suppose $\pi : X \to Y$ is a morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X, flat over Y. If $Y' \to Y$ is any morphism, and $p : X \times_Y Y' \to X$ is the projection, then $p^*\mathcal{F}$ is flat over Y'.) Also, flatness is transitive. (More explicitly: suppose $\pi : X \to Y$ and $\mathcal{F}$ is a quasicoherent sheaf on X, flat over Y. Suppose also that $\psi : Y \to Z$ is a flat morphism. Then $\mathcal{F}$ is flat over Z.)

We also have other statements easily. For example: open immersions are flat.

2.7. *Exercise.* If X is a scheme, and η is the generic point for an irreducible component, show that the natural morphism $\operatorname{Spec} \mathcal{O}_{X,\eta} \to X$ is flat. (Hint: localization is flat.)

We earlier proved the following important fact, although we did not have the language of flatness at the time.

2.8. *Theorem (cohomology commutes with flat base change). — Suppose*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a fiber diagram, and f (and thus f') is quasicompact and separated (so higher pushforwards exist). Suppose also that g is flat, and $\mathcal{F}$ is a quasicoherent sheaf on X. Then the natural morphisms $g^*R^if_*\mathcal{F} \to R^if'_*(g^*\mathcal{F})$ are isomorphisms.

A special case that is often useful is the case where Y' is the generic point of a component of Y. In other words, in light of Exercise 2.7, the stalk of the higher pushforward of

$\mathcal{F}$ at the generic point is the cohomology of $\mathcal{F}$ on the fiber over the generic point. This is a first example of something important: understanding cohomology of (quasicoherent sheaves on) fibers in terms of higher pushforwards. (We would certainly hope that higher pushforwards would tell us something about higher cohomology of fibers, but this is certainly not a priori clear!)

(I might dig up the lecture reference later, but I'll tell you now where proved it: where we described this natural morphism, I had a comment that if we had exactness of tensor product, them morphisms would be an isomorphism.)

We will spend the rest of our discussion on flatness as follows. First, we will ask ourselves: what are the flat modules over particularly nice rings? More generally, how can you check for flatness? And how should you picture it geometrically? We will then prove additional facts about flatness, and using flatness, answering the essential question: "why do we care?"

2.9. *Faithful flatness.* The notion of *faithful flatness* is handy, although we probably won't use it. We say that an extension of rings $B \to A$ is *faithfully flat* if for every A-module M, M is A-flat if and only if $M \otimes_A B$ is B-flat. We say that a morphism of schemes $X \to Y$ is *faithfully flat* if it is flat and surjective. These notions are the "same", as shown by the following exercise.

Exercise. Show that $B \to A$ is faithfully flat if and only if $\operatorname{Spec} A \to \operatorname{Spec} B$ is faithfully flat.

3. The "Tor" functors, and a "cohomological" criterion for flatness

In order to prove more facts about flatness, it is handy to have the notion of Tor. (Tor is short for "torsion". The reason for this name is that the 0th and/or 1st Tor-group measures common torsion in abelian groups (aka $\mathbb{Z}$-modules).) If you have never seen this notion before, you may want to just remember its properties, which are natural. But I'd like to prove everything anyway — it is surprisingly easy.

The idea behind Tor is as follows. Whenever we see a right-exact functor, we always hope that it is the end of a long-exact sequence. Informally, given a short exact sequence (1), we are hoping to see a long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_i^A(M, N') \longrightarrow \operatorname{Tor}_i^A(M, N) \longrightarrow \operatorname{Tor}_i^A(M, N'') \longrightarrow \cdots \tag{5}$$

$$\longrightarrow \operatorname{Tor}_1^A(M, N') \longrightarrow \operatorname{Tor}_1^A(M, N) \longrightarrow \operatorname{Tor}_1^A(M, N'')$$

$$\longrightarrow M \otimes_A N' \longrightarrow M \otimes_A N \longrightarrow M \otimes_A N'' \longrightarrow 0.$$

More precisely, we are hoping for *covariant functors* $\operatorname{Tor}_i^A(\cdot, N)$ from A-modules to A-modules (giving 2/3 of the morphisms in that long exact sequence), with $\operatorname{Tor}_0^A(M, N) \equiv$

$M \otimes_A N$, and natural δ morphisms $\operatorname{Tor}^A_{i+1}(M, N'') \to \operatorname{Tor}^A_i(M, N')$ for every short exact sequence (1) giving the long exact sequence. (In case you care, "natural" means: given a morphism of short exact sequences, the natural square you would write down involving the δ-morphism must commute. I'm not going to state this explicitly.)

It turns out to be not too hard to make this work, and this will later motivate derived functors. I'll now define $\operatorname{Tor}^A_i(M, N)$. Take any resolution $\mathcal{R}$ of N by free modules:

$$\cdots \longrightarrow A^{\oplus n_2} \longrightarrow A^{\oplus n_1} \longrightarrow A^{\oplus n_0} \longrightarrow N \to 0.$$

More precisely, build this resolution from right to left. Start by choosing generators of N as an A-module, giving us $A^{\oplus n_0} \to N \to 0$. Then choose generators of the kernel, and so on. Note that we are not requiring the n_i to be finite, although if N is a finitely-generated module and A is Noetherian (or more generally if N is coherent and A is coherent over itself), we can choose the n_i to be finite. Truncate the resolution, by stripping off the last term. Then tensor with M (which may lose exactness!). Let $\operatorname{Tor}^i_A(M, N)_{\mathcal{R}}$ be the homology of this complex at the ith stage ($i \geq 0$). The subscript $\mathcal{R}$ reminds us that our construction depends on the resolution, although we will soon see that it is independent of the resolution.

We make some quick observations.

- $\operatorname{Tor}^A_0(M, N)_{\mathcal{R}} \cong M \otimes_A N$ (and this isomorphism is canonical). Reason: as tensoring is right exact, and $A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0$ is exact, we have that $M^{\oplus n_1} \to M^{\oplus n_0} \to M \otimes_A N \to 0$ is exact, and hence that the homology of the truncated complex $M^{\oplus n_1} \to M^{\oplus n_0} \to 0$ is $M \otimes_A N$.
- If M is flat, then $\operatorname{Tor}^A_i(M, N)_{\mathcal{R}} = 0$ for all i.

Now given two modules N and N' and resolutions $\mathcal{R}$ and $\mathcal{R}'$ of N and N', we can "lift" any morphism $N \to N'$ to a morphism of the two resolutions:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & A^{\oplus n_i} & \longrightarrow & \cdots & \longrightarrow & A^{\oplus n_1} & \longrightarrow & A^{\oplus n_0} & \longrightarrow & N & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & A^{\oplus n'_i} & \longrightarrow & \cdots & \longrightarrow & A^{\oplus n'_1} & \longrightarrow & A^{\oplus n'_0} & \longrightarrow & N' & \longrightarrow & 0 \end{array}$$

Denote the choice of lifts by $\mathcal{R} \to \mathcal{R}'$. Now truncate both complexes and tensor with M. Maps of complexes induce maps of homology, so we have described maps (a priori depending on $\mathcal{R} \to \mathcal{R}'$)

$$\operatorname{Tor}^A_i(M, N)_{\mathcal{R}} \to \operatorname{Tor}^A_i(M, N')_{\mathcal{R}'}.$$

We say two maps of complexes $f, g : C_* \to C'_*$ are *homotopic* if there is a sequence of maps $w : C_i \to C'_{i+1}$ such that $f - g = dw + wd$. Two homotopic maps give the same map on homology. (Exercise: verify this if you haven't seen this before.)

Crucial Exercise: Show that any two lifts $\mathcal{R} \to \mathcal{R}'$ are homotopic.

We now pull these observations together.

(1) We get a covariant functor from $\operatorname{Tor}^A_i(M, N)_{\mathcal{R}} \to \operatorname{Tor}^A_i(M, N')_{\mathcal{R}'}$ (independent of the lift $\mathcal{R} \to \mathcal{R}'$).

(2) Hence for any two resolutions $\mathcal{R}$ and $\mathcal{R}'$ we get a canonical isomorphism $\mathrm{Tor}_i^A(M,N)_{\mathcal{R}} \cong \mathrm{Tor}_i^1(M,N)_{\mathcal{R}'}$. Here's why. Choose lifts $\mathcal{R} \to \mathcal{R}'$ and $\mathcal{R}' \to \mathcal{R}$. The composition $\mathcal{R} \to \mathcal{R}' \to \mathcal{R}$ is homotopic to the identity (as it is a lift of the identity map $N \to N$). Thus if $f_{\mathcal{R}\to\mathcal{R}'} : \mathrm{Tor}_i^A(M,N)_{\mathcal{R}} \to \mathrm{Tor}_i^1(M,N)_{\mathcal{R}'}$ is the map induced by $\mathcal{R} \to \mathcal{R}'$, and similarly $f_{\mathcal{R}'\to\mathcal{R}}$ is the map induced by $\mathcal{R} \to \mathcal{R}'$, then $f_{\mathcal{R}'\to\mathcal{R}} \circ f_{\mathcal{R}\to\mathcal{R}'}$ is the identity, and similarly $f_{\mathcal{R}\to\mathcal{R}'} \circ f_{\mathcal{R}'\to\mathcal{R}}$ is the identity.

(3) Hence the covariant functor doesn't depend on the resolutions!

Finally:

(4) For any short exact sequence (1) we get a long exact sequence of Tor's (5). Here's why: given a short exact sequence (1), choose resolutions of N' and N''. Then use these to get a resolution for N in the obvious way (see below; the map $A^{\oplus(n_0' \to n_0'')} \to N$ is the composition $A^{\oplus n_0'} \to N' \to N$ along with any lift of $A^{n_0''} \to N''$ to N) so that we have a short exact sequence of resolutions

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots & \longrightarrow & A^{\oplus n_1'} & \longrightarrow & A^{\oplus n_0'} & \longrightarrow & N' & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots & \longrightarrow & A^{\oplus(n_1'+n_1'')} & \longrightarrow & A^{\oplus(n_0'+n_0'')} & \longrightarrow & N & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots & \longrightarrow & A^{\oplus n_1''} & \longrightarrow & A^{\oplus n_0''} & \longrightarrow & N'' & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0 &
\end{array}$$

Then truncate (removing the column (1)), tensor with M (obtaining a short exact sequence of complexes) and take cohomology, yielding a long exact sequence.

We have thus established the foundations of Tor!

Note that if N is a free module, then $\mathrm{Tor}_i^A(M,N) = 0$ for all M and all $i > 0$, as N has itself as a resolution.

3.1. *Remark: Projective resolutions.* We used very little about free modules in the above construction; in fact we used only that free modules are *projective*, i.e. those modules M such that for any surjection $M' \to M''$, it is possible to lift any morphism $M \to M''$ to $M \to M'$. This is summarized in the following diagram.

$$\begin{array}{ccc}
 & \overset{\text{exists}}{\swarrow} & M \\
 & & \downarrow \\
M' & \twoheadrightarrow & M''
\end{array}$$

Equivalently, $\mathrm{Hom}(M,\cdot)$ is an *exact functor* ($\mathrm{Hom}(N,\cdot)$ is always left-exact for any N). (More generally, we can define the notion of a *projective object in any abelian category*.)

Hence (i) we can compute$\operatorname{Tor}_i^A(M, N)$ by taking any projective resolution of N, and (ii) $\operatorname{Tor}_i^A(M, N) = 0$ for any projective A-module N.

3.2. *Remark: Generalizing this construction.* The above description was low-tech, but immediately generalizes drastically. All we are using is that $M \otimes_A$ is a right-exact functor. In general, if F is *any* right-exact covariant functor from the category of A-modules to any abelian category, this construction will define a sequence of functors $L_i F$ (called left-derived functors of F) such that $L_0 F = F$ and the L_i's give a long-exact sequence. We can make this more general still. We say that an abelian category *has enough projectives* if for any object N there is a surjection onto it from a projective object. Then if F is any right-exact functor from an abelian category with enough projectives to any abelian category, then F has left-derived functors.

3.3. *Exercise.* The notion of an *injective object* in an abelian category is dual to the notion of a projective object. Define derived functors for (i) covariant left-exact functors (these are called right-derived functors), (ii) contravariant left-exact functors (also right-derived functors), and (iii) contravariant right-exact functors (these are called left-derived functors), making explicit the necessary assumptions of the category having enough injectives or projectives.

Here are two quick practice exercises, giving useful properties of Tor.

Important exercise. If B is A-flat, then we get isomorphism $B \otimes \operatorname{Tor}_i^A(M, N) \cong \operatorname{Tor}_i^B(B \otimes M, B \otimes N)$. (This is tricky rather than hard; it has a clever one-line answer. Here is a fancier fact that experts may want to try: if B is not A-flat, we don't get an isomorphism; instead we get a spectral sequence.)

Exercise- (not too important, but good practice if you haven't played with Tor *before).* If x is not a 0-divisor, show that $\operatorname{Tor}_i^A(A/x, M)$ is 0 for $i > 1$, and for $i = 0$, get M/xM, and for $i = 1$, get $(M : x)$ (those things sent to 0 upon multiplication by x).

3.4. "Symmetry" of Tor. The natural isomorphism $M \otimes N \to N \otimes M$ extends to the following.

3.5. *Theorem. — There is a natural isomorphism* $\operatorname{Tor}_i(M, N) \cong \operatorname{Tor}(N, M)$.

Proof. Take two resolutions of M and N:

$$\cdots \to A^{\oplus m_1} \to A^{\oplus m_0} \to M \to 0$$

and

$$\cdots \to A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0.$$

Consider the double complex obtained by tensoring their truncations.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus(m_2+n_2)} & \longrightarrow & A^{\oplus(m_1+n_2)} & \longrightarrow & A^{\oplus(m_0+n_2)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus(m_2+n_1)} & \longrightarrow & A^{\oplus(m_1+n_1)} & \longrightarrow & A^{\oplus(m_0+n_1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus(m_2+n_0)} & \longrightarrow & A^{\oplus(m_1+n_0)} & \longrightarrow & A^{\oplus(m_0+n_0)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

Apply our spectral sequence machinery. We compute the homology of this complex in two ways.

We start by using the vertical arrows. Notice that the ith column is precisely the truncated resolution of N, tensored with $A^{\oplus m_i}$. Thus the homology in the vertical direction in the ith column is 0 except in the bottom element of the column, where it is $N^{\oplus m_i}$. We next take homology in the horizontal direction. In the only non-zero row (the bottom row), we see precisely the complex computing $\mathrm{Tor}_i(N, M)$. After using these second arrows, the spectral sequence has converged. Thus the ith homology of the double complex is (naturally isomorphic to) $\mathrm{Tor}_i(N, M)$.

Similarly, if we began with the arrows in the horizontal direction, we would conclude that the ith homology of the double complex is $\mathrm{Tor}_i(M, N)$. □

This gives us a quick but very useful result. Recall that if $0 \to N' \to N \to N'' \to 0$ is exact, then so is the complex obtained by tensoring with M *if* M *is flat*. (Indeed that is the definition of flatness!) But in general we have an exact sequence

$$\mathrm{Tor}_1^A(M, N'') \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

Hence we conclude:

3.6. *Proposition. — If $0 \to N' \to N \to N'' \to 0$ is exact, and N'' is flat, then $0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$ is exact.*

Note that we would have cared about this result long before learning about Tor. This gives some motivation for learning about Tor. Presumably one can also show this directly by some sort of diagram chase. (Is there an easy proof?)

One important consequence is the following. Suppose we have a short exact sequence of sheaves on Y, and the rightmost element is flat (e.g. locally free). Then if we pull this exact sequence back to X, it remains exact. (I think we may have used this.)

3.7. An ideal-theoretic criterion for flatness. We come now to a useful fact. Observe that $\mathrm{Tor}_1(M, N) = 0$ for all N implies that M is flat; this in turn implies that $\mathrm{Tor}_i(M, N) = 0$ for all $i > 0$.

The following is a very useful variant on this.

3.8. *Key theorem. — M is flat if and only if $\mathrm{Tor}_1^A(M, A/I) = 0$ for all ideals I.*

(The interested reader can tweak the proof below a little to show that it suffices to consider *finitely generated ideals* I, but we won't use this fact.)

Proof. [The M's and N's are messed up in this proof.] We have already observed that if N is flat, then $\mathrm{Tor}_1^A(M, R/I) = 0$ for all I. So we assume that $\mathrm{Tor}_1^A(M, A/I) = 0$, and hope to prove that $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N, and hence that M is flat.

By induction on the number of generators of N, we can prove that $\mathrm{Tor}_1^A(M, N) = 0$ for all *finitely generated* modules N. (The base case is our assumption, and the inductive step is as follows: if N is generated by $a_1, \ldots, a_n$, then let N' be the submodule generated by $a_1, \ldots, a_{n-1}$, so $0 \to N' \to N \to A/I \to 0$ is exact, where I is some ideal. Then the long exact sequence for Tor gives us $0 = \mathrm{Tor}_1^A(M, N') \to \mathrm{Tor}_1^A(M, N) \to \mathrm{Tor}_1^A(M, A/I) = 0$.)

We conclude by noting that N is the union (i.e. direct limit) of its finitely generated submodules. As $\otimes$ commutes with direct limits, Tor_1 commutes with direct limits as well. (This requires some argument!)

Here is a sketch of an alternate conclusion. We wish to show that for any exact $0 \to N' \to N$, $0 \to M \otimes N' \to M \otimes N$ is also exact. Suppose $\sum m_i \otimes n_i' \mapsto 0$ in $M \otimes N$. Then that equality involves only finitely many elements of N. Work instead in the submodule generated by these elements of N. Within these submodules, we see that $\sum m_i \otimes n_i' = 0$. Thus this equality holds inside $M \otimes N'$ as well.

(I may try to write up a cleaner argument. Joe pointed out that the cleanest thing to do is to show that injectivity commutes with direct limits.) □

This has some cheap but important consequences.

Recall (or reprove) that flatness over a domain implies torsion-free.

3.9. *Corollary to Theorem 3.8. — Flatness over principal ideal domain is the* **same** *as torsion-free.*

This follows directly from the proposition.

3.10. *Important Exercise (flatness over the dual numbers).* This fact is important in deformation theory and elsewhere. Show that M is flat over $k[t]/t^2$ if and only if the natural map $M/tM \to tM$ is an isomorphism.

3.11. Flatness in exact sequences.

Suppose $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules.

3.12. *Proposition. — If* M *and* M'' *are both flat, then so is* M'*. If* M' *and* M'' *are both flat, then so is* M.

Proof. We use the characterization of flatness that N is flat if and only if $\mathrm{Tor}_i(N, N') = 0$ for all $i > 0$, N'. The result follows immediately from the long exact sequence for Tor. □

3.13. *Unimportant remark.* This begs the question: if M' and M are both flat, is M'' flat? (The argument above breaks down.) The answer is no: over $k[t]$, consider $0 \to tk[t] \to k[t] \to k[t]/t \to 0$ (geometrically: the closed subscheme exact sequence for a point on $\mathbb{A}^1$). The module on the right has torsion, and hence is not flat. The other two modules are free, hence flat.

3.14. *Easy exercise.* (We will use this shortly.) If $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ is an exact sequence, and M_i is flat for $i > 0$, show that M_0 is flat too. (Hint: break the exact sequence into short exact sequences.)

We now come to the next result about flatness that will cause us to think hard.

3.15. *Important Theorem (for coherent modules over Noetherian local rings, flat equals free). — Suppose* $(A, \mathfrak{m})$ *is a local ring, and* M *is a coherent* A*-module (e.g. if* A *is Noetherian, then* M *is finitely generated). Then* M *is flat if and only if it is free.*

(It is true more generally, although we won't use those facts: apparently we can replace coherent with finitely presented, which only non-Noetherian people care about; or we can give up coherent completely if A is Artinian, although I haven't defined this notion. Reference: Mumford p. 296. I may try to clean the proof up to work in these cases.)

Proof. Clearly we are going to be using Nakayama's lemma. Now $M/\mathfrak{m}M$ is a finite-dimensional vector space over the field $A/\mathfrak{m}$. Choose a basis, and lift it to elements $m_1, \dots, m_n \in M$. Then consider $A^n \to M$ given by $e_i \mapsto m_i$. We'll show this is an isomorphism. This is surjective by Nakayama's lemma: the image is all of M modulo the maximal ideal, hence is everything. Let K be the kernel, which is finitely generated by coherence:

$$0 \to K \to A^n \to M \to 0.$$

Tensor this with $A/\mathfrak{m}$. As M is flat, the result is still exact (Proposition 3.6):

$$0 \to K/\mathfrak{m}K \to (A/\mathfrak{m})^n \to M/\mathfrak{m}M \to 0.$$

But $(A/\mathfrak{m})^n \to M/\mathfrak{m}M$ is an isomorphism, so $K/\mathfrak{m}K = 0$. As K is finitely generated, $K = 0$. □

Here is an immediate corollary (or really just a geometric interpretation).

3.16. *Corollary. — Suppose $\mathcal{F}$ is coherent over a locally Noetherian scheme X. Then $\mathcal{F}$ is flat over X if and only if it is locally free.*

(Reason: we have shown that local-freeness can be checked at the stalks.)

This is a useful fact. Here's a consequence that we prove earlier by other means: if $C \to C'$ is a surjective map of nonsingular irreducible projective curves, then $\pi_* \mathcal{O}_C$ is locally free.

In general, this gives us a useful criterion for flatness: Suppose $X \to Y$ finite, and Y integral. Then f is flat if and only if $\dim_{FF(Y)} f_*(\mathcal{O}_X)_y \otimes FF(Y)$ is constant. So the normalization of a node is not flat (I drew a picture here).

3.17. A useful special case: flatness over nonsingular curves. When are morphisms to nonsingular curves flat? Local rings of nonsingular curves are discrete valuation rings, which are principal ideal domains, so for them flat = torsion-free (Prop. 3.9). Thus, any map from a scheme to a nonsingular curve where all associated points go to a generic point is flat. (I drew several pictures of this.)

Here's a version we've seen before: a map from an irreducible curve to a nonsingular curve.

Here is another important consequence, which we can informally state as: we can take flat limits over one-parameter families. More precisely: suppose A is a discrete valuation ring, and let 0 be the closed point of $\operatorname{Spec} A$ and η the generic point. Suppose X is a scheme over A, and Y is a scheme over $X|_\eta$. Let Y' be the scheme-theoretic closure of Y in X. Then Y' is flat over A. Then $Y'|_0$ is often called the *flat limit* of Y.

(Suppose A is a discrete valuation ring, and let η be the generic point of $\operatorname{Spec} A$. Suppose X is proper over A, and Y is a closed subscheme of X_η. *Exercise:* Show that there is only one closed subscheme Y' of X, proper over A, such that $Y'|_\eta = Y$, and Y' is flat over A. Aside for experts: For those of you who know what the Hilbert scheme is, by taking the case of X as projective space, this shows that the Hilbert scheme is proper, using the valuative criterion for properness.)

3.18. *Exercise (an interesting explicit example of a flat limit).* (Here the base is $\mathbb{A}^1$, not a discrete valuation ring. You can either restrict to the discrete valuation ring that is the stalk near 0, or generalize the above discussion appropriately.) Let $X = \mathbb{A}^3 \times \mathbb{A}^1 \to Y = \mathbb{A}^1$ over a field k, where the coordinates on $\mathbb{A}^3$ are x, y, and z, and the coordinates on $\mathbb{A}^1$ are t. Define X away from $t = 0$ as the union of the two lines $y = z = 0$ (the x-axis) and $x = z - t = 0$ (the y-axis translated by t). Find the flat limit at $t = 0$. (Hint: it is *not* the union of the two axes, although it includes it. The flat limit is non-reduced.)

3.19. Stray but important remark: flat morphisms are (usually) open. I'm discussing this here because I have no idea otherwise where to put it.

3.20. *Exercise.* Prove that flat and locally finite type morphisms of locally Noetherian schemes are open. (Hint: reduce to the affine case. Use Chevalley's theorem to show that the image is constructible. Reduce to a target that is the spectrum of a local ring. Show that the generic point is hit.)

3.21. I ended by stating an important consequence of flatness: flat plus projective implies constant Euler characteristic. I'll state this properly in next Tuesday's notes, where I will also give consequences and a proof.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 43 AND 44

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CONTENTS

This week: constancy of Euler characteristic in flat families. The semicontinuity theorem and consequences. Glimpses of the relative Picard scheme.

1. FLAT IMPLIES CONSTANT EULER CHARACTERISTIC

We come to an important consequence of flatness. We'll see that this result implies many answers and examples to questions that we would have asked before we even knew about flatness.

1.1. *Important Theorem. — Suppose* $f : X \to Y$ *is a projective morphism, and* $\mathcal{F}$ *is a coherent sheaf on* X*, flat over* Y*. Suppose* Y *is locally Noetherian. Then* $\sum(-1)^i h^i(X_y \mathcal{F}|_y)$ *is a locally constant function of* $y \in Y$*. In other words, the Euler characteristic of* $\mathcal{F}$ *is constant in the fibers.*

This is first sign that cohomology behaves well in families. (We'll soon see a second: the Semicontinuity Theorem 4.4.) Before getting to the proof, I'll show you some of its many consequences. (A second proof will be given after the semicontinuity discussion.)

The theorem also gives a necessary condition for flatness. It also sufficient if target is integral and locally Noetherian, although we won't use this. (Reference: You can translate Hartshorne Theorem III.9.9 into this.) I seem to recall that both the necessary and sufficient conditions are due to Serre, but I'm not sure of a reference. It is possible that integrality is not necessary, and that reducedness suffices, but I haven't checked.

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1.2. *Corollary.* — *Assume the same hypotheses and notation as in Theorem 1.1. Then the Hilbert polynomial of $\mathcal{F}$ is locally constant as a function of $y \in Y$.*

Thus for example a flat family of varieties in projective space will all have the same degree and genus (and the same dimension!). Another consequence of the corollary is something remarkably useful.

1.3. *Corollary.* — *An invertible sheaf on a flat projective family of connected nonsingular curves has locally constant degree on the fibers.*

Proof. An invertible sheaf $\mathcal{L}$ on a flat family of curves is always flat (as locally it is isomorphic to the structure sheaf). Hence $\chi(\mathcal{L}_y)$ is constant. From the Riemann-Roch formula $\chi(\mathcal{L}_y) = \deg(\mathcal{L}_y) - g(X_y) + 1$, using the local constancy of $\chi(\mathcal{L}_y)$, the result follows. $\square$

Riemann-Roch holds in more general circumstances, and hence the corollary does too. Technically, in the example I'm about to give, we need Riemann-Roch for the union of two $\mathbb{P}^1$'s, which I haven't shown. This can be shown in three ways. (i) I'll prove that Riemann-Roch holds for projective generically reduced curves later. (ii) You can prove it by hand, as an exercise. (iii) You can consider this curve C inside $\mathbb{P}^1 \times \mathbb{P}^1$ as the union of a "vertical fiber" and "horizontal fiber". Any invertible sheaf on C is the restriction of some $\mathcal{O}(a, b)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Use additivity of Euler characteristics on $0 \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a-1, b-1) \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b) \to \mathcal{O}_C(a, b) \to 0$, and note that we have earlier computed the $\chi(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(c, d))$.

This result has a lot of interesting consequences.

1.4. *Example of a proper non-projective surface.* We can use it to show that a certain proper surface is not projective. Here is how.

Fix any field with more than two elements. We begin with a flat projective family of curves whose $X \to \mathbb{P}^1$, such that the fiber X_0 over 0 is isomorphic to $\mathbb{P}^1$, and the fiber X_∞ over ∞ is isomorphic to two $\mathbb{P}^1$'s meeting at a point, $X_\infty = Y_\infty \cup Z_\infty$. For example, consider the family of conics in $\mathbb{P}^2$ (with projective coordinates x, y, z) parameterized by $\mathbb{P}^1$ (with projective coordinates λ and μ given by

$$\lambda xy + \mu z(x + y + z) = 0.$$

This family unfortunately is singular for $[\lambda; \mu] = [0; 1]$ (as well as $[1; 0]$ and one other point), so change coordinates on $\mathbb{P}^1$ so that we obtain a family of the desired form.

We now take a break from this example to discuss an occasionally useful construction.

1.5. *Gluing two schemes together along isomorphic closed subschemes.* Suppose X' and X'' are two schemes, with closed subschemes $W' \hookrightarrow X'$ and $W'' \hookrightarrow X''$, and an isomorphism $W' \to W''$. Then we can glue together X' and X'' along $W' \cong W''$. We define this more formally as the *coproduct*:

$$\begin{array}{ccc} W' \cong W'' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ X'' & \longrightarrow & ?. \end{array}$$

Exercise. Prove that this coproduct exists. Possible hint: work by analogy with our product construction. If the coproduct exists, it is unique up to unique isomorphism. Start with judiciously chosen affine open subsets, and glue.

Warning: You might hope that if you have a scheme X with two disjoint closed subschemes W' and W'', and an isomorphism $W' \to W''$, then you should be able to glue X to itself along $W' \to W''$. This is not always possible! I'll give an example shortly. You can still make sense of the quotient as an *algebraic space*, which I will not define here. If you want to know what it is, ask Jarod, or come to one of the three lectures he'll give later this quarter.

1.6. *Back to the non-projective surface.* Now take two copies of the X we defined above; call them X' and X''. Glue X' to X'' by identifying X'_0 with Y''_∞ (in any way you want) and Y'_∞ with X''_0. (Somewhat more explicitly: we are choosing an isomorphism $X'_0 \cup Y'_\infty$ with $X''_0 \cup Y''_\infty$ that "interchanges the components".) I claim that the resulting surface X is proper and not projective over the base field k. The first is an exercise.

Exercise. Show that X is proper over k. (Hint: show that the union of two proper schemes is also proper.)

Suppose now that X is projective, and is embedded in projective space by an invertible sheaf (line bundle) $\mathcal{L}$. Then the degree of $\mathcal{L}$ on each curve of X is non-negative. For any curve $C \subset X$, let $\deg C$ be the degree of $\mathcal{L}$ on C (or equivalently, the degree of C under this projective embedding). Pull $\mathcal{L}$ back to X'. Then this is a line bundle on a flat projective family, so the degree is constant in fibers. Thus

$$\deg X'_0 = \deg(Y'_\infty \cup Z'_\infty) = \deg Y'_\infty + \deg Z'_\infty > \deg Y'_\infty.$$

(Technically, we have not shown that the middle equality holds, so you should think about why that is clear.) Similarly $\deg X''_0 > \deg Y''_\infty$. But after gluing, $X'_0 = Y''_\infty$ and $X''_0 = Y'_\infty$, so we have a contradiction.

1.7. *Remark.* This is a stripped down version of Hironaka's example in dimension 3. Hironaka's example has the advantage of being nonsingular. I'll present that example (and show how this one comes from Hironaka) when we discuss blow-ups. (I think it is a fact that nonsingular proper surfaces over a field are always projective.)

1.8. *Unimportant remark.* You can do more fun things with this example. For example, we know that projective surfaces can be covered by three affine open sets. This can be used to give an example of (for any N) a proper surface that requires at least N affine open subsets to cover it (see my paper with Mike Roth on my preprints page, Example 4.9).

1.9. *Problematic nature of the notion of "projective morphism".* This example shows that the notion of being projective isn't a great notion. There are four possible definitions that might go with this notion. (1) We are following Grothendieck's definition. This notion is not local on the base. For example, by following the gluing above for the morphisms $X' \to \mathbb{P}^1$ and $X'' \to \mathbb{P}^1$, we obtain a morphism $\pi : X \to \mathbb{P}^1 \cup \mathbb{P}^1$, where the union on the right is obtained by gluing the 0 of the first $\mathbb{P}^1$ to the ∞ of the second, and vice versa. Then away from each node of the target, π is projective. (You could even give some explicit equations if you wanted.) However, we know that π is not projective, as $\rho : \mathbb{P}^1 \cup \mathbb{P}^1 \to \operatorname{Spec} k$ is projective, but we have already shown that $\rho \circ \pi : X \to \operatorname{Spec} k$ is not projective.

(2) Hartshorne's definition is designed for finite type k-schemes, and is definitely the wrong one for schemes in general.

(3) You could make our notion "local on the base" by also requiring more information: e.g. the notion of a projective morphism could be a morphism of schemes $X \to Y$ along with an invertible sheaf $\mathcal{L}$ on X that serves as an $\mathcal{O}(1)$. This is a little unpleasant; when someone says "consider a projective surface", they usually wouldn't want to have any particular projective embedding preferred.

(4) Another possible notion is that of *locally projective*: $\pi : X \to Y$ is locally projective if there is an open cover of Y by U_i such that over each U_i, π is projective (in our original sense (1)). The disadvantage is that this isn't closed under composition, as is shown by our example $X \to \mathbb{P}^1 \cup \mathbb{P}^1 \to \operatorname{Spec} k$.

1.10. *Example: You can't always glue a scheme to itself along isomorphic disjoint subschemes.* In class, we had an impromptu discussion of this, so it is a little rough. I'll use a variation of the above example. We'll see that you can't glue X to itself along an isomorphism $X_0 \cong Y_\infty$. (To make this a precise statement: there is no morphism $\pi : X \to W$ such that there is a curve $C \hookrightarrow W$ such that $\pi^{-1}(W - U) = X - X_0 - Y_\infty$, and π maps both X_0 and Y_∞ isomorphically to W.) A picture here is essential!

If there were such a scheme W, consider the point $\pi(Y_\infty \cap Z_\infty) \in W$. It has an affine neighborhood U; let K be its complement. Consider $\pi^{-1}(K)$. This is a closed subset of X, missing $Y_\infty \cap Z_\infty$. Note that it meets Y_∞ (as the affine open U can contain no $\mathbb{P}^1$'s) and Z_∞. Discard all components of $\pi^{-1}(K)$ that are dimension 0, and that contain components of fibers; call what's left K'. *Caution: I need to make sure that I don't end up discarding the points on Y_∞ and Z_∞. I could show that $\pi^{-1}(K)$ has pure codimension 1, but I'd like to avoid doing that. For now, assume that is the case; I may patch this later.* Then K' is an effective Cartier divisor, inducing an invertible sheaf on the surface X, which in turn is a flat projective family over $\mathbb{P}^1$. Thus the degree of K' is constant on fibers. Then we get the same sort of contradiction:

$$\deg_{K'} Y_\infty = \deg_{K'} X_0 = \deg_{K'} Y_\infty + \deg_{K'} Z_\infty > \deg_{K'} Y_\infty.$$

This led to a more wide-ranging discussion. A surprisingly easy theorem (which you can find in Mumford's Abelian Varieties for example) states that if X is a projective k-scheme with an action by a finite group G, then the quotient X/G exists, and is also a projective scheme. (One first has to define what one means by X/G!) If you are a little careful in choosing the isomorphisms used to build our nonprojective surface (picking

$X'_0 \to Y''_\infty$ and $X''_0 \to Y'_\infty$ to be the "same" isomorphisms), then there is a $\mathbb{Z}/2$-action on X ("swapping the $\mathbb{P}^1$'s"), we have shown that the quotient W does *not* exist as a scheme, hence giving another proof (modulo things we haven't shown) that X is not projective.

2. Proof of Important Theorem on constancy of Euler characteristic in flat families

Now you've seen a number of interesting results that seem to have nothing to do with flatness. I find this a good motivation for this motivation: using the concept, we can prove things that were interested in beforehand. It is time to finally prove Theorem 1.1.

Proof. The question is local on the base, so we may reduce to case Y is affine, say $Y = \operatorname{Spec} B$, so $X \hookrightarrow \mathbb{P}^n_B$ for some n. We may reduce to the case $X = \mathbb{P}^n_B$ (as we can consider $\mathcal{F}$ as a sheaf on $\mathbb{P}^n_B$). We may reduce to showing that Hilbert polynomial $\mathcal{F}(m)$ is locally constant for all $m \gg 0$ (as by Serre vanishing for $m \gg 0$, the Hilbert polynomial agrees with the Euler characteristic). Now consider the Cech complex $\mathcal{C}^*$ for $\mathcal{F}$. Note that all the terms in the Cech complex are flat. Twist by $\mathcal{O}(m)$ for $m \gg 0$, so that all the higher push-forwards vanish. Hence $\Gamma(\mathcal{C}^*(m))$ is exact except at the first term, where the cohomology is $\Gamma(\pi_*\mathcal{F}(m))$. We tack on this module to the front of the complex, so it is once again exact. Thus (by an earlier exercise), as we have an exact sequence in which all but the first terms are known to be flat, the first term is flat as well. As it is finitely generated, it is also free by an earlier fact (flat and finitely generated over a Noetherian local ring equals free), and thus has constant rank.

We're interested in the cohomology of the fibers. To obtain that, we tensor the Cech resolution with $k(y)$ (as y runs over Y) and take cohomology. Now the extended Cech resolution (with $\Gamma(\pi_*\mathcal{F}(m))$ tacked on the front) is an exact sequence of flat modules, and hence remains exact upon tensoring with $k(y)$ (or indeed anything else). (Useful translation: cohomology commutes with base change.) Thus $\Gamma(\pi_*\mathcal{F}(m)) \otimes k(y) \cong \Gamma(\pi_*\mathcal{F}(m)|_y)$. Thus the dimension of the Hilbert function is the rank of the locally free sheaf at that point, which is locally constant. □

3. Start of Thursday's class: Review

At this point, you've already seen a large number of facts about flatness. Don't be overwhelmed by them; keep in mind that you care about this concept because we have answered questions we cared about even before knowing about flatness. Here are three examples. (i) If you have a short exact sequence where the last is locally free, then you can tensor with anything and the exact sequence will remain exact. (ii) We described a morphism that is proper but not projective. (iii) We showed that you can't always glue a scheme to itself.

Here is a summary of what we know, highlighting the hard things.

- definition; basic properties (pullback and localization). flat base change commutes with higher pushforwards
- Tor: definition and symmetry. (Hence tensor exact sequences of flats with anything and keep exactness.)
- ideal-theoretic criterion: $\mathrm{Tor}_1(M, A/I) = 0$ for all I. (flatness over PID = torsion-free; over dual numbers) (important special case: DVR)
- for coherent modules over Noetherian local rings, flat=locally free
- flatness is open in good circumstances (flat + lft of lN is open; we should need only weaker hypotheses)
- euler characteristics behave well in projective flat families. In particular, the degree of an invertible sheaf on a flat projective family of curves is locally constant.

4. Cohomology and base change theorems

Here is the type of question we are considering. We'd like to see how higher pushforwards behave with respect to base change. For example, we've seen that higher pushforward commutes with *flat* base change. A special case of base change is the inclusion of a point, so this question specializes to the question: can you tell the cohomology of the fiber from the higher pushforward? The next group of theorems I'll discussed deal with this issue. I'll prove things for projective morphisms. The statements are true for proper morphisms of Noetherian schemes too; the one fact you'll see that I need is the following: that the higher direct image sheaves of coherent sheaves under proper morphisms are also coherent. (I'm largely following Mumford's *Abelian Varieties*. The geometrically interesting theorems all flow from the following neat but unmotivated result.

4.1. *Key theorem. — Suppose* $\pi : X \to \operatorname{Spec} B$ *is a projective morphism of Noetherian [needed?] schemes, and* $\mathcal{F}$ *is a coherent sheaf on* X*, flat over* $\operatorname{Spec} B$*. Then there is a* finite *complex*

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely generated projective B*-modules and an isomorphism of functors*

$$H^p(X \times_B A, \mathcal{F} \otimes_B A) \cong H^p(K^* \otimes_B A) \qquad (1)$$

for all $p \geq 0$ *in the category of* B*-algebras* A*.*

In fact, K^i will be free for $i > 0$. For $i = 0$, it is projective hence flat hence locally free (by an earlier theorem) on Y.

Translation/idea: Given $\pi : X \to \operatorname{Spec} B$, we will have a complex of vector bundles on the target that computes cohomology (higher-pushforwards), "universally" (even after any base change). The idea is as follows: take the Cech complex, produce a "quasiisomorphic" complex (a complex with the same cohomology) of free modules. For those taking derived category class: we have an isomorphic object in the derived category which is easier to deal with as a complex. We'll first construct the complex so that (1) holds for $B = A$, and then show the result for general A later. Let's put this into practice.

4.2. *Lemma. — Let* C^* *be a complex of* B*-modules such that* $H^i(C^*)$ *are finitely generated* B*-modules, and that* $C^p \neq 0$ *only if* $0 \leq p \leq n$*. Then there exists a complex* K^* *of finitely generated* B*-modules such that* $K^p \neq 0$ *only if* $0 \leq p \leq n$ *and* K^p *is free for* $p \geq 1$*, and a homomorphism of complexes* $\phi : K^* \to C^*$ *such that* ϕ *induces isomorphisms* $H^i(K^*) \to H^i(C^*)$ *for all* i*.*

Note that K^i *is* B*-flat for* $i > 0$*. Moreover, if* C^p *are* B*-flat, then* K^0 *is* B*-flat too.*

For all of our purposes except for a side remark, I'd prefer a cleaner statement, where C^* is a complex of B-modules, with $C^p \neq 0$ only if $p \leq n$ (in other words, there could be infinitely many non-zero C^p's). The proof is then about half as long

Proof. Step 1. We'll build this complex inductively, and worry about K^0 when we get there.

$$\begin{array}{ccccccccc} & & K^m & \xrightarrow{\delta^m} & K^{m+1} & \xrightarrow{\delta^{m+1}} & K^{m+2} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \phi_m} & & \downarrow{\scriptstyle \phi_{m+1}} & & \downarrow{\scriptstyle \phi_{m+2}} & & \\ \cdots \longrightarrow C^{m-1} & \longrightarrow & C^m & \xrightarrow[\delta^m]{} & C^{m+1} & \xrightarrow[\delta^{m+1}]{} & C^{m+2} & \longrightarrow & \cdots . \end{array}$$

We assume we've defined (K^p, ϕ_p, δ^p) for $p \geq m+1$ such that these squares commute, and the top row is a complex, and ϕ^p defines an isomorphism of cohomology $H^q(K^*) \to H^q(C^*)$ for $q \geq m+2$ and a surjection $\ker \delta^{m+1} \to H^{m+1}(C^*)$, and the K^p are finitely generated B-modules.

We'll adjust the complex to make ϕ_{m+1} an isomorphism of cohomology, and then again to make ϕ_m a surjection on cohomology. Let $B^{m+1} = \ker(\delta^{m+1} : H^{m+1}(K^*) \to H^{m+1}(C^*))$. Then we choose generators, and make these K_1^m. We have a new complex. We get the 0-maps on cohomology at level m. We then add more in to surject on cohomology on level m.

Now what happens when we get to $m = 0$? We have maps of complexes, where everything in the top row is free, and we have an isomorphism of cohomology everywhere except for K^0, where we have a surjection of cohomology. Replace K^0 by $K^0 / \ker \delta^0 \cap \ker \phi_0$. Then this gives an isomorphism of cohomology.

Step 2. We need to check that K^0 is B-flat. Note that everything else in this quasiisomorphism is B-flat. Here is a clever trick: construct the *mapping cylinder* (call it M^*):

$$0 \to K^0 \to C^0 \oplus K^1 \to C^1 \oplus K^2 \to \cdots \to C^{n-1} \oplus K^n \to C^n \to 0.$$

Then we have a short exact sequence of complexes

$$0 \to C^* \to M^* \to K^*[1] \to 0$$

(where $K^*[1]$ is just the same complex as K^*, except slid over by one) yielding isomorphisms of cohomology $H^*(K^*) \to H^*(C^*)$, from which $H^*(M^*) = 0$. (This was an earlier exercise: given a map of complexes induces an isomorphism on cohomology, the mapping cylinder is exact.) Now look back at the mapping cylinder M^*, which we now realize is an exact sequence. All terms in it are flat except possibly K^0. Hence K^0 is flat too (also by an earlier exercise)! □

4.3. *Lemma. — Suppose* $K^* \to C^*$ *is a morphism of finite complexes of* **flat** B*-modules inducing isomorphisms of cohomology (a "quasiisomorphism"). Then for every* B*-algebra* A*, the maps* $H^p(C^* \otimes_B A) \to H^p(K^* \otimes_B A)$ *are isomorphisms.*

Proof. Consider the mapping cylinder M^*, which we know is exact. Then $M^* \otimes_B A$ is still exact! (The reason was our earlier exercise that any exact sequence of flat modules tensored with anything remains flat.) But $M^* \otimes_B A$ is the mapping cylinder of $K^* \otimes_B A \to C^* \otimes_B A$, so this is a quasiisomorphism too. □

Now let's prove the theorem!

Proof of theorem 4.1. Choose a finite covering (e.g. the standard covering). Take the Cech complex C^* for $\mathcal{F}$. Apply the first lemma to get the nicer version K^* of the same complex C^*. Apply the second lemma to see that if you tensor with B and take cohomology, you get the same answer whether you use K^* or C^*. □

We are now ready to put this into use. We will use it to discuss a trio of facts: the Semicontinuity Theorem, Grauert's Theorem, and the Cohomology and Base Change Theorem. (We'll prove the first two.) The theorem of constancy of euler characteristic in flat families also fits in this family.

These theorems involve the following situation. Suppose $\mathcal{F}$ is a coherent sheaf on X, $\pi : X \to Y$ projective, Y (hence X) Noetherian, and $\mathcal{F}$ flat over Y.

Here are two related questions. Is $R^p\pi_*\mathcal{F}$ locally free? Is $\phi^p : R^p\pi_*\mathcal{F}\otimes k(y) \to H^p(X_y, \mathcal{F}_y)$ an isomorphism?

We have shown Key theorem 4.1, that if Y is affine, say $Y = \operatorname{Spec} B$, then we can compute the pushforwards of $\mathcal{F}$ by a complex of locally free modules

$$0 \to M^0 \to M^1 \to \cdots \to M^n \to 0$$

where in fact M^p is free for $p > 1$. Moreover, this computes pushforwards "universally": after a base change, this remains true.

Now the dimension of the left is uppersemicontinuous by uppersemicontinuity of fiber dimension of coherent sheaves. The semicontinuity theorem states that the dimension of the right is also uppersemicontinuous. More formally:

4.4. *Semicontinuity theorem. — Suppose* $X \to Y$ *is a projective morphisms of Noetherian schemes, and* $\mathcal{F}$ *is a coherent sheaf on* X *flat over* Y*. Then for each* $p \geq 0$*, the function* $Y \to \mathbb{Z}$ *given by* $y \mapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$ *is upper semicontinuous on* Y*.*

So "cohomology groups jump in projective flat families". Again, we can replace projective by proper once we've shown finite-dimensionality of higher pushforwards (which we haven't). For pedants: can the Noetherian hypotheses be excised?

Here is an example of jumping in action. Let C be a positive genus nonsingular projective irreducible curve, and consider the projection $\pi : E \times E \to E$. Let $\mathcal{L}$ be the invertible sheaf (line bundle) corresponding to the divisor that is the diagonal, minus the section $p_0 \in E$. then $\mathcal{L}_{p_0}$ is trivial, but $\mathcal{L}_p$ is non-trivial for any $p \neq p_0$ (as we've shown earlier in the "fun with curves" section). Thus $h^0(E, \mathcal{L}_p)$ is 0 in general, but jumps to 1 for $p = p_0$.

Remark. Deligne showed that in the smooth case, at least over $\mathbb{C}$, there is no jumping of cohomology of the structure sheaf.

Proof. The result is local on Y, so we may assume Y is affine. Let K^* be a complex as in the key theorem 4.1. By localizing further, we can assume K^* is locally free. *So we are computing cohomology on any fiber using a complex of vector bundles.*

Then for $y \in Y$

$$\begin{aligned}\dim_{k(y)} H^p(X_y, \mathcal{F}_y) &= \dim_{k(y)} \ker(d^p \otimes_A k(y)) - \dim_{k(y)} \operatorname{im}(d^{p-1} \otimes_A k(y)) \\ &= \dim_{k(y)}(K^p \otimes k(y)) - \dim_{k(y)} \operatorname{im}(d^p \otimes_A k(y)) - \dim_{k(y)} \operatorname{im}(d^{p-1} \otimes_A k(y))\end{aligned}$$

(Side point: by taking alternating sums of these terms, we get a second proof of Theorem 1.1 that $\chi(X_y, \mathcal{F}_y) = \sum(-1)^i h^i(X_y, \mathcal{F}_y)$ is a constant function of y. I mention this because if extended the fact that higher cohomology of coherents is coherent under proper pushforwards, we'd also have Theorem 1.1 in this case.)

Now $\dim_{k(y)} \operatorname{im}(d^p \otimes_A k(y))$ is a lower semicontinuous function on Y. Reason: the locus where the dimension is less than some number q is obtained by setting all $q \times q$ minors of the matrix $K^p \to K^{p+1}$ to 0. So we're done! □

5. Line bundles are trivial in a Zariski-closed locus, and glimpses of the relative Picard scheme

(This was discussed on Thursday May 4, but fits in well here.)

5.1. *Proposition. — Suppose $\mathcal{L}$ is an invertible sheaf on an integral projective scheme X such that both $\mathcal{L}$ and $\mathcal{L}^\vee$ have non-zero sections. Then $\mathcal{L}$ is the trivial sheaf.*

As usual, "projective" may be replaced by "proper". The only fact we need (which we haven't proved) is that the only global functions on proper schemes are constants. (We haven't proved that. It follows easily from the valuative criterion of properness — but we haven't proved that either!)

Proof. Suppose s and t are the non-zero sections of $\mathcal{L}$ and $\mathcal{L}^\vee$. Then they are both non-zero at the generic point (or more precisely, in the stalk at the generic point). (Otherwise, they would be the zero-section — this is where we are using the integrality of X.) Under the map $\mathcal{L} \otimes \mathcal{L}^\vee \to \mathcal{O}$, $s \otimes t$ maps to st, which is also non-zero. But the only global functions (global sections of $\mathcal{O}_X$) are the constants, so st is a non-zero constant. But then s

is nowhere 0 (or else st would be somewhere zero), so $\mathcal{L}$ has a nowhere vanishing section, and hence is trivial (isomorphic to $\mathcal{O}_X$). $\square$

Now suppose $X \to Y$ is a flat projective morphism with integral fibers. (It is a "flat family of geometrically integral schemes".) Suppose that $\mathcal{L}$ is an invertible sheaf. Then the locus of $y \in Y$ where $\mathcal{L}_y$ is trivial on X_y is a closed set. Reason: the locus where $h^0(X_y, \mathcal{L}_y) \geq 1$ is closed by the Semicontinuity Theorem 4.4, and the same holes for the locus where $h^0(X_y, \mathcal{L}_y^\vee) \geq 1$.

(Similarly, if $\mathcal{L}'$ and $\mathcal{L}''$ are two invertible sheaves on the family X, the locus of points y where $\mathcal{L}'_y \cong \mathcal{L}''$ is a closed subset: just apply the previous paragraph to $\mathcal{L} := \mathcal{L}' \otimes (\mathcal{L}'')^\vee$.)

In fact, we can jazz this up: for any $\mathcal{L}$, there is in a natural sense a closed *subscheme* where $\mathcal{L}$ is trivial. More precisely, we have the following theorem.

5.2. *Seesaw Theorem. — Suppose $\pi : X \to Y$ is a projective flat morphism to a Noetherian scheme, all of whose fibers are geometrically integral schemes, and $\mathcal{L}$ is an invertible sheaf on X. Then there is a unique closed subscheme $Y' \hookrightarrow Y$ such that for any fiber diagram*

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{g} & X \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\pi} \\ Z & \xrightarrow{f} & Y \end{array}$$

such that $g^*\mathcal{L} \cong \rho^*\mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on Z, then f factors (uniquely) through $Y' \to Y$.

I want to make three comments before possibly proving this.

• I have no idea why it is called the seesaw theorem.

• As a special case, there is a "largest closed subscheme" on which the invertible sheaf is the pullback of a trivial invertible sheaf.

• Also, this is precisely the statement that the functor is representable $Y' \to Y$, and that this morphism is a closed immersion.

I'm not going to use this, so I won't prove it. But a slightly stripped down version of this appears in Mumford (p. 89), and you should be able to edit his proof so that it works in this generality.

There is a lesson I want to take away from this: this gives evidence for existence of a very important moduli space: the Picard scheme. The Picard scheme $\operatorname{Pic} X/Y \to Y$ is a scheme over Y which represents the following functor: Given any $T \to Y$, we have the set of invertible sheaves on $X \times_Y T$, modulo those invertible sheaves pulled back from T. In other words, there is a natural bijection between diagrams of the form

$$\begin{array}{ccc} \mathcal{L} & & \\ \downarrow & & \\ X \times_T Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Y \end{array}$$

and diagrams of the form

$$\begin{array}{ccc} & & \operatorname{Pic}_{X/Y} \\ & \nearrow & \downarrow \\ T & \longrightarrow & Y. \end{array}$$

It is a hard theorem (due to Grothendieck) that (at least if Y is reasonable, e.g. locally Noetherian — I haven't consulted the appropriate references) $\operatorname{Pic} X/Y \to Y$ exists, i.e. that this functor is representable. In fact $\operatorname{Pic} X/Y$ is of finite type.

We've seen special cases before when talking about curves: if C is a geometrically integral curve over a field k, of genus g, $\operatorname{Pic} C = \operatorname{Pic} C/k$ is a dimension g projective nonsingular variety.

Given its existence, it is easy to check that $\operatorname{Pic}_{X/Y}$ is a group scheme over Y, using our functorial definition of group schemes.

5.3. *Exercise.* Do this!

The group scheme has a zero-section $0 : Y \to \operatorname{Pic}_{X/Y}$. This turns out to be a closed immersion. The closed subscheme produced by the Seesaw theorem is precisely the pullback of the 0-section. I suspect that you can use the Seesaw theorem to show that the zero-section *is* a closed immersion.

5.4. *Exercise.* Show that the Picard scheme for $X \to Y$ (with our hypotheses: the morphism is flat and projective, and the fibers are geometrically integral) is separated over Y by showing that it satisfies the valuative criterion of separatedness.

Coming up soon: Grauert's Theorem and Cohomology and base change!

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 45 AND 46

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CONTENTS

This week: Grauert's theorem and the Cohomology and base change theorem, and applications. The Rigidity Lemma. Proof of Grauert's theorem. Dimensions behave well for flat morphisms. Associated points go to associated points.

1. COHOMOLOGY AND BASE CHANGE THEOREMS

We're in the midst of discussing a family of theorems involving the following situation. Suppose $\mathcal{F}$ is a coherent sheaf on X, $\pi : X \to Y$ projective, Y (hence X) Noetherian, and $\mathcal{F}$ flat over Y.

Here are two related questions. Is $R^p\pi_*\mathcal{F}$ locally free? Is $\phi^p : R^p\pi_*\mathcal{F} \otimes k(y) \to H^p(X_y, \mathcal{F}_y)$ an isomorphism?

We have shown a key intermediate result, that if Y is affine, say $Y = \operatorname{Spec} B$, then we can compute the pushforwards of $\mathcal{F}$ by a complex of locally free modules

$$0 \to M^0 \to M^1 \to \cdots \to M^n \to 0$$

where in fact M^p is free for $p > 1$. Moreover, this computes pushforwards "universally": after a base change, this remains true.

We have already shown the constancy of Euler characteristic, and the semicontinuity theorem. I'm now going to discuss two big theorems, Grauert's theorem and the Cohomology and base change theorem, that are in some sense the scariest in Hartshorne, coming at the end of Chapter III (along with the semicontinuity theorem). I hope you

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agree that semicontinuity isn't that scary (given the key fact). I'd like to discuss applications of these two theorems to show you why you care; then given time I'll give proofs. I've found the statements worth remembering, even though they are a little confusing.

Note that if $R^p\pi_*\mathcal{F}$ is locally free and ϕ^p is an isomorphism, then the right side is locally constant. The following is a partial converse.

1.1. *Grauert's Theorem. — If Y is reduced, then h^p locally constant implies $R^p\pi_*\mathcal{F}$ is locally free and ϕ^p is an isomorphism.*

1.2. *Cohomology and base change theorem. — Assume ϕ^p is surjective. Then the following hold.*

(a) *ϕ^p is an isomorphism, and the same is true nearby. [Note: The hypothesis is trivially satisfied in the common case $H^p = 0$. If $H^p = 0$ at a point, then it is true nearby by semicontinuity.]*
(b) *ϕ^{p-1} is surjective (=isomorphic) if and only if $R^p\pi_*\mathcal{F}$ is locally free. [This in turn implies that h^p is locally constant.]*

Notice that (a) is about just what happens over the reduced scheme, but (b) has a neat twist: you can check things over the reduced scheme, and it has implications over the scheme as a whole!

Here are a couple of consequences.

1.3. *Exercise.* Suppose $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$. Show that ϕ^{p-1} is an isomorphism for all $y \in Y$. (Hint: cohomology and base change (b).)

1.4. *Exercise.* Suppose $R^p\pi_*\mathcal{F} = 0$ for $p \geq p_0$. Show that $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$, $k \geq k_0$. (Same hint. You can also do this directly from the key theorem above.)

2. When the pushforward of the functions on X are the functions on Y

Many fun applications happen when a certain hypothesis holds, which I'll now describe.

We say that π satisfies (*) if it is projective, and the natural morphism $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is an isomorphism. Here are two statements that will give you a feel for this notion. First:

2.1. *Important Exercise.* Suppose π is a projective flat family, each of whose fibers are (nonempty) integral schemes, or more generally whose fibers satisfy $h^0(X_y) = 1$. Then (*) holds. (Hint: consider

$$\mathcal{O}_Y \otimes k(y) \longrightarrow (\pi_*\mathcal{O}_X) \otimes k(y) \xrightarrow{\phi^0} H^0(X_y, \mathcal{O}_{X_y}) \cong k(y) .$$

The composition is surjective, hence ϕ^0 is surjective, hence it is an isomorphism (by the Cohomology and base change theorem 1.2 (a)). Then thanks to the Cohomology and base change theorem 1.2 (b), $\pi_*\mathcal{O}_X$ is locally free, thus of rank 1. If I have a map of invertible sheaves $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ that is an isomorphism on closed points, it is an isomorphism (everywhere) by Nakayama.)

Note in the previous exercise: we are obtaining things not just about closed points!

Second: we will later prove a surprisingly hard result, that given any projective (proper) morphism of Noetherian schemes satisfying (*) (without any flatness hypotheses!), the fibers are all connected ("Zariski's connectedness lemma").

2.2. *Exercise (the Hodge bundle; important in Gromov-Witten theory).* Suppose $\pi : X \to Y$ is a projective flat family, all of whose geometric fibers are connected reduced curves of arithmetic genus g. Show that $R^1\pi_*\mathcal{O}_X$ is a locally free sheaf of rank g. This is called the *Hodge bundle*. [Hint: use cohomology and base change (b) twice, once with $p = 2$, and once with $p = 1$.]

Here is the question we'll address in this section. Given an invertible sheaf $\mathcal{L}$ on X, we wonder when it is the pullback of an invertible sheaf $\mathcal{M}$ on Y. Certainly it is necessary for it to be trivial on the fibers. We'll see that (*) holds, then this basically suffices. Here is the idea: given $\mathcal{L}$, how can we recover $\mathcal{M}$? Thanks to the next exercise, it must be $\pi_*\mathcal{L}$.

2.3. *Exercise.* Suppose $\pi : X \to Y$ satisfies (*). Show that if $\mathcal{M}$ is any invertible sheaf on Y, then the natural morphism $\mathcal{M} \to \pi_*\pi^*\mathcal{M}$ is an isomorphism. In particular, we can recover $\mathcal{M}$ from $\pi^*\mathcal{M}$ by pushing forward. (Hint: projection formula.)

2.4. *Proposition. — Suppose $\pi : X \to Y$ is a morphism of locally Noetherian integral schemes with geometrically integral fibers (hence by Exercise 2.1 satisfying (*)). Suppose also that Y is reduced, and $\mathcal{L}$ is an invertible sheaf on X that is trivial on the fibers of π (i.e. $\mathcal{L}_y$ is a trivial invertible sheaf on X_y). Then $\pi_*\mathcal{L}$ is an invertible sheaf on Y (call it $\mathcal{M}$), and $\mathcal{L} = \pi^*\mathcal{M}$.*

Proof. To show that there exists such an invertible sheaf $\mathcal{M}$ on Y with $\pi^*\mathcal{M} \cong \mathcal{L}$, it suffices to show that $\pi_*\mathcal{L}$ is an invertible sheaf (call it $\mathcal{M}$) and the natural homomorphism $\pi^*\mathcal{M} \to \mathcal{L}$ is an isomorphism.

Now by Grauert's theorem 1.1, $\pi_*\mathcal{L}$ is locally free of rank 1 (again, call it $\mathcal{M}$), and $\mathcal{M} \otimes_{\mathcal{O}_Y} k(y) \to H^0(X_y, \mathcal{L}_y)$ is an isomorphism. We have a natural map of invertible sheaves $\pi^*\mathcal{M} = \pi^*\pi_*\mathcal{L} \to \mathcal{L}$. To show that it is an isomorphism, we need only show that it is surjective, i.e. show that it is surjective on the fibers, which is done. □

Here are some consequences.

A first trivial consequence: if you have two invertible sheaves on X that agree on the fibers of π, then they differ by a pullback of an invertible sheaf on Y.

2.5. *Exercise.* Suppose X is an integral Noetherian scheme. Show that $\operatorname{Pic}(X \times \mathbb{P}^1) \cong \operatorname{Pic} X \times \mathbb{Z}$. (Side remark: If X is non-reduced, this is still true, see Hartshorne Exercise III.12.6(b). It need only be connected of finite type over k. Presumably locally Noetherian suffices.) Extend this to $X \times \mathbb{P}^n$. Extend this to any $\mathbb{P}^n$-bundle over X.

2.6. *Exercise.* Suppose $X \to Y$ is the projectivization of a vector bundle $\mathcal{F}$ over a reduced locally Noetherian scheme (i.e. $X = \underline{\operatorname{Proj}} \operatorname{Sym}^* \mathcal{F}$). Then I think we've already shown in an exercise that it is also the projectivization of $\mathcal{F} \otimes \mathcal{L}$. If Y is reduced and locally Noetherian, show that these are the only ways in which it is the projectivization of a vector bundle. (Hint: note that you can recover $\mathcal{F}$ by pushing forward $\mathcal{O}(1)$.)

2.7. *Exercise.* Suppose $\pi : X \to Y$ is a projective flat morphism over a Noetherian integral scheme, all of whose geometric fibers are isomorphic to $\mathbb{P}^n$ (over the appropriate field). Show that this is a projective bundle if and only if there is an invertible sheaf on X that restricts to $\mathcal{O}(1)$ on all the fibers. (One direction is clear: if it is a projective bundle, then it has a projective $\mathcal{O}(1)$. In the other direction, the candidate vector bundle is $\pi_* \mathcal{O}(1)$. Show that it is indeed a locally free sheaf of the desired rank. Show that its projectivization is indeed $\pi : X \to Y$.)

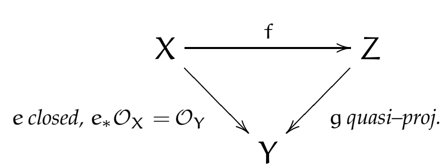
2.8. *Exercise (An example of a Picard scheme).* Show that the Picard scheme of $\mathbb{P}^1_k$ over k is isomorphic to $\mathbb{Z}$.

2.9. *Harder but worthwhile Exercise (An example of a Picard scheme).* Show that if E is an elliptic curve over k (a geometrically integral and nonsingular genus 1 curve with a marked k-point), then $\operatorname{Pic} E$ is isomorphic to $E \times \mathbb{Z}$. Hint: Choose a marked point p. (You'll note that this isn't canonical.) Describe the candidate universal invertible sheaf on $E \times \mathbb{Z}$. Given an invertible sheaf on $E \times X$, where X is an arbitrary Noetherian scheme, describe the morphism $X \to E \times \mathbb{Z}$.

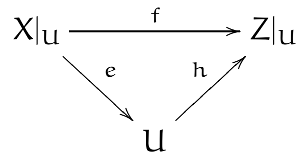
3. The rigidity lemma

The rigidity lemma is another useful fact about morphisms $\pi : X \to Y$ such that $\pi_* \mathcal{O}_X$ (condition (*) of the previous section). It is quite powerful, and quite cheap to prove, so we may as well do it now. (During class, the hypotheses kept on dropping until there was almost nothing left!)

3.1. *Rigidity lemma (first version). — Suppose we have a commutative diagram*



where Y is locally Noetherian, where f takes X_y for some $y \in Y$. Then there is a neighborhood $U \subset Y$ of y on which this is true. Better: over U, f factors through the projection to Y, i.e. the *following diagram commutes for some choice of* h*:*



.

Proof. This proof is very reminiscent of an earlier result, when we showed that a projective morphisms with finite fibers is a finite morphism.

We can take g to be projective. We can take Y to be an affine neighborhood of y. Then $Z \hookrightarrow \mathbb{P}^n_Y$ for some n. Choose a hyperplane of $\mathbb{P}^n_y$ missing $f(X_y)$, and extend it to a hyperplane H of $\mathbb{P}^n_Y$. (If $Y = \operatorname{Spec} B$, and $y = [\mathfrak{n}]$, then we are extending a linear equation with coefficients in $B/\mathfrak{n}$ to an equation with coefficients in B.) Pull back this hyperplane to X; the preimage is a closed subset. The image of this closed subset in Y is also a closed set $K \subset Y$, as e is a closed map. But $y \notin K$, so let $U = Y - K$. Over U, $f(X_y)$ misses our hyperplane H. Thus the map $X_y \to \mathbb{P}^n_U$ factors through $X_y \to \mathbb{A}^n_U$. Thus the map is given by n functions on $X|_U$. But $e_*\mathcal{O}_X \cong \mathcal{O}_Y$, so these are precisely the pullbacks of functions on U, so we are done. □

3.2. *Rigidity lemma (second version). — Same thing, with the condition on* g *changed from "projective" to simply "finite type".*

Proof. Shrink Y so that it is affine. Choose an open affine subset Z' of Z containing the $f(X_y)$. Then the complement the pullback of $K = Z - Z'$ to X is a closed subset of X whose image in Y is thus closed (as again e is a closed map), and misses y. We shrink Y further such that $f(X)$ lies in Z'. But $Z' \to Y$ is quasiprojective, so we can apply the previous version. □

Here is another mild strengthening.

3.3. *Rigidity lemma (third version). — If* X *is reduced and* g *is separated, and* Y *is connected, and there is a section* $Y \to X$*, then we can take* $U = Y$*.*

Proof. We have two morphisms $X \to Z$: f and $f \circ s \circ e$ which agree on the open set U. But we've shown earlier that any two morphisms from a reduced scheme to a separated scheme agreeing on a dense open set are the same. □

Here are some nifty consequences.

3.4. *Corollary (abelian varieties are abelian). — Suppose* A *is a projective integral group variety (an abelian variety) over a field* k*. Then the multiplication map* $m : A \times A \to A$ *is commutative.*

Proof. Consider the commutator map $c : A \times A \to A$ that corresponds to $(x, y) \mapsto xyx^{-1}y^{-1}$. We wish to show that this map sends $A \times A$ to the identity in A. Consider $A \times A$ as a family over the first factor. Then over $x = e$, c maps the fiber to e. Thus by the rigidity lemma (third version), the map c is a function only of the first factor. But then $c(x, y) = c(x, e) = e$. $\square$

3.5. *Exercise.* By a similar argument show that any map $f : A \to A'$ from one abelian variety to another is a group homomorphism followed by a translation. (Hint: reduce quickly to the case where f sends the identity to the identity. Then show that "$f(x + y) - f(x) - f(y) = e$".)

4. Proof of Grauert's theorem

I'll prove Grauert, but not Cohomology and Base Change. It would be wonderful if Cohomology and Base Change followed by just mucking around with maps of free modules over a ring.

4.1. *Exercise++.* Find such an argument.

We'll need a preliminary result.

4.2. *Lemma. — Suppose $Y = \operatorname{Spec} B$ is a reduced Noetherian scheme, and $f : M \to N$ is a homomorphism of coherent free (hence projective, flat) B-modules. If $\dim_{k(y)} \operatorname{im}(f \otimes k(y))$ is locally constant, then there are splittings $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$ with f killing M_1, and sending M_2 isomorphically to N_1.*

Proof. Note that $f(M) \otimes k \cong f(M \otimes k)$ from that surjection. From $0 \to f(M) \to N \to N/f(M) \to 0$ we have

$$\begin{array}{ccccccc} f(M) \otimes k & \longrightarrow & N \otimes k & \longrightarrow & N/f(M) \otimes k & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ f(M \otimes k) & \longrightarrow & N \otimes k & \longrightarrow & N \otimes k / f(M \otimes k) & \longrightarrow & 0 \end{array}$$

from which $(N/f(M)) \otimes k \cong (N \otimes k)/f(M \otimes k)$. Now the one on the right has locally constant rank, so the one on the left does too, hence is locally free, and flat, and projective. Hence $0 \to f(M) \to N \to N/f(M) \to 0$ splits, so let $N_2 = N/f(M)$, $N_1 = f(M)$. Also, N and $N/f(M)$ are flat and coherent, hence so is $f(M)$.

We now play the same game with

$$0 \to \ker f \to M \to f(M) \to 0.$$

$f(M)$ is projective, hence this splits. Let $\ker f = M_1$. $\square$

Now let's prove Grauert's theorem 1.1. We can use this lemma to rewrite

$$M^{p-1} \xrightarrow{d^{p-1}} M^p \xrightarrow{d^p} M^{p+1}$$

as $Z^{p-1} \oplus K^{p-1} \longrightarrow B^p \oplus H^p \oplus K^p \longrightarrow B^{p+1} \oplus K^{p+1}$ where d^{p-1} sends K^{p-1} isomorphically onto B^p (and is otherwise 0), and d^p sends K^p isomorphically onto B_{p+1}. Here H^p is a projective module, so we have local freeness. Thus when we tensor with some other ring, this structure is preserved as well; hence we have isomorphism. □

5. Dimensions behave well for flat morphisms

There are a few easier statements about flatness that I could have said much earlier.

Here's a basic statement about how dimensions behave in flat families.

5.1. *Proposition. — Suppose $f : X \to Y$ is a flat morphism of schemes all of whose stalks are localizations of finite type k-algebras, with $f(x) = y$. (For example, X and Y could be finite type k-schemes.) Then the dimension of X_y at x plus the dimension of Y and y is the dimension X at x.*

In other words, there can't be any components contained in a fibers; and you can't have any dimension-jumping.

In class, I first incorrectly stated this with the weaker hypotheses that X and Y are just locally Noetherian. Kirsten pointed out that I used the fact that height = codimension, which is not true for local Noetherian rings in general. However, we have shown it for local rings of finite type k-schemes. Joe suggested that one could work around this problem.

Proof. This is a question about local rings, so we can consider $\operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \mathcal{O}_{Y,y}$. We may assume that Y is reduced. We prove the result by induction on $\dim Y$. If $\dim Y = 0$, the result is immediate, as $X_y = X$ and $\dim_y Y = 0$.

Now for $\dim Y > 0$, I claim there is an element $t \in \mathfrak{m}$ that is not a zero-divisor, i.e. is not contained in any associated prime, i.e. (as Y is reduced) is not contained in any minimal prime. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the (finite number of) minimal primes. If $\mathfrak{m} \subset \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_n$, then in the first quarter we showed (in an exercise) that $\mathfrak{m} \subset \mathfrak{p}_i$ for some i. But as $\mathfrak{m}$ is maximal, and $\mathfrak{p}_i$ is minimal, we must have $\mathfrak{m} = \mathfrak{p}_i$, and $\dim Y = 0$.

Now by flatness t is not a zero-divisor of $\mathcal{O}_{X,x}$. (Recall that non-zero-divisors pull back to non-zero-divisors.) $\dim \mathcal{O}_{Y,y}/t = \dim \mathcal{O}_{Y,y} - 1$ by Krull's principal ideal theorem (here we use the fact that codimension = height), and $\dim \mathcal{O}_{X,x}/t = \dim \mathcal{O}_{X,x} - 1$ similarly. □.

5.2. *Corollary. — Suppose $f : X \to Y$ is a flat finite-type morphism of locally Noetherian schemes, and Y is irreducible. Then the following are equivalent.*

- *Every irreducible component of* X *has dimension* $\dim Y + n$.
- *For any point* $y \in Y$ *(not necessarily closed!), every irreducible component of the fiber* X_y *has dimension* n.

5.3. *Exercise.* Prove this.

Important definition: If these conditions hold, we say that π is *flat of relative dimension* n. This definition will come up when we define *smooth of relative dimension* n.

5.4. *Exercise.*
(a) Suppose $\pi : X \to Y$ is a finite-type morphism of locally Noetherian schemes, and Y is irreducible. Show that the locus where π is flat of relative dimension n is an open condition.
(b) Suppose $\pi : X \to Y$ is a *flat* finite-type morphism of locally Noetherian schemes, and Y is irreducible. Show that X can be written as the disjoint union of schemes $X_0 \cup X_1 \cup \cdots$ where $\pi|_{X_n} : X_n \to Y$ is flat of relative dimension n.

5.5. *Important Exercise.* Use a variant of the proof of Proposition 5.1 to show that if $f : X \to Y$ is a flat morphism of finite type k-schemes (or localizations thereof), then any associated point of X must map to an associated point of Y. (I find this an important point when visualizing flatness!)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 47 AND 48

RAVI VAKIL

CONTENTS

This week: Local criteria for flatness (statement), (relatively) base-point-free, (relatively) ample, very ample, every ample on a proper has a tensor power that is very ample, Serre's criterion for ampleness, Riemann-Roch for generically reduced curves.

1. THE LOCAL CRITERION FOR FLATNESS

I'll end our discussion of flatness with the statement of two results which can be quite useful. (Translation: I've seen them used.) They are both called the local criterion for flatness.

In both situations, assume that $(B, \mathfrak{n}) \to (A, \mathfrak{m})$ is a local morphism of local Noetherian rings (i.e. a ring homomorphism with $\mathfrak{n}A \subset \mathfrak{m}$), and that M is a finitely generated A-module. Of course we picture this in terms of geometry:

$$\begin{array}{c} \tilde{M} \\ \Big| \\ \operatorname{Spec}(A, \mathfrak{m}) \\ \Big\downarrow \\ \operatorname{Spec}(B, \mathfrak{n}). \end{array}$$

The local criteria for flatness are criteria for when M is flat over A. In practice, these are used in two circumstances: to check when a morphism to a locally Noetherian scheme is flat, or when a coherent sheaf on a locally Noetherian scheme is flat.

We've shown that to check if M is flat, we need check if $\operatorname{Tor}_1^B(B/I, M) = 0$ for all ideals I. The (first) local criterion says we need only deal with the maximal ideal.

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1.1. *Theorem (local criterion for flatness). —* M *is* B*-flat if and only if* $\mathrm{Tor}_1^B(B/\mathfrak{n}, M) = 0$*.*

(You can see a proof in Eisenbud, p. 168.)

An even more useful variant is the following. Suppose t is a non-zero-divisor of B in $\mathfrak{m}$ (geometrically: a Cartier divisor on the target passing through the generic point). If M is flat over B, then t is not a zero-divisor of M (we've checked this before: tensor $0 \longrightarrow B \xrightarrow{\times t} B \longrightarrow B/(t) \to 0$ with M). Also, M/tM is a flat B/tB-module (flatness commutes with base change). The next result says that this is a characterization.

1.2. *Theorem (local slicing criterion for flatness). —* M *is* B*-flat if and only if* M/tM *is flat over* $B/(t)$*.*

This is also sometimes called the local criterion for flatness. The proof is short (given the first local criterion). You can read it in Eisenbud (p. 169).

1.3. *Exercise (for those who know what a Cohen-Macaulay scheme is).* Suppose $\pi : X \to Y$ is a map of locally Noetherian schemes, where both X and Y are equidimensional, and Y is nonsingular. Show that if any two of the following hold, then the third does as well:

- π is flat.
- X is Cohen-Macaulay.
- Every fiber X_y is Cohen-Macaulay of the expected dimension.

I concluded the section on flatness by reviewing everything we have learned about flatness, in a good order.

2. Base-point-free, ample, very ample

My goal is to discuss properties of invertible sheaves on schemes (an "absolute" notion), and properties of invertible sheaves on a scheme with a morphism to another scheme (a "relative" notion, meaning that it makes sense in families). The notions fit into this table:

absolute	relative
base-point-free	relatively base-point-free
ample	relatively ample
very ample over a ring	very ample

This is admittedly horrible terminology. Warning: my definitions may have some additional hypotheses not used in EGA. The additional hypotheses exclude some nasty behavior which tends not to come up in nature; indeed, I have only seen these notions used in the circumstances in which I will describe them. There are very few facts to know, and there is fairly little to prove.

2.1. *Definition of base-point-free and relative base-point-free (review from class 22 and class 24, respectively).* Recall that if $\mathcal{F}$ is a quasicoherent sheaf on a scheme X, then $\mathcal{F}$ is generated by global sections if for any $x \in X$, the global sections generate the stalk $\mathcal{F}_x$. Equivalently: $\mathcal{F}$ is the quotient of a free sheaf. If $\mathcal{F}$ is a finite type quasicoherent sheaf, then we just need to check that for any x, the global sections generate the fiber of $\mathcal{F}$, by Nakayama's lemma. If furthermore $\mathcal{F}$ is invertible, we need only check that for any x there is a global section not vanishing there. In the case where $\mathcal{F}$ is invertible, we give "generated by global sections" a special name: *base-point-free*.

2.2. *Exercise (generated $\otimes$ generated = generated for finite type sheaves).* Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves on a scheme X that are generated by global sections. Show that $\mathcal{F} \otimes \mathcal{G}$ is also generated by global sections. In particular, if $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a scheme X, and both $\mathcal{L}$ and $\mathcal{M}$ are base-point-free, then so is $\mathcal{L} \otimes \mathcal{M}$. (This is often summarized as "base-point-free + base-point-free = base-point-free". The symbols $+$ is used rather than $\otimes$, because Pic is an abelian group.)

If $\pi : X \to Y$ is a morphism of schemes *that is quasicompact and quasiseparated* (so push-forwards of quasicoherent sheaves are quasicoherent sheaves), and $\mathcal{F}$ is a quasicoherent sheaf on X, we say that $\mathcal{F}$ is *relatively generated by global sections* (or *relatively generated* for short) if $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is a surjection of sheaves (class 24). As this is a morphism of quasicoherent sheaves, this can be checked over any affine open subset of the target, and corresponds to "generated by global sections" above each affine. In particular, this notion is affine-local on the target. If $\mathcal{F}$ is locally free, this notion is called *relatively base-point-free*.

2.3. *Definition of very ample.* Suppose $X \to Y$ is a projective morphism. Then $X = \underline{\operatorname{Proj}}\, \mathcal{S}_*$ for some graded algebra, locally generated in degree 1; given this description, X comes with $\mathcal{O}(1)$. Then any invertible sheaf on X of this sort is said to be *very ample* (for the morphism π). The notion of very ample is local on the base. (This is "better" than the notion of projective, which isn't local on the base, as we've seen in class 43/44 p. 4. Recall why: a morphism is projective if there *exists* an $\mathcal{O}(1)$. Thus a morphism $X \to Y \cup Y'$ could be projective over Y and over Y', but not projective over $Y \cup Y'$, as the "$\mathcal{O}(1)$" above Y need not be the same as the "$\mathcal{O}(1)$" above Y'. On the other hand, the notion is "very ample" is precisely the data of "an $\mathcal{O}(1)$".) You'll recall that given such an invertible sheaf, then $X = \underline{\operatorname{Proj}}\, \pi_*\mathcal{L}^{\otimes n}$, where the algebra on the right has the desired form. (It isn't necessarily the same graded algebra as you originally used to construct X.)

Notational remark: If Y is implicit, it is often omitted from the terminology. For example, if X is a complex projective scheme, the phrase "$\mathcal{L}$ is very ample on X" often means that "$\mathcal{L}$ is very ample for the structure morphism $X \to \operatorname{Spec} \mathbb{C}$".

2.4. *Exercise (very ample + very ample = very ample).* If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a scheme X, and both $\mathcal{L}$ and $\mathcal{M}$ are base-point-free, then so is $\mathcal{L} \otimes \mathcal{M}$. Hint: Segre. In particular, tensor powers of a very ample invertible sheaf are very ample.

2.5. *Tricky exercise+ (very ample + relatively generated = very ample).* Suppose $\mathcal{L}$ is very ample, and $\mathcal{M}$ is relatively generated, both on $X \to Y$. Show that $\mathcal{L} \otimes \mathcal{M}$ is very ample.

(Hint: Reduce to the case where the target is affine. $\mathcal{L}$ induces a map to $\mathbb{P}^n_A$, and this corresponds to $n+1$ sections $s_0, \ldots, s_n$ of $\mathcal{L}$. We also have a finite number m of sections $t_1, \ldots, t_m$ of $\mathcal{M}$ which generate the stalks. Consider the $(n+1)m$ sections of $\mathcal{L} \otimes \mathcal{M}$ given by $s_i t_j$. Show that these sections are base-point-free, and hence induce a morphism to $\mathbb{P}^{(n+1)m-1}$. Show that it is a closed immersion.)

2.6. *Definition of ample and relatively ample.* Suppose X is a quasicompact scheme. We say an invertible sheaf $\mathcal{L}$ on X is *ample* if for all finite type sheaves $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for $n \gg 0$. ("After finite twist, it is generated by global sections.") This is an *absolute* notion, not depending on a morphism.

2.7. *Example.* (a) If X is an affine scheme, and $\mathcal{L}$ is any invertible sheaf on X, then $\mathcal{L}$ is ample.
(b) If $X \to \operatorname{Spec} B$ is a projective morphism and $\mathcal{L}$ is a very ample invertible sheaf on X, then $\mathcal{L}$ is ample (by Serre vanishing, Theorem 4.2(ii), class 29, p. 5). (We may need B Noetherian here.)

We now give the relative version of this notion. Suppose $\pi : X \to Y$ is a morphism, and $\mathcal{L}$ is an invertible sheaf on X. Suppose that for every affine open subset $\operatorname{Spec} B$ of Y there is an n_0 such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ restricted to the preimage of $\operatorname{Spec} B$ is relatively generated by global sections for $n \geq n_0$. (In particular, π is quasicompact and quasiseparated — that was a hypothesis for relatively generated.) Then we say that $\mathcal{L}$ is *relatively ample* (with respect to π; although the reference to the morphism is often suppressed when it is clear from the context). It is also sometimes called π-ample. Warning: the n_0 depends on the affine open; we may not be able to take a single n_0 for all affine opens. We can, however, if Y is quasicompact, and hence we'll see this quasicompactness hypothesis on Y often.

Example. The examples of 2.7 naturally generalize.
(a) If $X \to Y$ is an affine morphism, and $\mathcal{L}$ is any invertible sheaf on X, then $\mathcal{L}$ is relatively ample.
(b) If $X \to Y$ is a projective morphism and $\mathcal{L}$ is a very ample invertible sheaf on X, then $\mathcal{L}$ is relatively ample. (We may need Y locally Noetherian here.)

2.8. *Easy Lemma. — Fix a positive integer* n.

(a) *If $\mathcal{L}$ is an invertible sheaf on a scheme X, then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes n}$ is ample.*
(b) *If $\pi : X \to Y$ is a morphism, and $\mathcal{L}$ is an invertible sheaf on X, then $\mathcal{L}$ is relatively ample if and only if $\mathcal{L}^{\otimes n}$ is relatively ample.*

In general, statements about ample sheaves (such as (a) above) will have immediate analogues for statements about relatively ample sheaves where the target is quasicompact (such as (b) above), and I won't spell them out in the future. [I'm not sure what I meant by this comment about (b); I'll think about it.]

Proof. We prove (a); (b) is then immediate.

Suppose $\mathcal{L}$ is ample. Then for any finite type sheaf $\mathcal{F}$ on X, there is some m_0 such that for $m \geq m_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes m_0}$ is generated by global sections. Thus for $m' \geq m_0/n$, $\mathcal{F} \otimes (\mathcal{L}^{\otimes n})^{m'}$ is generated by global sections, so $\mathcal{L}^{\otimes n}$ is ample.

Suppose next that $\mathcal{L}^{\otimes n}$ is ample, and let $\mathcal{F}$ by any finite type sheaf. Then there is some m_0 such that $(\mathcal{F}) \otimes (\mathcal{L}^n)^m$, $(\mathcal{F} \otimes \mathcal{L}) \otimes (\mathcal{L}^n)^m$, $(\mathcal{F} \otimes \mathcal{L}^{\otimes 2}) \otimes (\mathcal{L}^n)^m$, $\dots (\mathcal{F} \otimes \mathcal{L}^{\otimes(m-1)}) \otimes (\mathcal{L}^n)^m$, are all generated by global sections for $m \geq m_0$. In other words, for $m' \geq nm_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes m'}$ is generated by global sections. Hence $\mathcal{L}$ is ample. □

Example: any positive degree invertible sheaf on a curve is ample. Reason: a high tensor power (such that the degree is at least $2g + 1$) is very ample.

2.9. *Proposition. — In each of the following,* X *is a scheme,* $\mathcal{L}$ *is an ample invertible sheaf (hence* X *is quasicompact), and* $\mathcal{M}$ *is an invertible sheaf.*

(a) *(ample + generated = ample) If $\mathcal{M}$ is generated by global sections, then $\mathcal{L} \otimes \mathcal{M}$ is ample.*
(b) *(ample + ample = ample) If $\mathcal{M}$ is ample, then $\mathcal{L} \otimes \mathcal{M}$ is ample.*

Similar statements hold for quasicompact and quasiseparated morphisms and relatively ample and relatively generated.

Proof. (a) Suppose $\mathcal{F}$ is any finite type sheaf. Then by ampleness of $\mathcal{L}$, there is an n_0 such that for $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections. Hence $\mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{M}^{\otimes n}$ is generated by global sections. Thus there is an n_0 such that for $n \geq n_0$, $\mathcal{F} \otimes (\mathcal{L} \otimes \mathcal{M})^{\otimes n}$ is generated by global sections. Hence $\mathcal{L} \otimes \mathcal{M}$ is ample.

(b) As $\mathcal{M}$ is ample, $\mathcal{M}^{\otimes n}$ is base-point-free for some $n > 0$. But $\mathcal{L}^{\otimes n}$ is ample, so by (a) $(\mathcal{L} \otimes \mathcal{M})^{\otimes n}$ is ample, so by Lemma 2.8, $\mathcal{L} \otimes \mathcal{M}$ is ample. □

3. Every ample on a proper has a tensor power that is very ample

We'll spend the rest of our discussion of ampleness considering consequences of the following very useful result.

3.1. *Theorem. — Suppose* $\pi : X \to Y$ *is proper and* $Y = \operatorname{Spec} B$ *is affine. If* $\mathcal{L}$ *is ample, then some tensor power of* $\mathcal{L}$ *is very ample.*

The converse follows from our earlier discussion, that very ample implies ample, Example 2.7(b).

Proof. I hope to type in a short proof at some point. For now, I'll content myself with referring to Hartshorne Theorem II.7.6. (He has more hypotheses, but his argument essentially applies in this more general situation.)

3.2. *Exercise.* Suppose $\pi : X \to Y$ is proper and Y is quasicompact. Show that if $\mathcal{L}$ is relatively ample on X, then some tensor power of $\mathcal{L}$ is very ample.

Serre vanishing holds for any relatively ample invertible sheaf for a proper morphism to a Noetherian base. More precisely:

3.3. *Corollary (Serre vanishing, take two).* — *Suppose $\pi : X \to Y$ is a proper morphism, Y is quasicompact, and $\mathcal{L}$ is a π-ample invertible sheaf on X. Then for any coherent sheaf $\mathcal{F}$ on X, for $m \gg 0$, $R^i\pi_*\mathcal{F} \otimes \mathcal{L}^{\otimes m} = 0$ for all $i > 0$.*

Proof. By Theorem 3.1, $\mathcal{L}^{\otimes n}$ very ample for some n, so π is projective. Apply Serre vanishing to $\mathcal{F} \otimes \mathcal{L}^{\otimes i}$ for $0 \leq i < n$. □

The converse holds, i.e. this in fact characterizes ampleness. For convenience, we state it for the case of an affine target.

3.4. *Theorem (Serre's criterion for ampleness).* — *Suppose that $\pi : X \to Y = \operatorname{Spec} B$ is a proper morphism, and $\mathcal{L}$ is an invertible sheaf on X such that for any finite type sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections for $n \gg 0$. Then $\mathcal{L}$ is ample.*

Essentially the same statement holds for relatively ample and quasicompact target. *Exercise.* Give and prove the statement. **Whoops! Ziyu and Rob point out that I used Serre's criterion as the *definition* of ampleness (and similarly, relative ampleness). Thus this exercise is nonsense.**

3.5. *Proof of Serre's criterion.* I hope to type in a better proof before long, but for now I'll content myself with referring to Hartshorne, Proposition III.5.3.

3.6. *Exercise.* Use Serre's criterion for ampleness to prove that the pullback of ample sheaf on a projective scheme by a finite morphism is ample. Hence if a base-point-free invertible sheaf on a proper scheme induces a morphism to projective space that is finite onto its image, then it is is ample.

3.7. *Key Corollary.* — *Suppose $\pi : X \to \operatorname{Spec} B$ is proper, and $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on X with $\mathcal{L}$ ample. Then $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is very ample for $n \gg 0$.*

3.8. *Exercise.* Give and prove the corresponding statement for a relatively ample invertible sheaf over a quasicompact base.

Proof. The theorem says that $\mathcal{L}^{\otimes n}$ is very ample for $n \gg 0$. By the definition of ampleness, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is generated for $n \gg 0$. Tensor these together, using the above. □

A key implication of the key corollary is:

3.9. *Corollary. — Any invertible sheaf on a projective* $X \to \operatorname{Spec} B$ *is a difference of two very ample invertible sheaves.*

Proof. If $\mathcal{M}$ is any invertible sheaf, choose $\mathcal{L}$ very ample. Corollary 3.7 states that $\mathcal{M} \otimes \mathcal{L}^{\otimes n}$ is very ample. As $\mathcal{L}^{\otimes n}$ is very ample (Exercise 2.4), we can write $\mathcal{M}$ as the difference of two very ample sheaves: $\mathcal{M} \cong (\mathcal{M} \otimes \mathcal{L}^{\otimes n}) \otimes (\mathcal{L}^{\otimes n})^*$.

As always, we get a similar statement for relatively ample sheaves over a quasicompact base.

Here are two interesting consequences of Corollary 3.9.

3.10. *Exercise.* Suppose X a projective k-scheme. Show that every invertible sheaf is the difference of two *effective* Cartier divisors. Thus the groupification of the semigroup of effective Cartier divisors is the Picard group. Hence if you want to prove something about Cartier divisors on such a thing, you can study effective Cartier divisors.

(This is false if projective is replaced by proper — ask Sam Payne for an example.)

3.11. *Important exercise.* Suppose C is a generically reduced projective k-curve. Then we can define degree of an invertible sheaf $\mathcal{M}$ as follows. Show that $\mathcal{M}$ has a meromorphic section that is regular at every singular point of C. Thus our old definition (number of zeros minus number of poles, using facts about discrete valuation rings) applies. Prove the Riemann-Roch theorem for generically reduced projective curves. (Hint: our original proof essentially will carry through without change.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 49 AND 50

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CONTENTS

At the start of class 49, I gave an informal discussion on other criteria for ampleness, and other adjectives for divisors. We discussed the following notions: Kleiman's criterion for ampleness, numerical equivalence, Neron-Severi group, Picard number, nef, the nef cone and the ample cone, Nakai's criterion, the Nakai-Moishezon criterion, big, $\mathbb{Q}$-Cartier, Snapper's theorem.)

1. BLOWING UP A SCHEME ALONG A CLOSED SUBSCHEME

We'll next discuss an important construction in algebraic geometry (and especially the geometric side of the subject), the blow-up of a scheme along a closed subscheme (cut out by a finite type ideal sheaf). We'll start with a motivational example that will give you a picture of the construction in a particularly important case (and the historically earliest case), in Section 2. I'll then give a formal definition, in terms of universal property, Section 3. This definition won't immediately have a clear connection to the motivational example! We'll deduce some consequences of the definition (assuming that the blow-up actually exists). We'll prove that the blow-up always exists, by describing it quite explicitly, in Section 4. As a consequence, the blow-up morphism is projective, and we'll deduce more consequences from this. In Section 5, we'll do a number of explicit computations, and see that in practice, it is possible to compute many things by hand. I'll then mention a couple of useful facts: (i) the blow-up a nonsingular variety in a nonsingular variety is still nonsingular, something we'll have observed in our explicit examples, and (ii) Castelnuovo's criterion, that on a smooth surface, "(-1)-curves" ($\mathbb{P}^1$'s with normal bundle $\mathcal{O}(-1)$) can be "blown down".

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We're going to generalize the following notion, which will correspond to "blowing up" the origin of $\mathbb{A}_k^2$ (over an algebraically closed field k). Because this is just motivation, I'll be informal. Consider the subset of $\mathbb{A}^2 \times \mathbb{P}^1$ corresponding to the following. We interpret $\mathbb{P}^1$ as the lines through the origin. Consider the subset $\{(p \in \mathbb{A}^2, [\ell] \in \mathbb{P}^1) : p \in \ell)\}$. (I showed you a model in class, admittedly over the non-algebraically-closed field $k = \mathbb{R}$.)

I'll now try to convince you that this is nonsingular (informally). Now $\mathbb{P}^1$ is smooth, and for each point $[\ell]$ in $\mathbb{P}^1$, we have a smooth choice of points on the line ℓ. Thus we are verifying smoothness by way of the fibration over $\mathbb{P}^1$.

Let's make this more algebraic. Let x and y be coordinates on $\mathbb{A}^2$, and X and Y be projective coordinates on $\mathbb{P}^1$ ("corresponding" to x and y); we'll consider the subset $\mathrm{Bl}_{(0,0)} \mathbb{A}^2$ of $\mathbb{A}^2 \times \mathbb{P}^1$ corresponding to $xY - yX = 0$. We could then verify that this is nonsingular (by looking at two covering patches).

Notice that the preimage of $(0, 0)$ is a curve and hence a divisor (an effective Cartier divisor, as the blown-up surface is nonsingular). Also, note that if we have some curve singular at the origin, this could be partially desingularized. (A *desingularization* or a *resolution of singularities* of a variety X is a proper birational morphism $\tilde{X} \to X$ from a nonsingular scheme. We are interested in desingularizations for many reasons. For example, we understand nonsingular curves quite well. and we could hope to understand other curves through their desingularizations. This philosophy holds true in higher dimension as well.) For example, the curve $y^2 = x^3 + x^2$, which is nonsingular except for a node at the origin, then we can take the preimage of the curve minus the origin, and take the closure of this locus in the blow-up, and we'll obtain a nonsingular curve; the two branches of the node downstairs are separated upstairs. (This will later be an exercise, once we've defined things properly. The result will be called the *proper transform* of the curve.)

Let's generalize this. First, we can blow up $\mathbb{A}^n$ at the origin (or more informally, "blow up the origin"), getting a subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$. More algebraically, If $x_1, \ldots, x_n$ are coordinates on $\mathbb{A}^n$, and $X_1, \ldots, X_n$ are projective coordinates on $\mathbb{P}^{n-1}$, then the blow-up $\mathrm{Bl}_{\vec{0}} \mathbb{A}^n$ is given by the equations $x_i X_j - x_j X_i = 0$. Once again, this is smooth: $\mathbb{P}^{n-1}$ is smooth, and for each point $[\ell] \in \mathbb{P}^{n-1}$, we have a smooth choice of $p \in \ell$.

We can extend this further, by blowing up $\mathbb{A}^{n+m}$ along a coordinate m-plane $\mathbb{A}^n$ by adding m more variables $x_{n+1}, \ldots, x_{n+m}$ to the previous example; we get a subset of $\mathbb{A}^{n+m} \times \mathbb{P}^{n-1}$.

Then intuitively, we could extend this to blowing up a nonsingular subvariety of a nonsingular variety. We'll make this more precise. In the course of doing so, we will accidentally generalize this notion greatly, defining the blow-up of any finite type sheaf of ideals in a scheme. In general, blowing up may not have such an intuitive description as in the case of blowing up something nonsingular inside something nonsingular — it does great violence to the scheme — but even then, it is very useful (for example, in developing intersection theory). The result will be very powerful, and will touch on many other useful notions in algebra (such as the Rees algebra) that we won't discuss here.

Our description will depend only the closed subscheme being blown up, and not on coordinates. That remedies a defect was already present in the first baby example, blowing up the plane at the origin. It is not obvious that if we picked different coordinates for the plane (preserving the origin as a closed subscheme) that we wouldn't have two different resulting blow-ups.

As is often the case, there are two ways of understanding this notion, and each is useful in different circumstances. The first is by universal property, which lets you show some things without any work. The second is an explicit construction, which lets you get your hands dirty and compute things (and implies for example that the blow-up morphism is projective).

3. Blowing up, by universal property

I'll start by defining the blow-up using the universal property. The disadvantage of starting here is that this definition won't obviously be the same as the examples I just gave. It won't even look related!

Suppose $X \hookrightarrow Y$ is a closed subscheme corresponding to a finite type sheaf of ideals. (If Y is locally Noetherian, the "finite type" hypothesis is automatic, so Noetherian readers can ignore it.)

The blow-up $X \hookrightarrow Y$ is a *fiber diagram*

$$\begin{array}{ccc} E_X Y & \hookrightarrow & \mathrm{Bl}_X Y \\ \downarrow & & \downarrow \beta \\ X & \hookrightarrow & Y \end{array}$$

such that $E_X Y$ is an *effective Cartier divisor* on $\mathrm{Bl}_X Y$ (and is the scheme-theoretical pullback of X on Y), such any other such fiber diagram

(1)
$$\begin{array}{ccc} D & \hookrightarrow & W \\ \downarrow & & \downarrow \\ X & \hookrightarrow & Y, \end{array}$$

where D is an effective Cartier divisor on W, factors uniquely through it:

$$\begin{array}{ccc} D & \hookrightarrow & W \\ \downarrow & & \downarrow \\ E_X Y & \hookrightarrow & \mathrm{Bl}_X Y \\ \downarrow & & \downarrow \\ X & \hookrightarrow & Y. \end{array}$$

(Recall that an effective Cartier divisor is locally cut out by one equation that is not a zero-divisor; equivalently, it is locally cut out by one equation, and contains no associated points. This latter description will prove crucial.) $\mathrm{Bl}_X Y$ is called *the blow-up* (of Y along X, or of Y with center X). $E_X Y$ is called the *exceptional divisor*. (Bl and β stand for "blow-up", and E stands for "exceptional".)

By a universal property argument, if the blow-up exists, it is unique up to unique isomorphism. (We can even recast this more explicitly in the language of Yoneda's lemma: consider the category of diagrams of the form (1), where morphisms are of the form

$$\begin{array}{ccc} D & \hookrightarrow & W \\ \downarrow & & \downarrow \\ D' & \hookrightarrow & W' \\ \downarrow & & \downarrow \\ X & \hookrightarrow & Y. \end{array}$$

Then the blow-up is a final object in this category, if one exists.)

If $Z \hookrightarrow Y$ is any closed subscheme of Y, then the (scheme-theoretic) pullback $\beta^{-1}Z$ is called the *total transform* of Z. We will soon see that β is an isomorphism away from X (Observation 3.4). $\overline{\beta^{-1}(Z - X)}$ is called the *proper transform* or *strict transform* of Z. (We will use the first terminology. We will also define it in a more general situation.) We'll soon see that the proper transform is naturally isomorphic to $\mathrm{Bl}_{Z \cap X} Z$, where by $Z \cap X$ we mean the scheme-theoretic intersection (the blow-up closure lemma 3.7).

We will soon show that the blow-up always exists, and describe it explicitly. But first, we make a series of observations, assuming that the blow up exists.

3.1. *Observation.* If X is the empty set, then $\mathrm{Bl}_X Y = Y$. More generally, if X is a Cartier divisor, then the blow-up is an isomorphism. (Reason: $\mathrm{id}_Y : Y \to Y$ satisfies the universal property.)

3.2. *Exercise.* If U is an open subset of Y, then $\mathrm{Bl}_{U \cap X} U \cong \beta^{-1}(U)$, where $\beta : \mathrm{Bl}_X Y \to Y$ is the blow-up. (Hint: show $\beta^{-1}(U)$ satisfies the universal property!)

Thus "we can compute the blow-up locally."

3.3. *Exercise.* Show that if Y_α is an open cover of Y (as α runs over some index set), and the blow-up of Y_α along $X \cap Y_\alpha$ exists, then the blow-up of Y along X exists.

3.4. *Observation.* Combining Observation 3.1 and Exercise 3.2, we see that the blow-up is an isomorphism away from the locus you are blowing up:

$$\beta|_{\mathrm{Bl}_X Y - E_X Y} : \mathrm{Bl}_X Y - E_X Y \to Y - X$$

is an isomorphism.

3.5. *Observation.* If $X = Y$, then the blow-up is the empty set: the only map $W \to Y$ such that the pullback of X is a Cartier divisor is $\emptyset \hookrightarrow Y$. In this case we have "blown Y out of existence"!

3.6. *Exercise (blow-up preserves irreducibility and reducedness).* Show that if Y is irreducible, and X doesn't contain the generic point of Y, then $\mathrm{Bl}_X Y$ is irreducible. Show that if Y is reduced, then $\mathrm{Bl}_X Y$ is reduced.

The following blow-up closure lemma is useful in several ways. At first, it is confusing to look at, but once you look closely you'll realize that it is not so unreasonable.

Suppose we have a fibered diagram

$$\begin{array}{ccc} W & \stackrel{\text{cl. imm.}}{\hookrightarrow} & Z \\ \downarrow & & \downarrow \\ X & \stackrel{\text{cl. imm.}}{\hookrightarrow} & Y \end{array}$$

where the bottom closed immersion corresponds to a finite type ideal sheaf (and hence the upper closed immersion does too). The first time you read this, it may be helpful to consider the special case where $Z \to Y$ is a closed immersion.

Then take the fiber product of this square by the blow-up $\beta : \mathrm{Bl}_X Y \to Y$, to obtain

$$\begin{array}{ccc} Z \times_Y E_X Y & \hookrightarrow & Z \times_Y \mathrm{Bl}_X Y \\ \downarrow & & \downarrow \\ E_X Y & \stackrel{\text{Cartier}}{\hookrightarrow} & \mathrm{Bl}_X Y. \end{array}$$

The bottom closed immersion is locally cut out by one equation, and thus the same is true of the top closed immersion as well. However, it need not be a non-zero-divisor, and thus the top closed immersion is not necessarily an effective Cartier divisor.

Let $\overline{Z}$ be the scheme-theoretic closure of $Z \times_Y \mathrm{Bl}_X Y - W \times_Y \mathrm{Bl}_X Y$ in $Z \times_Y \mathrm{Bl}_X Y$. Note that in the special case where $Z \to Y$ is a closed immersion, $\overline{Z}$ is the proper transform, as defined in §3. For this reason, it is reasonable to call $\overline{Z}$ the proper transform of Z even if Z *isn't* a closed immersion. Similarly, it is reasonable to call $Z \times_Z \mathrm{Bl}_X Y$ the total transform even if Z isn't a closed immersion.

Define $E_{\overline{Z}} \hookrightarrow \overline{Z}$ as the pullback of $E_X Y$ to $\overline{Z}$, i.e. by the fibered diagram

$$\begin{array}{cccl} E_{\overline{Z}} & \hookrightarrow & \overline{Z} & \quad \text{proper transform} \\ \downarrow{\scriptstyle \text{cl. imm.}} & & \downarrow{\scriptstyle \text{cl. imm.}} & \\ Z \times_Y E_X Y & \hookrightarrow & Z \times_Y \mathrm{Bl}_X Y & \quad \text{total transform} \\ \downarrow & & \downarrow & \\ E_X Y & \stackrel{\text{Cartier}}{\hookrightarrow} & \mathrm{Bl}_X Y. & \end{array}$$

Note that $E_{\overline{Z}}$ is Cartier on $\overline{Z}$ (as it is locally the zero-scheme of a single function that does not vanish on any associated points of $\overline{Z}$).

3.7. *Blow-up closure lemma.* — $(\mathrm{Bl}_Z W, E_Z W)$ *is canonically isomorphic to* $(\overline{Z}, E_{\overline{Z}})$.

This is very handy.

The first three comments apply to the special case where $Z \to W$ is a closed immersion, and the fourth basically tells us we shouldn't have concentrated on this special case.

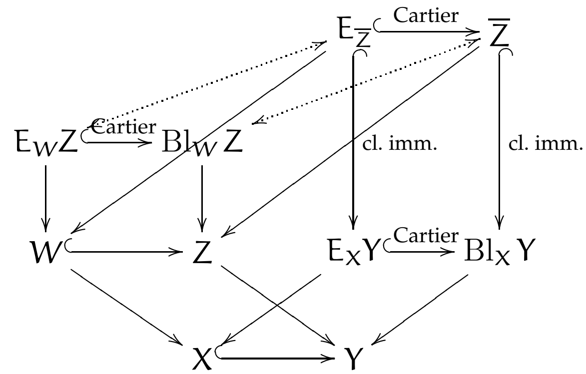
(1) First, note that if $Z \to Y$ is a closed immersion, then this states that the proper transform (as defined in §3) is the blow-up of Z along the scheme-theoretic intersection $W = X \cap Z$.

(2) In particular, it lets you actually compute blow-ups, and we'll do lots of examples soon. For example, suppose C is a plane curve, singular at a point p, and we want to blow up C at p. Then we could instead blow up the plane at p (which we have already described how to do, even if we haven't yet proved that it satisfies the universal property of blowing up), and then take the scheme-theoretic closure of $C - p$ in the blow-up.

(3) More generally, if W is some nasty subscheme of Z that we wanted to blow-up, and Z were a finite type k-scheme, then the same trick would work. We could work locally (Exercise 3.2), so we may assume that Z is affine. If W is cut out by r equations $f_1, \dots, f_r \in \Gamma(\mathcal{O}_Z)$, then complete the f's to a generating set $f_1, \dots, f_n$ of $\Gamma(\mathcal{O}_Z)$. This gives a closed immersion $Y \hookrightarrow \mathbb{A}^n$ such that W is the scheme-theoretic intersection of Y with a coordinate linear space $\mathbb{A}^r$.

3.8. (4) Most generally still, this reduces the existence of the blow-up to a specific special case. (If you prefer to work over a fixed field k, feel free to replace $\mathbb{Z}$ by k in this discussion.) Suppose that for each n, $\mathrm{Bl}_{(x_1,\dots,x_n)} \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$ exists. Then I claim that the blow-up always exists. Here's why. We may assume that Y is affine, say $\operatorname{Spec} B$, and $X = \operatorname{Spec} B/(f_1, \dots, f_n)$. Then we have a morphism $Y \to \mathbb{A}^n_{\mathbb{Z}}$ given by $x_i \mapsto f_i$, such that X is the scheme-theoretic pullback of the origin. Hence by the blow-up closure lemma, $\mathrm{Bl}_X Y$ exists.

3.9. *Tricky Exercise+.* Prove the blow-up closure lemma. Hint: obviously, construct maps in both directions, using the universal property. The following diagram may or may not help.



3.10. *Exercise.* If Y and Z are closed subschemes of a given scheme X, show that $\mathrm{Bl}_Y Y \cup Z \cong \mathrm{Bl}_{Y \cap Z} Z$. (In particular, if you blow up a scheme along an irreducible component, the irreducible component is blown out of existence.)

4. The blow-up exists, and is projective

It is now time to show that the blow up always exists. I'll give two arguments, because I find them enlightening in two different ways. Both will imply that the blow-up morphism is projective. Hence the blow-up morphism is projective, hence quasicompact, proper, finite type, separated. In particular, if $Y \to Z$ is projective (resp. quasiprojective, quasicompact, proper, finite type, separated), so is $\mathrm{Bl}_X Y \to Z$. The blow-up of a k-variety is a k-variety (using the fact that irreducibility, reducedness are preserved, Exercise 3.6).

Approach 1. As explained above (§3.8), it suffices to show that $\mathrm{Bl}_{(x_1,\dots,x_n)} \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$ exists. But we know what it is supposed to be: the locus in

$$\operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n] \times \operatorname{Proj} \mathbb{Z}[X_1, \dots, X_n]$$

such that $x_i X_j - x_j X_i = 0$. We'll show this soon.

Approach 2. We can describe the blow-up all at once as a $\underline{\operatorname{Proj}}$.

4.1. *Theorem ($\underline{\operatorname{Proj}}$ description of the blow-up). — Suppose $X \hookrightarrow Y$ is a closed subscheme cut out by a finite type sheaf of ideals $\mathcal{I} \hookrightarrow \mathcal{O}_Y$. Then*

$$\underline{\operatorname{Proj}} \left(\mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots \right) \to Y$$

satisfies the universal property of blowing up.

We'll prove this soon (Section 4.2), after seeing what this gives us. (The reason we had a finite type requirement is that I wanted this $\underline{\operatorname{Proj}}$ to exist; we needed the sheaf of algebras to satisfy the conditions stated earlier.)

But first, we should make sure that the preimage of X is indeed an effective Cartier divisor. We can work affine-locally (Exercise 3.2), so I'll assume that $Y = \operatorname{Spec} B$, and X is cut out by the finitely generated ideal I. Then

$$\mathrm{Bl}_X Y = \mathrm{Proj}\left(B \oplus I \oplus I^2 \oplus \cdots\right).$$

(We are slightly abusing notation by using the notation $\mathrm{Bl}_X Y$, as we haven't yet shown that this satisfies the universal property. But I hope that by now you trust me.)

The preimage of X isn't just any effective Cartier divisor; it corresponds to the invertible sheaf $\mathcal{O}(1)$ on this $\underline{\mathrm{Proj}}$. Indeed, $\mathcal{O}(1)$ corresponds to taking our graded ring, chopping off the bottom piece, and sliding all the graded pieces to the left by 1; it is the invertible sheaf corresponding to the graded module

$$I \oplus I^2 \oplus I^3 \oplus \cdots$$

(where that first summand I has grading 0). But this can be interpreted as the scheme-theoretic pullback of X, which corresponds to the ideal I of B:

$$I\left(B \oplus I \oplus I^2 \oplus \cdots\right) \hookrightarrow B \oplus I \oplus I^2 \oplus \cdots.$$

Thus the scheme-theoretic pullback of $X \hookrightarrow Y$ to $\underline{\mathrm{Proj}}\,\mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \cdots$, the invertible sheaf corresponding to $\mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots$, is an effective Cartier divisor in class $\mathcal{O}(1)$. Once we have verified that this construction is indeed the blow-up, this divisor will be our exceptional divisor $E_X Y$.

Moreover, we see that the exceptional divisor can be described beautifully as a $\underline{\mathrm{Proj}}$ over X:

$$E_X Y = \underline{\mathrm{Proj}}_X\, B/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots. \tag{2}$$

We'll later see that in good circumstances (if X is a local complete intersection in something nonsingular, or more generally a local complete intersection in a Cohen-Macaulay scheme) this is a projective bundle (the "projectivized normal bundle").

4.2. Proof of the universal property, Theorem 4.1. Let's prove that this $\underline{\mathrm{Proj}}$ construction satisfies the universal property. Then approach 1 will also follow, as a special case of approach 2. You may ask why I bothered with approach 1. I have two reasons: one is that you may find it more comfortable to work with this one nice ring, and the picture may be geometrically clearer to you (in the same way that thinking about the blow-up closure lemma in the case where $Z \to Y$ is a closed immersion is more intuitive). The second reason is that, as you'll find in the exercises, you'll see some facts more easily in this explicit example, and you can then pull them back to more general examples.

Proof. Reduce to the case of affine target R with ideal I. Reduce to the case of affine source, with principal effective Cartier divisor t. (A principal effective Cartier divisor is cut out by a single non-zero-divisor. Recall that an effective Cartier divisor is cut out only *locally* by a single non-zero divisor.) Thus we have reduced to the case $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$, corresponding to $f : R \to S$. Say $(x_1, \dots, x_n) = I$, with $(f(x_1), \dots, f(x_n)) = (t)$. We'll describe *one* map $\mathrm{Spec}\, S \to \mathrm{Proj}\, R[I]$ that will extend the map on the open set $\mathrm{Spec}\, S_t \to \mathrm{Spec}\, R$. It is then unique: a map to a separated R-scheme is determined by its behavior away from the associated points (proved earlier). We map R[I] to S as follows: the degree one part is $f : R \to S$, and $f(X_i)$ (where X_i corresponds to x_i, except it is in degree 1) goes to $f(x_i)/t$. Hence an element X of degree d goes to $X/(t^d)$. On the open set $D_+(X_1)$, we get the map $R[X_2/X_1, \dots, X_n/X_1]/(x_2 - X_2/X_1 x_1, \dots, x_i X_j - x_j X_i, \dots) \to S$ (where there may be many relations) which agrees with f away from $D(t)$. Thus this map does extend away from $V(I)$. $\square$

Here are some applications and observations arising from this construction of the blow-up.

4.3. *Observation.* We can verify that our initial motivational examples are indeed blow-ups. For example, blowing up $\mathbb{A}^2$ (with co-ordinates x and y) at the origin yields: $B = k[x,y]$, $I = (x,y)$, and $\operatorname{Proj} B \oplus I \oplus I^2 = \operatorname{Proj} B[X,Y]$ where the elements of B have degree 0, and X and Y are degree 1 and correspond to x and y.

4.4. *Observation.* Note that the normal bundle to a Cartier divisor D is the invertible sheaf $\mathcal{O}(D)|_D$, the invertible sheaf corresponding to the D on the total space, then restricted to D. (This was discussed earlier in the section on differentials.) (Reason: if D corresponds to the ideal sheaf $\mathcal{I}$, then recall that $\mathcal{I} = \mathcal{O}(D)^\vee$, and that the conormal sheaf was $\mathcal{I}/\mathcal{I}^2 = \mathcal{I}|_D$.) The ideal sheaf corresponding to the exceptional divisor is $\mathcal{O}(1)$, so the invertible sheaf corresponding to the exceptional divisor is $\mathcal{O}(-1)$. (I prefer to think of this in light of approach 1, but there is no real difference.) Thus for example in the case of the blow-up of a point in the plane, the exceptional divisor has normal bundle $\mathcal{O}(-1)$. In the case of the blow-up of a nonsingular subvariety of a nonsingular variety, the blow up turns out to be nonsingular (a fact discussed soon in §6.1), and the exceptional divisor is a projective bundle over X, and the normal bundle to the exceptional divisor restricts to $\mathcal{O}(-1)$.

4.5. More serious application: dimensional vanishing for quasicoherent sheaves on quasiprojective schemes. Here is something promised long ago. I want to point out something interesting here: in proof I give below, we will need to potentially blow up arbitrary closed schemes. We won't need to understand precisely what happens when we do so; all we need is the fact that the exceptional divisor is indeed a (Cartier) divisor.

5. Explicit computations

In this section you will do a number of explicit of examples, to get a sense of how blow-ups behave, how they are useful, and how one can work with them explicitly. For convenience, all of the following are over an algebraically closed field k of characteristic 0.

5.1. *Example: Blowing up the plane along the origin.* Let's first blow up the plane $\mathbb{A}^2_k$ along the origin, and see that the result agrees with our discussion in §2. Let x and y be the coordinates on $\mathbb{A}^2_k$. The the blow-up is $\operatorname{Proj} k[x,y,X,Y]$ where $xY - yX = 0$. This is naturally a closed subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$, cut out (in terms of the projective coordinates X and Y on $\mathbb{P}^1_k$) by $xY - yX = 0$. We consider the two usual patches on $\mathbb{P}^1_k$: $[X;Y] = [s;1]$ and $[1;t]$. The first patch yields $\operatorname{Spec} k[x,y,s]/(sy - x)$, and the second gives $\operatorname{Spec} k[x,y,t]/(y -$

xt). Notice that both are nonsingular: the first is naturally $\operatorname{Spec} k[y,s] \cong \mathbb{A}^2_k$, the second is $\operatorname{Spec} k[x,t] \cong \mathbb{A}^2_k$.

Let's describe the exceptional divisor. We first consider the first (s) patch. The ideal is generated by (x,y), which in our ys-coordinates is $(ys, y) = (y)$, which is indeed principal. Thus on this patch the exceptional divisor is generated by y. Similarly, in the second patch, the exceptional divisor is cut out by x. (This can be a little confusing, but there is no contradiction!)

5.2. *The proper transform of a nodal curve.* Consider next the curve $y^2 = x^3 + x^2$ inside the plane $\mathbb{A}^2_k$. Let's blow up the origin, and compute the total and proper transform of the curve. (By the blow-up closure lemma, the latter is the blow-up of the nodal curve at the origin.) In the first patch, we get $y^2 - s^2y^2 - s^3y^3 = 0$. This factors: we get the exceptional divisor y with multiplicity two, and the curve $1 - s^2 - y^3 = 0$. Easy exercise: check that the proper transform is nonsingular. Also, notice where the proper transform meets the exceptional divisor: at two points, $s = \pm 1$. This corresponds to the two tangent directions at the origin. (Notice that $s = y/x$.)

5.3. *Exercise.* Describe both the total and proper transform of the curve C given by $y = x^2 - x$ in $\operatorname{Bl}_{(0,0)} \mathbb{A}^2$. Verify that the proper transform of C is isomorphic to C. Interpret the intersection of the proper transform of C with the exceptional divisor E as the slope of C at the origin.

5.4. *Exercise: blowing up a cuspidal plane curve.* Describe the proper transform of the cuspidal curve C' given by $y^2 = x^3$ in the plane $\mathbb{A}^2_k$. Show that it is nonsingular. Show that the proper transform of C meets the exceptional divisor E at one point, and is tangent to E there.

5.5. *Exercise.* (a) Desingularize the tacnode $y^2 = x^4$ by blowing up the plane at the origin (and taking the proper transform), and then blowing up the resulting surface once more.
(b) Desingularize $y^8 - x^5 = 0$ in the same way. How many blow-ups do you need?
(c) Do (a) instead in one step by blowing up (y, x^2).

5.6. *Exercise.* Blowing up a nonreduced subscheme of a nonsingular scheme can give you something singular, as shown in this example. Describe the blow up of the ideal (x, y^2) in $\mathbb{A}^2_k$. What singularity do you get? (Hint: it appears in a nearby exercise.)

5.7. *Exercise.* Blow up the cone point $z^2 = x^2 + y^2$ at the origin. Show that the resulting surface is nonsingular. Show that the exceptional divisor is isomorphic to $\mathbb{P}^1$.

5.8. *Harder but enlightening exercise.* If $X \hookrightarrow \mathbb{P}^n$ is a projective scheme, show that the exceptional divisor of the blow up the affine cone over X at the origin is isomorphic to X, and that its normal bundle is $\mathcal{O}_X(-1)$. (I prefer approach 1 here, but both work.)

In the case $X = \mathbb{P}^1$, we recover the blow-up of the plane at a point. In particular, we again recover the important fact that the normal bundle to the exceptional divisor is $\mathcal{O}(-1)$.

5.9. *Exercise.* Show that the multiplicity of the exceptional divisor in the total transform of a subscheme of $\mathbb{A}^n$ when you blow up the origin is the lowest degree that appears in a defining equation of the subscheme. (For example, in the case of the nodal and cuspidal curves above, Example 5.2 and Exercise 5.4 respectively, the exceptional divisor appears with multiplicity 2.) This is called the *multiplicity* of the singularity.

5.10. *Exercise.* Suppose Y is the cone $x^2 + y^2 = z^2$, and X is the ruling of the cone $x = 0$, $y = z$. Show that $\mathrm{Bl}_X Y$ is nonsingular. (In this case we are blowing up a codimension 1 locus that is not a Cartier divisor. Note that it *is* Cartier away from the cone point, so you should expect your answer to be an isomorphism away from the cone point.)

5.11. *Harder but useful exercise (blow-ups resolve base loci of rational maps to projective space).* (I find this easier via method 1.) Suppose we have a scheme Y, an invertible sheaf $\mathcal{L}$, and a number of sections $s_0, \ldots, s_n$ of $\mathcal{L}$. Then away from the closed subscheme X cut out by $s_0 = \cdots = s_n = 0$, these sections give a morphism to $\mathbb{P}^n$. Show that this morphism extends to a morphism $\mathrm{Bl}_X Y \to \mathbb{P}^n$, where this morphism corresponds to the invertible sheaf $(\pi^* \mathcal{L})(-E_X Y)$, where $\pi : \mathrm{Bl}_X Y \to Y$ is the blow-up morphism. In other words, "blowing up the base scheme resolves this rational map". (Hint: it suffices to consider an affine open subset of Y where $\mathcal{L}$ is trivial.)

6. Two stray facts

There are two stray facts I want to mention.

6.1. Blowing up a nonsingular in a nonsingular. The first is that if you blow up a nonsingular subscheme of a nonsingular locally Noetherian scheme, the result is nonsingular. I didn't have the time to prove this, but I discussed some of the mathematics behind it. (This is harder than our previous discussion. Also, it uses a flavor of argument that in general I haven't gotten to, about local complete intersections and Cohen-Macaulayness.) Moreover, for a local complete intersection $X \hookrightarrow Y$ cut out by ideal sheaf $\mathcal{I}$, I/I^2 is locally free (class 39/40, Theorem 2.20, p. 10). Then it is a fact (unproved here) that for a local complete intersection, the natural map $\mathrm{Sym}^n \mathcal{I}/\mathcal{I}^2 \to \mathcal{I}^n/\mathcal{I}^{n+1}$ is an isomorphism. Of course it suffices to prove this for affine open sets. More generally, if A is Cohen-Macaulay (recall that I've stated that nonsingular schemes are Cohen-Macaulay), and $x_1, \ldots, x_r \in \mathfrak{m}$ is a regular sequence, with $I = (x_1, \ldots, x_r)$, then the natural map is an isomorphism. You can read about this at p. 110 of Matsumura's Commutative Algebra.

Assuming this fact, we conclude that if $X \hookrightarrow Y$ is a complete intersection in a nonsingular scheme (or more generally cut out by a regular sequence in a Cohen-Macaulay scheme), the exceptional divisor is the projectivized normal bundle (by (2)). (Exercise:

Blow up (xy, z) in $\mathbb{A}^3$, and verify that the exceptional divisor is indeed the projectivized normal bundle.)

In particular, in the case where we blow up a nonsingular subvariety in a nonsingular variety, the exceptional divisor is nonsingular. We can then show that the blow-up is nonsingular as follows. The blow-up $\mathrm{Bl}_X Y$ remains nonsingular away from $E_X Y$, as it is here isomorphic to the nonsingular space $Y - X$. Thus we need check only the exceptional divisor. Fix any point of the exceptional divisor p. Then the dimension of $E_X Y$ at p is precisely the dimension of the Zariski tangent space (by nonsingularity). Moreover, the dimension of $\mathrm{Bl}_X Y$ at p is one more than that of $E_X Y$ (by Krull's Principal Ideal Theorem), as the latter is an effective Cartier divisor), and the dimension of the Zariski tangent space of $\mathrm{Bl}_X Y$ at p is at most one more than that of $E_X Y$. But the first of these is at most as big as the second, so we must have equality, which means that $\mathrm{Bl}_X Y$ is nonsingular at p.

6.2. *Exercise.* Suppose X is an irreducible nonsingular subvariety of a nonsingular variety Y, of codimension at least 2. Describe a natural isomorphism $\mathrm{Pic}\, \mathrm{Bl}_X Y \cong \mathrm{Pic}\, Y \oplus \mathbb{Z}$. (Hint: compare divisors on $\mathrm{Bl}_X Y$ and Y. Show that the exceptional divisor $E_X Y$ gives a non-torsion element of $\mathrm{Pic}(\mathrm{Bl}_X Y)$ by describing a $\mathbb{P}^1$ on $\mathrm{Bl}_X Y$ which has intersection number -1 with $E_X Y$.)

(If I had more time, I would have used this to give Hironaka's example of a nonprojective proper nonsingular threefold. If you are curious and have ten minutes, please ask me! It includes our nonprojective proper surface as a closed subscheme, and indeed that is how we can show nonprojectivity.)

6.3. Castelnuovo's criterion.

A curve in a nonsingular surface that is isomorphic to $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1)$ is called a (-1)-curve. We've shown that if we blow up a nonsingular point of a surface at a (reduced) point, the exceptional divisor is a (-1)-curve. Castelnuovo's criterion is the converse: if we have a quasiprojective surface containing a (-1)-curve, that surface is obtained by blowing up another surface at a reduced nonsingular point. (We say that we can "blow down" the (-1)-curve.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 51 AND 52

RAVI VAKIL

CONTENTS

1. SMOOTH, ÉTALE, UNRAMIFIED

We will next describe analogues of some important notions in differential geometry — the following particular types of maps of manifolds. They naturally form a family of three.

- *Submersions* are maps that induce surjections of tangent spaces everywhere. They are useful in the notion of a fibration.
- *Covering spaces* are maps that induce isomorphisms of tangent spaces, or equivalently, are local isomorphisms.
- *Immersions* are maps that induce injections of tangent spaces.

Warning repeated from earlier: "immersion" is often used in algebraic geometry with a different meaning. We won't use this word in an algebro-geometric context (without an adjective such as "open" or "closed") in order to avoid confusion. I drew pictures of the three. (A fourth notion is related to these three: a map of manifolds is an *embedding* if it is an immersion that is an inclusion of sets, where the source has the subspace topology. This is analogous to *locally closed immersion* in algebraic geometry.)

We will define algebraic analogues of these three notions: smooth, étale, and unramified. In the case of nonsingular varieties over an algebraically closed field, we could take the differential geometric definition. We would like to define these notions more generally. Indeed, one of the points of algebraic geometry is to generalize "smooth" notions to singular situations. Also, we'll want to make arguments by "working over" the generic point, and also over nonreduced subschemes. We may even want to do things over non-algebraically closed fields, or over the integers.

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Our definitions will be combinations of notions we've already seen, and thus we'll see that they have many good properties. We'll see (§2.1) that in the category of nonsingular varieties over algebraically closed fields, we recover the differential geometric definition. Our three definitions won't be so obviously a natural triplet, but I'll mention the definition given in EGA (§4.1), and in this context once again the definitions are very similar.

Let's first consider some examples of things we want to be analogues of "covering space" and "submersion", and see if they help us make good definitions.

We'll start with something we would want to be a covering space. Consider the parabola $x = y^2$ projecting to the x-axis, over the complex numbers. (This example has come up again and again!) We might reasonably want this to be a covering space away from the origin. We might also want the notion of covering space to be an open condition: the locus where a morphism is a covering space should be open on the source. This is true for the differential geometric definition. (More generally, we might want this notion to be preserved by base change.) But then this should be a "covering space" over the generic point, and here we get a non-trivial residue field extension ($\mathbb{C}(y)/\mathbb{C}(y^2)$), not an isomorphism. Thus we are forced to consider (the Spec's of) certain finite extensions of fields to be covering spaces. (We'll see soon that we just want separable extensions.)

Note also in this example there are no (non-empty) Zariski-open subsets $U \subset X$ and $V \subset V$ where the map sends U into V isomorphically. This will later lead to the notion of the étale topology, which is a bizarre sort of topology (not even a topology in the usual sense, but a "Grothendieck topology").

1.1. Here is an issue with smoothness: we would certainly want the fibers to be smooth, so reasonably we would want the fibers to be nonsingular. But we know that nonsingularity over a field does not behave well over a base change (consider $\operatorname{Spec} k(t)[u]/(u^p - t) \to \operatorname{Spec} k(t)$ and base change by $\operatorname{Spec} k(t)[v]/(v^p - t) \to \operatorname{Spec} k(t)$, where $\operatorname{char} k = p$). We can patch that by noting that nonsingularity behaves well over algebraically closed fields, and hence we could require that all the geometric fibers are nonsingular. But that isn't quite enough. For example, a horrible map from a scheme X to a curve Y that maps a different nonsingular variety to a each point Y (X is an infinite disjoint union of these) should not be considered a submersion in any reasonable sense. Also, we might reasonably not want to consider $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]/\epsilon^2$ to be a submersion (for example, this isn't surjective on tangent spaces, and more generally the picture "doesn't look like a fibration"). (I drew pictures of these two pathologies.) Both problems are failures of $\pi : X \to Y$ to be a nice, "continuous" family. Whenever we are looking for some vague notion of "niceness" we know that "flatness" will be in the definition. (This is the reason we waited so long before introducing the notion of smoothness — we needed to develop flatness first!)

One last issue: we will require the geometric fibers to be varieties, so we can think of them as "smooth" in the old-fashioned intuitive sense. We could impose this by requiring our morphisms to be locally of finite type, or (a stronger condition) locally of finite presentation.

I should have defined "locally of finite presentation" back when we defined "locally of finite type" and the many other notions satisfying the affine covering lemma. It isn't any harder. A morphism of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ is *locally of finite presentation* if it corresponds to $B \to B[x_1, \dots, x_n]/(f_1, \dots, f_r)$ — A should be finitely generated over B, and also have a finite number of relations. This notion satisfies the hypotheses of the affine covering lemma. A morphism of schemes $\pi : X \to Y$ is *locally of finite presentation* if every map of affine open sets $\operatorname{Spec} A \to \operatorname{Spec} B$ induced by π is locally of finite presentation. If you work only with locally Noetherian schemes, then these two notions are the same.

I haven't thought through why Grothendieck went with the stricter condition of "locally of finite presentation" in his definition of smooth etc., rather than "locally of finite type".

Finally, we define our three notions!

1.2. Definition. A morphism $\pi : X \to Y$ is *smooth of relative dimension* n provided that it is locally of finite presentation and flat of relative dimension n, and $\Omega_{X/Y}$ is locally free of rank n.

A morphism $\pi : X \to Y$ is *étale* provided that it is locally of finite presentation and flat, and $\Omega_{X/Y} = 0$.

A morphism $\pi : X \to Y$ is *unramified* provided that it is locally of finite presentation, and $\Omega_{X/Y} = 0$.

1.3. Examples.

- $\mathbb{A}^n_Y \to Y$, $\mathbb{P}^n_Y \to Y$ are smooth morphisms of relative dimension n.
- Locally finitely presented open immersions are étale.
- *Unramified.* Locally finitely presented locally closed immersions are unramified.

1.4. Quick observations and comments.

1.5. All three notions are local on the target, and local on the source, and are preserved by base change. That's because all of the terms arising in the definition have these properties. *Exercise.* Show that all three notions are open conditions. State this rigorously and prove it. (Hint: Given $\pi : X \to Y$, then there is a largest open subset of X where π is smooth of relative dimension n, etc.)

1.6. Note that π is étale if and only if π is smooth and unramified, if and only if π is flat and unramified.

1.7. *Jacobian criterion.* The smooth and étale definitions are perfectly set up to use a Jacobian criterion. *Exercise.* Show that $\operatorname{Spec} B[x_1, \dots, x_n]/(f_1, \dots, f_r) \to \operatorname{Spec} B$ is smooth of relative dimension n (resp. étale) if it is flat of relative dimension n (resp. flat) and the corank of Jacobian matrix is n (resp. the Jacobian matrix is full rank).

1.8. *Exercise: smoothness etc. over an algebraically closed field.* Show that if k is an algebraically closed field, $X \to \operatorname{Spec} k$ is smooth of relative dimension n if and only if X is a disjoint union of nonsingular k-varieties of dimension n. (Hint: use the Jacobian criterion.) Show that $X \to \operatorname{Spec} k$ is étale if and only if it is unramified if and only if X is a union of points isomorphic to $\operatorname{Spec} k$. More generally, if k is a field (not necessarily algebraically closed), show that $X \to \operatorname{Spec} k$ is étale if and only if it is unramified if and only if X is the disjoint union of Spec's of finite separable extensions of k.

1.9. A morphism $\pi : X \to Y$ is *smooth* if it is locally of finite presentation and flat, and in an open neighborhood of every point $x \in X$ in which π is of constant relative dimension, $\Omega_{X/Y}$ is locally free of that relative dimension. (I should have shown earlier that the locus where a locally of finite presentation morphism is flat of a given relative dimension is open, but I may not have. We indeed showed the fact without the "relative dimension" statement, and the argument is essentially the same with this condition added.) (*Exercise.* Show that π is smooth if X can be written as a disjoint union $X = \coprod_{n \geq 0} X_n$ where $\pi|_{X_n}$ is smooth of relative dimension n.) This notion isn't really as "clean" as "smooth of relative dimension n", but people often use the naked adjective "smooth" for simplicity.

Exercise. Show that étale is the same as smooth of relative dimension 0. In other words, show that étale implies relative dimension 0. (Hint: if there is a point $x \in X$ where π has positive relative dimension, show that $\Omega_{X/Y}$ is not 0 at x. You may want to base change, to consider just the fiber above $\pi(x)$.)

1.10. Note that unramified doesn't have a flatness hypothesis, and indeed we didn't expect it, as we would want the inclusion of the origin into $\mathbb{A}^1$ to be unramified. Thus seemingly pathological things of the sort we excluded from the notion of "smooth" and "unramified" morphisms are unramified. For example, if $X = \coprod_{z \in \mathbb{C}} \operatorname{Spec} \mathbb{C}$, then the morphism $X \to \mathbb{A}^1_{\mathbb{C}}$ sending the point corresponding to z to the point $z \in \mathbb{A}^1_{\mathbb{C}}$ is unramified. Such is life.

Exercise. Suppose
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h = g \circ f} \searrow \quad \swarrow {\scriptstyle g} & \\ & Z & \end{array}$$
are locally finitely presented morphisms.

(a) Show that if h is unramified, then so is f. (Hint: property P exercise.)
(b) Suppose g is étale. Show that f is smooth (resp. étale, unramified) if and only if h is. (Hint: Observe that $\Omega_{X/Y} \to \Omega_{X/Y}$ is an isomorphism from the relative cotangent sequence, see 2.3 for a reminder.)

Regularity vs. smoothness. Suppose $\operatorname{char} k = p$, and consider the morphism $\operatorname{Spec} k(u) \to \operatorname{Spec} k(u^p)$. Then the source is nonsingular, but the morphism is not étale (or smooth, or unramified).

In fact, if k is not algebraically closed, "nonsingular" isn't a great notion, as we saw in the fall when we had to work hard to develop the theory of nonsingularity. Instead, "smooth (of some dimension)" over a field is much better. You should almost go back in your notes and throw out our discussion of nonsingularity. But don't — there were a couple of key concepts that have been useful: discrete valuation rings (nonsingularity in codimension 1) and nonsingularity at closed points of a variety (nonsingularity in top codimension).

2. Harder facts

I want to segregate three facts which require more effort, to emphasize that the earlier facts are automatic given what we know.

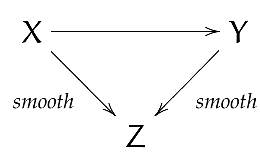
2.1. Connection to differential-geometric notion of smoothness.

The following exercise makes the connection to the differential-geometric notion of smoothness. Unfortunately, we will need this fact in the next section on generic smoothness.

2.2. *Trickier Exercise.* Suppose $\pi : X \to Y$ is a morphism of smooth (pure-dimensional) varieties over a field k. Let $n = \dim X - \dim Y$. Suppose that for each closed point $x \in X$, the induced map on the Zariski tangent space $T_f : T_x \to T_y$ is surjective. Show that f is smooth of relative dimension n. (Hint: The trickiest thing is to show flatness. Use the (second) local criterion for flatness.)

I think this is the easiest of the three "harder" facts, and it isn't so bad.

For pedants: I think the same argument works over a more arbitrary base. In other words, suppose in the following diagram of pure-dimensional Noetherian schemes, Y is reduced.



Let $n = \dim X - \dim Y$. Suppose that for each closed point $x \in X$, the induced map on the Zariski tangent space $T_f : T_x \to T_y$ is surjective. Show that f is smooth of relative dimension n. I think the same argument works, with a twist at the end using Exercise 1.10(b). Please correct me if I'm wrong!

2.3. The relative cotangent sequence is left-exact in good circumstances.

Recall the relative cotangent sequence. Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes. Then there is an exact sequence of quasicoherent sheaves on X

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

We have been always keeping in mind that if you see a right-exact sequence, you should expect that this is the tail end of a long exact sequence. In this case, you should expect that the next term to the left (the "H_1 term") should depend just on X/Y, and not on Z, because the last term on the right does. Indeed this is the case: these "homology" groups are called André-Quillen homology groups. You might also hope then that in some mysteriously "good" circumstances, this first "H_1" on the left should vanish, and hence the relative cotangent sequence should be exact on the left. Indeed that is the case, as is hinted by the following exercise.

2.4. *Exercise on differentials.* If $X \to Y$ is a smooth morphism, show that the relative cotangent sequence is exact on the left as well.

This exercise is the reason this discussion is in the "harder" section — the rest is easy. Can someone provide a clean proof of this fact?

2.5. *Unimportant exercise.* Predict a circumstance in which the relative conormal sequence is left-exact.

2.6. *Corollary.* Suppose f is étale. Then the pullback of differentials $f^*\Omega_{Y/Z} \to \Omega_{X/Z}$ is an isomorphism. (This should be very believable to you from the picture you should have in your head!)

2.7. *Exercise.* Show that all three notions are preserved by composition. (More precisely, in the smooth case, smooth of relative dimension m composed with smooth of relative dimension n is smooth of relative dimension $n+m$.) You'll need Exercise 2.4 in the smooth case.

2.8. *Easy exercise.* Show that all three notions are closed under products. (More precisely, in the case of smoothness: If $X, Y \to Z$ are smooth of relative dimension m and n respectively, then $X \times_Z Y \to Z$ is smooth of relative dimension $m+n$.) (Hint: This is a consequence of base change and composition, as we have discussed earlier. Consider $X \times_Z Y \to Y \to Z$.)

2.9. *Exercise: smoothness implies surjection of tangent sheaves.* Continuing the terminology of the above, Suppose $X \to Y$ is a smooth morphism of Z-schemes. Show that $0 \to T_{X/Y} \to T_{X/Z} \to f^*T_{Y/Z} \to 0$ is an exact sequence of sheaves, and in particular, $T_{X/Z} \to f^*T_{Y/Z}$ is surjective, paralleling the notion of submersion in differential geometry. (Recall $T_{X/Y} = \underline{\mathrm{Hom}}(\Omega_{X/Y}, \mathcal{O}_X)$ and similarly for $T_{X/Z}$, $T_{Y/Z}$.)

2.10. Characterization of smooth and etale in terms of fibers.

By Exercise 1.8, we know what the fibers look like for étale and unramified morphisms; and what the geometric fibers look like for smooth morphisms. There is a good characterization of these notions in terms of the geometric fibers, and this is a convenient way of thinking about the three definitions.

2.11. *Exercise: characterization of etale and unramified morphisms in terms of fibers.* Suppose $\pi : X \to Y$ is a morphism locally of finite presentation. Prove that π is étale if and only if it is flat, and the geometric fibers (above $\operatorname{Spec} \overline{k} \to Y$, say) are unions of Spec's of fields (with discrete topology), each a finite separable extension of the field $\overline{k}$. Prove that π is unramified if and only if the geometric fibers (above $\operatorname{Spec} \overline{k} \to Y$, say) are unions of Spec's of fields (with discrete topology), each a finite separable extension of the field $\overline{k}$. (Hint: a finite type sheaf that is 0 at all points must be the 0-sheaf.)

There is an analogous statement for smooth morphisms, that is harder. (That's why this discussion is in the "harder" section.)

2.12. *Harder exercise.* Suppose $\pi : X \to Y$ is locally of finite presentation. Show that π is smooth of relative dimension n if and only if π is flat, and the geometric fibers are disjoint unions of n-dimensional nonsingular varieties (over the appropriate field).

3. Generic smoothness in characteristic 0

We will next see a number of important results that fall under the rubric of "generic smoothness". All will require working over a field of characteristic 0 in an essential way. So far in this course, we have had to add a few caveats here and there for people encountering positive characteristic. This is probably the first case where positive characteristic people should just skip this section.

Our first result is an algebraic analog of Sard's theorem.

3.1. *Proposition (generic smoothness in the source). — Let k be a field of characteristic 0, and let $\pi : X \to Y$ be a dominant morphism of integral finite-type k-schemes. Then there is a non-empty (=dense) open set $U \subset X$ such that $\pi|_U$ is smooth.*

We've basically seen this argument before, when we showed that a variety has an open subset that is nonsingular.

Proof. Define $n = \dim X - \dim Y$ (the "relative dimension"). Now $FF(X)/FF(Y)$ is a finitely generated field extension of transcendence degree n. It is separably generated by n elements (as we are in characteristic 0). Thus Ω has rank n at the generic point. Its rank is at least n everywhere. By uppersemicontinuity of fiber rank of a coherent sheaf, it is rank n for every point in a dense open set. Recall that on a reduced scheme, constant rank implies locally free of that rank (Class 15, Exercise 5.2); hence Ω is locally free of rank n on that set. Also, by openness of flatness, it is flat on a dense open set. Let U be the intersection of these two open sets. □

For pedants: In class, I retreated to this statement above. However, I think the following holds. Suppose $\pi : X \to Y$ is a dominant finite type morphism of integral schemes, where $\operatorname{char} FF(Y) = 0$ (and hence $\operatorname{char} FF(X) = 0$ from $FF(Y) \hookrightarrow FF(X)$). Then there is a non-empty open set $U \subset X$ such that $\pi|_U$ is smooth.

The proof above needs the following tweak. Define $n = \dim X - \dim Y$. Let η be the generic point of Y, and let X_η be fiber of π above η; it is non-empty by the dominant hypothesis. Then X_η is a finite type scheme over $FF(Y)$. I claim $\dim X_\eta = n$. Indeed, π is flat near X_η (everything is flat over a field, and flatness is an open condition), and we've shown for a flat morphism the dimension of the fiber is the dimension of the source minus the dimension of the target. Then proceed as above.

Please let me know if I've made a mistake!

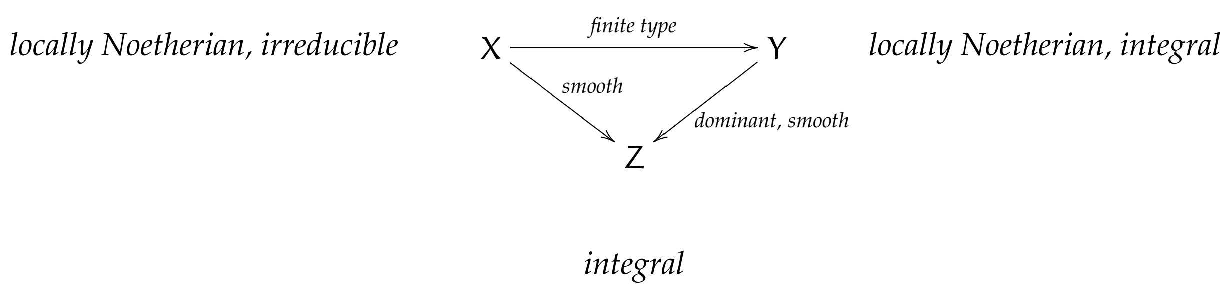
3.2. In §1.1, we saw an example where this result fails in positive characteristic, involving an inseparable extension of fields. Here is another example, over an *algebraically closed* field of characteristic p: $\mathbb{A}^1_k = \operatorname{Spec} k[t] \to \operatorname{Spec} k[u] = \mathbb{A}^1_k$, given by $u \mapsto t^p$. The earlier example (§1.1) is what is going on at the generic point.

If the source of π is smooth over a field, the situation is even nicer.

3.3. *Theorem (generic smoothness in the target). — Suppose $f : X \to Y$ is a morphism of k-varieties, where $\operatorname{char} k = 0$, and X is smooth over k. Then there is a dense open subset of Y such that $f|_{f^{-1}(U)}$ is a smooth morphism.*

(Note: $f^{-1}(U)$ may be empty! Indeed, if f is not dominant, we will have to take such a U.)

For pedants: I think the following generalization holds, assuming that my earlier notes to pedants aren't bogus. Generalize the above hypotheses to the following morphisms of $\mathbb{Q}$-schemes. (Requiring a scheme to be defined over $\mathbb{Q}$ is precisely the same as requiring it to "live in characteristic 0", i.e. the morphism to $\operatorname{Spec} \mathbb{Z}$ has image precisely $[(0)]$.)



To prove this, we'll use a neat trick.

3.4. *Lemma. — Suppose $\pi : X \to Y$ is a morphism of schemes that are finite type over k, where $\operatorname{char} k = 0$. Define*

$$X_r = \{\textit{closed points } x \in X : \textit{rank } T_{\pi,x} \leq r\}.$$

Then $\dim f(X_r) \leq r$. *(Note that* X_r *is a closed subset; it is cut out by determinantal equations. Hence by Chevalley's theorem, its image is constructible, and we can take its dimension.)*

For pedants: I think the only hypotheses we need are that π *is a finite type morphism of locally Noetherian schemes over* $\mathbb{Q}$*. The proof seems to work as is, after an initial reduction to verifying it on an arbitrary affine open subset of* Y*.*

Here is an example of the lemma, to help you find it believable. Suppose X is a nonsingular surface, and Y is a nonsingular curve. Then for each $x \in x$, the tangent map $T_{\pi,x} : T_x \to T_{\pi(x)}$ is a map from a two-dimensional vector space to a one-dimensional vector space, and thus has rank 1 or 0. I then drew some pictures. If π is dominant, then we have a picture like this [omitted]. The tangent map has rank 0 at this one point. The image is indeed rank 0. The tangent map has rank at most 1 everywhere. The image indeed has rank 1.

Now imagine that π contracted X to a point. Then the tangent map has rank 0 everywhere, and indeed the image has dimension 0.

Proof of lemma. We can replace by X by an irreducible component of X_r, and Y by the closure of that component's image of X in Y. (The resulting map will have all of X contained in X_r. This boils down to the following linear algebra observation: if a linear map $\rho : V_1 \to V_2$ has rank at most r, and V_i' is a subspace of V_i, with ρ sending V_1' to V_2', then the restriction of ρ to V_1' has rank at most that of ρ itself.) Thus we have a dominant morphism $f : X \to Y$, and we wish to show that $\dim Y \leq r$. By generic smoothness on the source (Proposition 3.1), there is a nonempty open subset $U \subset X$ such that $f : U \to Y$ is smooth. But then for any $x \in X$, the tangent map $T_{x,X} \to T_{\pi(x),Y}$ is surjective (by smoothness), and has rank at most r, so $\dim Y = \dim_{\pi(x)} Y \leq \dim T_{\pi(x),Y} \leq r$. ☐

There's not much left to prove the theorem.

Proof of Theorem 3.3. Reduce to the case Y smooth over k (by restricting to a smaller open set, using generic smoothness of Y, Proposition 3.1). Say $n = \dim Y$. $\dim \overline{f(X_{n-1})} \leq n-1$ by the lemma, so remove this as well. Then the rank of T_f is at least r for each closed point of X. But as Y is nonsingular of dimension r, we have that T_f is surjective for every closed point of X, hence surjective. Thus f is smooth by Hard Exercise 2.2. ☐

3.5. The Kleiman-Bertini theorem. The Kleiman-Bertini theorem is elementary to prove, and extremely useful, for example in enumerative geometry.

Throughout this discussion, we'll work in the category of k-varieties, where k is an algebraically closed field of characteristic 0. The definitions and results generalize easily to the non-algebraically closed case, and I'll discuss this parenthetically.

3.6. Suppose G is a group variety. Then I claim that G is smooth over k. Reason: It is generically smooth (so it has a dense open set U that is smooth), and G acts transitively on itself (so we can cover G with translates of U).

We can generalize this. We say that a G-action $a : G \times X \to X$ on a variety X is *transitive* if it is transitive on closed points. (If k is not algebraically closed, we replace this by saying that it is transitive on $\overline{k}$-valued points. In other words, we base change to the algebraic closure, and ask if the resulting action is transitive. Note that in characteristic 0, reduced = geometrically reduced, so G and X both remain reduced upon base change to $\overline{k}$.)

In other words, if U is a non-empty open subset of X, then we can cover X with translates of X. (Translation: $G \times U \to X$ is surjective.) Such an X (with a transitive G-action) is called a *homogeneous space* for G.

3.7. *Exercise.* Paralleling §3.6, show that a homogeneous space X is smooth over k.

3.8. *The Kleiman-Bertini theorem. — Suppose* X *is homogeneous space for group variety* G *(over an algebraically closed field* k *of characteristic* 0*). Suppose* $f : Y \to X$ *and* $g : Z \to X$ *be morphisms from smooth* k*-varieties* Y, Z*. Then there is a nonempty open subset* $V \subset G$ *such that for every* $\sigma \in V(k)$, $Y \times_X Z$ *defined by*

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Z \\ \downarrow & & \downarrow g \\ Y & \xrightarrow{\sigma \circ f} & X \end{array}$$

(i.e. Y *is "translated by* σ*") is smooth over* k *of dimension exactly* $\dim Y + \dim Z - \dim X$*. Better: there is an open subset of* $V \subset G$ *such that*

$$(G \times_k Y) \times_X Z \to G \tag{1}$$

is a smooth morphism of relative dimension $\dim Y + \dim Z - \dim X$.

(The statement and proof will carry through even if k is not algebraically closed.)

The first time you hear this, you should think of the special case where $Y \to X$ and $Z \to X$ are closed immersions (hence "smooth subvarieties"). In this case, the Kleiman-Bertini theorem says that the second subvariety will meet a "general translate" of the first transversely.

Proof. It is more pleasant to describe this proof "backwards", by considering how we would prove it ourselves. We will end up using generic smoothness twice, as well as many facts we now know and love.

In order to show that the morphism (1) is generically smooth on the target, it would suffice to apply Theorem 3.3), so we wish to show that $(G \times_k Y) \times_X Z$ is a smooth k-variety. Now Z is smooth over k, so it suffices to show that $(G \times_k Y) \times_X Z \to Z$ is a smooth morphism (as the composition of two smooth morphisms is smooth). But this is obtained by base changed from $G \times_k Y \to X$, so it suffices to show that this latter morphism is smooth (as smoothness is preserved by base change).

This is a G-equivariant morphism $G \times_k Y \xrightarrow{a \circ f} X$. (By "G-equivariant", we mean that G action on both sides respects the morphism.) By generic smoothness of the target (Theorem 3.3), this is smooth over a dense open subset X. But then by transitivity of the G action, this morphism is smooth (everywhere). (*Exercise:* verify the relative dimension statement.) □

3.9. *Corollary (Bertini's theorem, improved version). Suppose* X *is a smooth* k*-variety, where* k *is algebraically closed of characteristic* 0*. Let* δ *be a finite-dimensional base-point-free linear system, i.e. a finite vector space of sections of some invertible sheaf* $\mathcal{L}$*. Then almost every element of* δ*, considered as a closed subscheme of* X*, is nonsingular. (More explicitly: each element* $s \in H^0(X, \mathcal{L})$ *gives a closed subscheme of* X*. For a general* s*, considered as a point of* $\mathbb{P}H^0(X, \mathcal{L})$*, the closed subscheme is smooth over* k*.)*

(Again, the statement and proof will carry through even if k is not algebraically closed.)

This is a good improvement on Bertini's theorem. For example, we don't actually need $\mathcal{L}$ to be very ample, or X to be projective.

3.10. *Exercise.* Prove this!

3.11. *Easy Exercise.* Interpret the old version of Bertini's theorem (over a characteristic 0 field) as a corollary of this statement.

Note that this fails in positive characteristic, as shown by the one-dimensional linear system $\{pP : P \in \mathbb{P}^1\}$. This is essentially Example 3.2.

4. Formal interpretations

For those of you who like complete local rings, or who want to make the connection to complex analytic geometry, here are some useful reformulations, which I won't prove.

Suppose $(B, \mathfrak{n}) \to (A, \mathfrak{m})$ is a map of Noetherian local rings, inducing an isomorphism of residue fields, and a morphism of completions at the maximal ideals $\hat{B} \to \hat{A}$ (the "hat" terminology arose first in class 13, immediately after the statement of Theorem 2.2). Then the induced map of schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ is:

- *étale* if $\hat{B} \to \hat{A}$ is a bijection.
- *smooth* if $\hat{B} \to \hat{A}$ is isomorphic to $\hat{B} \to \hat{B}[[x_1, \ldots, x_n]]$. In other words, formally, smoothness involves adding some free variables. (In case I've forgotten to say this before: "Formally" means "in the completion".)
- *unramified* if $\hat{B} \to \hat{A}$ is surjective.

4.1. Formally unramified, smooth, and étale. EGA has defines these three notions differently. The definitions there make clear that these three definitions form a family, in a way that is quite similar to the differential-geometric definition. (You should largely ignore what follows, unless you find later in life that you really care. I won't prove anything.) We say that $\pi : X \to Y$ is *formally smooth* (resp. *formally unramified*, *formally étale*) if for all affine schemes Z, and every closed subscheme Z_0 defined by a nilpotent ideal, and every morphism $Z \to Y$, the canonical map $\mathrm{Hom}_Y(Z, X) \to \mathrm{Hom}_Y(Z_0, X)$ is surjective (resp. injective, bijective). This is summarized in the following diagram, which is reminiscent of the valuative criteria for separatedness and properness.

$$\begin{array}{ccc} \operatorname{Spec} Z_0 & \longrightarrow & X \\ {\scriptstyle \text{nilpotent ideal}} \Big\downarrow & \overset{?}{\nearrow} & \Big\downarrow {\scriptstyle \pi} \\ \operatorname{Spec} Z & \longrightarrow & Y \end{array}$$

(Exercise: show that this is the same as the definition we would get by replacing "nilpotent" by "square-zero". This is sometimes an easier formulation to work with.)

EGA defines smooth as morphisms that are formally smooth and locally of finite presentation (and similarly for the unramified and étale).

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 53 AND 54

RAVI VAKIL

CONTENTS

1. SERRE DUALITY

Our last topic is Serre duality. Recall that Serre duality arose in our section on "fun with curves" (classes 33–36). We'll prove the statement used there, and generalize it greatly.

Our goal is to rigorously prove everything we needed for curves, and to generalize the statement significantly. Serre duality can be generalized beyond belief, and we'll content ourselves with the version that is most useful. For the generalization, we will need a few facts that we haven't proved, but that we came close to proving.

(i) The existence (and behavior) of the cup product in (Cech) cohomology. For any quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is a natural map $H^i(X, \mathcal{F}) \times H^j(X, \mathcal{G}) \to H^{i+j}(X, \mathcal{F} \otimes \mathcal{G})$ satisfying all the properties you might hope. From the Cech cohomology point of view this isn't hard. For those of you who prefer derived functors, I haven't thought through why it is true. For $i = 0$ or $j = 0$, the meaning of the cup product is easy. (For example, if $i = 0$, the map involves the following. The j-cocycle of $\mathcal{G}$ is the data of sections of $\mathcal{G}$ of $(j+1)$-fold intersections of affine open sets. The cup product corresponds to "multiplying

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each of these by the (restriction of the) global section of $\mathcal{F}''$.) This version is all we'll need for nonsingular projective curves (as if $i, j > 0$, $i + j > 1$).

(ii) The Cohen-Macaulay/flatness theorem. I never properly defined Cohen-Macaulay, so I didn't have a chance to prove that nonsingular schemes are Cohen-Macaulay, and if $\pi : X \to Y$ is a morphism from a pure-dimensional Cohen-Macaulay scheme to a pure-dimensional nonsingular scheme, then π is flat if all the fibers are of the expected dimension. (I stated this, however.)

We'll take these two facts for granted.

Here now is the statement of Serre duality.

Suppose X is a Cohen-Macaulay projective k-scheme of pure dimension n. A *dualizing sheaf* for X over k is a coherent sheaf ω_X (or $\omega_{X/k}$) on X along with a *trace map* $H^n(X, \omega_W) \to k$, such that for all finite rank locally free sheaves $\mathcal{F}$ on X,

$$H^i(X, \mathcal{F}) \times H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k \tag{1}$$

is a perfect pairing. In terms of the cup product, the first map in (1) is the composition

$$H^i(X, \mathcal{F}) \times H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X) \to H^n(X, (\mathcal{F} \otimes \mathcal{F}^\vee) \otimes \omega_X) \to H^n(X, \omega_X).$$

1.1. *Theorem (Serre duality). — A dualizing sheaf always exists.*

We will proceed as follows.

- We'll partially extend this to coherent sheaves in general: $\operatorname{Hom}(\mathcal{F}, \omega_X) \to H^n(\mathcal{F})^\vee$ is an isomorphism for all $\mathcal{F}$.
- Using this, we'll show by a Yoneda argument that (ω_X, t) is unique up to unique isomorphism.
- We will then prove the Serre duality theorem 1.1. This will take us some time. We'll first prove that the dualizing sheaf exists for projective space. We'll then prove it for anything admitting a finite flat morphism to projective space. Finally we'll show that every projective Cohen-Macaulay k-scheme admits a finite flat morphism to projective space.
- We'll prove the result in families (i.e. we'll define a "relative dualizing sheaf" in good circumstances). This is useful in the theory of moduli of curves, and Gromov-Witten theory.
- The existence of a dualizing sheaf will be straightforward to show — surprisingly so, at least to me. However, it is also surprisingly slippery — getting a hold of it in concrete circumstances is quite difficult. For example, on the open subset where X is smooth, ω_X is an invertible sheaf. We'll show this. Furthermore, on this locus, $\omega_X = \det \Omega_X$. (Thus in the case of curves, $\omega_X = \Omega_X$. In the "fun with curves" section, we needed the fact that Ω_X is dualizing because we wanted to prove the Riemann-Hurwitz formula.)

1.2. *Warm-up trivial exercise.* Show that if $h^0(X, \mathcal{O}_X) = 1$ (e.g. if X is geometrically integral), then the trace map is an isomorphism, and conversely.

2. Extension to coherent sheaves; uniqueness of the dualizing sheaf

2.1. *Proposition. — If (ω_X, t) exists, then for any coherent sheaf $\mathcal{F}$ on X, the natural map $\operatorname{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \to k$ is a perfect pairing.*

In other words, (1) holds for $i = n$ and any coherent sheaf (not just locally free coherent sheaves). You might reasonably ask if it holds for general i, and it is true that these other cases are very useful, although not as useful as the case we're proving here. In fact the naive generalization does not hold. The correct generalization involves Ext groups, which we have not defined. The precise statement is the following. For any quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is a natural map $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}) \times H^j(X, \mathcal{F}) \to H^{i+j}(\mathcal{G})$. Via this morphism,

$$\operatorname{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k$$

is a perfect pairing.

Proof of Proposition 2.1. Given any coherent $\mathcal{F}$, take a partial locally free resolution

$$\mathcal{E}^1 \to \mathcal{E}^0 \to \mathcal{F} \to 0.$$

(Recall that we find a locally free resolution as follows. $\mathcal{E}^0$ is a direct sum of line bundles. We then find $\mathcal{E}^1$ that is also a direct sum of line bundles that surjects onto the kernel of $\mathcal{E}^0 \to \mathcal{F}$.)

Then applying the left-exact functor $\operatorname{Hom}(\cdot, \omega_X)$, we get

$$0 \to \operatorname{Hom}(\mathcal{F}, \omega_X) \to \operatorname{Hom}(\mathcal{E}^0, \omega_X) \to \operatorname{Hom}(\mathcal{E}^1, \omega_X)$$
$$\text{i.e.} \quad 0 \to \operatorname{Hom}(\mathcal{F}, \omega_X) \to (\mathcal{E}^0)^\vee \otimes \omega_X \to (\mathcal{E}^1)^\vee \otimes \omega_X$$

Also

$$H^n(\mathcal{E}^1) \to H^n(\mathcal{E}^0) \to H^n(\mathcal{F}) \to 0$$

from which

$$0 \to H^n(\mathcal{F})^\vee \to H^n(\mathcal{E}^0)^\vee \to H^n(\mathcal{E}^1)^\vee$$

There is a natural map $\operatorname{Hom}(\mathcal{H}, \omega_X) \times H^n(\mathcal{H}) \to H^n(\omega_X) \to k$ for all coherent sheaves, which by assumption (that ω_X is dualizing) is an isomorphism when $\mathcal{H}$ is locally free. Thus we have morphisms (where all squares are commuting)

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Hom}(\mathcal{F}, \omega) & \longrightarrow & (\mathcal{E}^0)^\vee(\omega) & \longrightarrow & (\mathcal{E}^1)^\vee(\omega) \\
\downarrow \sim & & \downarrow & & \downarrow \sim & & \downarrow \sim \\
0 & \longrightarrow & H^n(\mathcal{F})^\vee & \longrightarrow & H^n(\mathcal{E}^0)^\vee & \longrightarrow & H^n(\mathcal{E}^1)^\vee
\end{array}$$

where all vertical maps but one are known to be isomorphisms. Hence by the Five Lemma, the remaining map is also an isomorphism. □

We can now use Yoneda's lemma to prove:

2.2. *Proposition.* — If a dualizing sheaf (ω_X, t) exists, it is unique up to unique isomorphism.

Proof. Suppose we have two dualizing sheaves, (ω_X, t) and (ω'_X, t'). From the two morphisms

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k \tag{2}$$

$$\mathrm{Hom}(\mathcal{F}, \omega'_X) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega'_X) \xrightarrow{t'} k,$$

we get a natural bijection $\mathrm{Hom}(\mathcal{F}, \omega_X) \cong \mathrm{Hom}(\mathcal{F}, \omega'_X)$, which is functorial in $\mathcal{F}$. By Yoneda's lemma, this induces a (unique) isomorphism $\omega_X \cong \omega'_X$. From (2), under this isomorphism, the two trace maps must be the same too. ☐

3. Proving Serre duality for projective space over a field

3.1. *Exercise.* Prove (1) for $\mathbb{P}^n$, and $\mathcal{F} = \mathcal{O}(m)$, where $\omega_{\mathbb{P}^n} = \mathcal{O}(-n-1)$. (Hint: do this by hand!) Hence (1) holds for direct sums of $\mathcal{O}(m)$'s.

3.2. *Proposition.* — *Serre duality (Theorem 1.1) holds for projective space.*

Proof. We now prove (1) for any locally free $\mathcal{F}$ on $\mathbb{P}^n$. As usual, take

$$0 \to \mathcal{K} \to \oplus\mathcal{O}(m) \to \mathcal{F} \to 0. \tag{3}$$

Note that $\mathcal{K}$ is flat (as $\mathcal{O}(m)$ and $\mathcal{F}$ are flat and coherent), and hence $\mathcal{K}$ is also locally free of finite rank (flat coherent sheaves on locally Noetherian schemes are locally free — this was one of the important facts about flatness). For convenience, set $\mathcal{G} = \oplus\mathcal{O}(m)$.

Take the long exact sequence in cohomology, and dualize, to obtain

$$0 \to H^n(\mathbb{P}^n, \mathcal{F})^\vee \to H^n(\mathbb{P}^n, \mathcal{G})^\vee \to \cdots \to H^0(\mathbb{P}^n, \mathcal{H})^\vee \to 0. \tag{4}$$

Now instead take (3), tensor with $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$ (which preserves exactness, as $\mathcal{O}_{\mathbb{P}^n}(-n-1)$ is locally free), and take the corresponding long exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) \longrightarrow H^0(\mathbb{P}^n, \mathcal{G}^\vee \otimes \omega_{\mathbb{P}^n}) \longrightarrow H^0(\mathbb{P}^n, \mathcal{H}^\vee \otimes \omega_{\mathbb{P}^n})$$

$$\longrightarrow H^1(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) \longrightarrow \cdots$$

Using the trace morphism, this exact sequence maps to the earlier one (4):

$$\begin{array}{ccccccc}
H^i(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{G}^\vee \otimes \omega_{\mathbb{P}^n}) & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{H}^\vee \otimes \omega_{\mathbb{P}^n}) & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) \\
\downarrow \alpha^i_{\mathcal{F}} & & \downarrow \alpha^i_{\mathcal{G}} & & \downarrow \alpha^i_{\mathcal{H}} & & \downarrow \alpha^{i+1}_{\mathcal{F}} \\
H^{n-i}(\mathbb{P}^n, \mathcal{F})^\vee & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{G})^\vee & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{H})^\vee & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{F})^\vee
\end{array}$$

(At some point around here, I could simplify matters by pointing out that $H^i(\mathcal{G}) = 0$ for all $i \neq 0, n$, as $\mathcal{G}$ is the direct sum of line bundles, but then I'd still need to deal with the ends, so I'll prefer not to.) All squares here commute. This is fairly straightforward check for those not involving the connecting homomorphism. (*Exercise.* Check this.) It is longer and more tedious (but equally straightforward) to check that

$$\begin{array}{ccc}
H^i(\mathbb{P}^n, \mathcal{H}^\vee \otimes \omega_{\mathbb{P}^n}) & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) \\
\downarrow \alpha^i_{\mathcal{H}} & & \downarrow \alpha^{i+1}_{\mathcal{F}} \\
H^i(\mathbb{P}^n, \mathcal{H})^\vee & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{F})^\vee
\end{array}$$

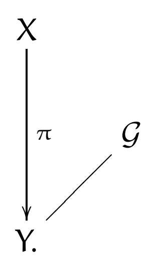
commutes. This requires the definition of the cup product, which we haven't done, so this is one of the arguments I promised to omit.

We then induct our way through the sequence as usual: $\alpha^{-1}_{\mathcal{G}}$ is surjective (vacuously), and $\alpha^{-1}_{\mathcal{H}}$ and $\alpha^0_{\mathcal{G}}$ are injective, hence by the "subtle" Five Lemma (class 32, page 10), $\alpha^0_{\mathcal{F}}$ is injective for all locally free $\mathcal{F}$. In particular, $\alpha^0_{\mathcal{H}}$ is injective (as $\mathcal{H}$ is locally free). But then $\alpha^0_{\mathcal{H}}$ is injective, and $\alpha^{-1}_{\mathcal{H}}$ and $\alpha^0_{\mathcal{G}}$ are surjective, hence $\alpha^0_{\mathcal{F}}$ is surjective, and thus an isomorphism for all locally free $\mathcal{F}$. Thus $\alpha^0_{\mathcal{H}}$ is also an isomorphism, and we continue inductively to show that $\alpha^i_{\mathcal{F}}$ is an isomorphism for all i. $\square$

4. Proving Serre duality for finite flat covers of other spaces for which duality holds

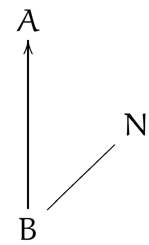
We're now going to make a new construction. It will be relatively elementary to describe, but the intuition is very deep. (Caution: here "cover" doesn't mean covering space as in differential geometry; it just means "surjective map". The word "cover" is often used in this imprecise way in algebraic geometry.)

Suppose $\pi : X \to Y$ is an *affine* morphism, and $\mathcal{G}$ is a quasicoherent sheaf on Y:



Observe that $\underline{\mathrm{Hom}}_Y(\pi_*\mathcal{O}_X, \mathcal{G})$ is a sheaf of $\pi_*\mathcal{O}_X$-modules. (The subscript Y is included to remind us where the sheaf lives.) The reason is that affine-locally on Y, over an affine open set $\mathrm{Spec}\, B$ (on which $\mathcal{G}$ corresponds to B-module N, and with preimage $\mathrm{Spec}\, A \subset X$)

(5)



this is the statement that $\mathrm{Hom}_B(A, N)$ is naturally an A-module (i.e. the A-module structure behaves well with respect to localization by $b \in B$, and hence these modules glue together to form a quasicoherent sheaf).

In our earlier discussion of affine morphisms, we saw that quasicoherent $\pi_* \mathcal{O}_X$-modules correspond to quasicoherent sheaves on X. Hence $\underline{\mathrm{Hom}}_Y(\pi_* \mathcal{O}_X, \mathcal{G})$ corresponds to some quasicoherent sheaf $\pi' \mathcal{G}$ on X.

Notational warning. This notation π' is my own, and solely for the purposes of this section. If π is finite, then this construction is called $\pi^!$ (pronounced "upper shriek"). You may ask why I'm introducing this extra notation "upper shriek". That's because this notation is standard, while my π' notation is just made up. $\pi^!$ is one of the "six operations" on sheaves defined Grothendieck. It is the most complicated one, and is complicated to define for general π. Those of you attending Young-Hoon Kiem's lectures on the derived category may be a little perplexed, as there he defined $\pi^!$ for elements of the derived category of sheaves, not for sheaves themselves. In the finite case, we can define this notion at the level of sheaves, but we can't in general.

Here are some important observations about this notion.

4.1. By construction, we have an isomorphism of quasicoherent sheaves on Y

$$\pi_* \pi' \mathcal{G} \cong \underline{\mathrm{Hom}}_Y(\pi_* \mathcal{O}_X, \mathcal{G}).$$

4.2. π' is a covariant functor from the category of quasicoherent sheaves on Y to quasicoherent sheaves on X.

4.3. If π is a finite morphism, and Y (and hence X) is locally Noetherian, then π' is a covariant functor from the category of *coherent* sheaves on Y to *coherent* sheaves on X. We show this affine locally, see (5). As A and N are both coherent B-modules, $\mathrm{Hom}_B(A, N)$ is a coherent B-module, hence a finitely generated B-module, and hence a finitely generated A-module, hence a coherent A-module.

4.4. If $\mathcal{F}$ is a quasicoherent sheaf on X, then there is a natural map

$$\pi_* \underline{\mathrm{Hom}}_X(\mathcal{F}, \pi' \mathcal{G}) \to \underline{\mathrm{Hom}}_Y(\pi_* \mathcal{F}, \mathcal{G}). \tag{6}$$

Reason: if M is an A-module, we have a natural map

$$\mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N)) \to \mathrm{Hom}_B(M, N) \tag{7}$$

defined as follows. Given $m \in M$, and an element of $\mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N))$, send m to $\phi_m(1)$. This is clearly a homomorphism of B-modules. Moreover, this morphism behaves well with respect to localization of B with respect to an element of B, and hence this description yields a morphism of quasicoherent sheaves.

4.5. *Lemma. The morphism* (6) *is an isomorphism.*

Is there an obvious reason why the map is an isomorphism? There should be...

Proof. We will show that the natural map (7) is an isomorphism. Fix a presentation of M:

$$A^{\oplus m} \to A^{\oplus n} \to M \to 0$$

(where the direct sums needn't be finite). Applying $\mathrm{Hom}_A(\cdot, \mathrm{Hom}_B(A, N))$ to this sequence yields the top row of the following diagram, and applying $\mathrm{Hom}_B(\cdot, N)$ yields the bottom row, and the vertical morphisms arise from the morphism (7).

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N)) & \longrightarrow & \mathrm{Hom}_A(A, \mathrm{Hom}_B(A, N))^{\oplus n} & \longrightarrow & \mathrm{Hom}_A(A, \mathrm{Hom}_B(A, N))^{\oplus m} \\
\downarrow \sim & & \downarrow & & \downarrow \sim & & \downarrow \sim \\
0 & \longrightarrow & \mathrm{Hom}_B(M, N) & \longrightarrow & \mathrm{Hom}_B(A, N)^{\oplus n} & \longrightarrow & \mathrm{Hom}_B(A, N)^{\oplus m}
\end{array}$$

(The squares clearly commute.) Be sure to convince yourself that

$$\mathrm{Hom}_B(A, N)^{\oplus n} \cong \mathrm{Hom}_B(A^{\oplus n}, N)$$

even when n isn't finite (and ditto for the three similar terms)! Then all but one of the vertical homomorphisms are isomorphisms, and hence by the Five Lemma the remaining morphism is an isomorphism. □

Hence π' is right-adjoint to π_* for affine morphisms and quasicoherent sheaves. (Also, by Observation 4.3, it is right-adjoint for finite morphisms and coherent sheaves on locally Noetherian schemes.) In particular, there is a natural morphism $\pi_* \pi^! \mathcal{G} \to \mathcal{G}$.

4.6. *Proposition. — Suppose* $X \to Y$ *is a finite flat morphism of projective* k*-schemes of pure dimension* n*, and* (ω_Y, t_Y) *is a dualizing sheaf for* Y*. Then* $\pi^! \omega_Y$ *along with the trace morphism*

$$t_X : H^n(X, \pi^! \omega_Y) \xrightarrow{\sim} H^n(Y, \pi_* \pi^! \omega_Y) \longrightarrow H^n(Y, \omega_Y) \xrightarrow{t_Y} k$$

is a dualizing sheaf for X.

(That first isomorphism arises because $X \to Y$ is affine.)

Proof.

$$\begin{aligned} H^{n-i}(X, \mathcal{F}^\vee(\pi^! \omega_Y)) &\cong H^{n-i}(Y, \pi_*(\mathcal{F}^\vee \otimes \pi^! \omega_Y)) \quad \text{as } \pi \text{ is affine} \\ &\cong H^{n-i}(Y, \pi_*(\underline{\mathrm{Hom}}(\mathcal{F}, \pi^! \omega_Y))) \\ &\cong H^{n-i}(Y, \underline{\mathrm{Hom}}(\pi_* \mathcal{F}, \omega_Y)) \quad \text{by 4.5} \\ &\cong H^{n-i}(Y, (\pi_* \mathcal{F})^\vee(\omega_Y)) \\ &\cong H^i(Y, \pi_* \mathcal{F})^\vee \quad \text{by Serre duality for } Y \\ &\cong H^i(X, \mathcal{F})^\vee \quad \text{as } \pi \text{ is affine} \end{aligned}$$

At the third-last and second-last steps, we are using the fact that $\pi_* \mathcal{F}$ is locally free, and it is here that we are using flatness! □

5. All projective Cohen-Macaulay k-schemes of pure dimension n are finite flat covers of $\mathbb{P}^n$

We conclude the proof of the Serre duality theorem 1.1 by establishing the result in the title of this section.

Assume $X \hookrightarrow \mathbb{P}^N$ is projective Cohen-Macaulay of pure dimension n (e.g. smooth).

First assume that k is an infinite field. Then long ago in an exercise that I promised would be important (and has repeatedly been so), we showed that there is a linear space of dimension $N - n - 1$ (one less than complementary dimension) missing X. Project from that linear space, to obtain $\pi : X \to \mathbb{P}^n$. Note that the fibers are finite (the fibers are all closed subschemes of affine space), and hence π is a finite morphism. I've stated the "Cohen-Macaulay/flatness theorem" that a morphism from a equidimensional Cohen-Macaulay scheme to a smooth k-scheme is flat if and only if the fibers are of the expected dimension. Hence π is flat.

5.1. *Exercise.* Prove the result in general, if k is not necessarily infinite. Hint: show that there is some d such that there is an intersection of $N - n - 1$ degree d hypersurfaces missing X. Then try the above argument with the dth Veronese of $\mathbb{P}^N$.

6. Serre duality in families

6.1. *Exercise: Serre duality in families.* Suppose $\pi : X \to Y$ is a flat projective morphism of locally Noetherian schemes, of relative dimension n. Assume all of the geometric fibers are Cohen-Macaulay. Then there exists a coherent sheaf $\omega_{X/Y}$ on X, along with a trace map $R^n \pi_* \omega_{X/Y} \to \mathcal{O}_Y$ such that, for every finite rank locally free sheaves $\mathcal{F}$ on X, each of whose higher pushforwards are locally free on Y,

$$R^i \pi_* \mathcal{F} \times R^{n-i} \pi_* (\mathcal{F}^\vee \otimes \omega_X) \longrightarrow R^n \pi_* \omega_X \xrightarrow{t} \mathcal{O}_Y \tag{8}$$

is a perfect pairing. (Hint: follow through the same argument!)

Note that the hypothesis, that all higher pushforwards are locally free on Y, is the sort of thing provided by the cohomology and base change theorem. (In the solution to Exercise 6.1, you will likely show that $R^{n-i}\pi_*(\mathcal{F}^\vee \otimes \omega_X)$ is a locally free sheaf for all $\mathcal{F}$ such that $R^i\pi_*\mathcal{F}$ is a locally free sheaf.)

You will need the *fibral flatness theorem* (EGA IV(3).11.3.10–11), which you should feel free to use: if $g : X \to S$, $h : Y \to S$ are locally of finite presentation, and $f : X \to Y$ is an S-morphism, then the following are equivalent:

(a) g is flat and $f_s : X_s \to Y_s$ is flat for all $s \in s$,
(b) h is flat at all points of $f(X)$ and f is flat.

7. WHAT WE STILL WANT

There are three or four more facts I want you to know.

- On the locus of X where k is smooth, there is an isomorphism $\omega_{X/k} \cong \det \Omega_{X/k}$. (Note for experts: it isn't canonical!) We define $\det \Omega_{X/k}$ to be $\mathcal{K}_X$. We used this in the case of smooth curves over k (proper, geometrically integral). This is surprisingly hard, certainly harder than the mere existence of the canonical sheaf!
- *The adjunction formula.* If D is a Cartier divisor on X (so D is also Cohen-Macaulay, by one of the facts about Cohen-Macaulayness I've mentioned), then $\omega_{D/k} = (\omega_{X/k} \otimes \mathcal{O}_X(D))\,|_D$.

On can show this using Ext groups, but I haven't established their existence or properties. So instead, I'm going to go as far as I can without using them, and then I'll tell you a little about them.

But first, here are some exercises *assuming* that ω is isomorphic to $\det \Omega$ on the smooth locus.

7.1. *Exercise (Serre duality gives a symmetry of the Hodge diamond).* Suppose X is a smooth projective k-variety of dimension n. Define $\Omega^p_X = \wedge^p \Omega X$. Show that we have a natural isomorphism $H^q(X, \Omega^p) \cong H^{n-q}(X, \Omega^{n-p})^\vee$.

7.2. *Exercise (adjunction for smooth subvarieties of smooth varieties).* Suppose X is a smooth projective k-scheme, and and D is a smooth effective Cartier divisor. Show that $\mathcal{K}_D \cong \mathcal{K}_X(D)|_D$. Hence if we knew that $\mathcal{K}_X \cong \omega_X$ and $\mathcal{K}_D \cong \omega_D$, this would let us compute ω_D in terms of ω_X. We will use this shortly.

7.3. *Exercise.* Compute $\mathcal{K}$ for a smooth complete intersection in $\mathbb{P}^N$ of hypersurfaces of degree $d_1, \ldots, d_n$. Compute ω for a complete intersection in $\mathbb{P}^N$ of hypersurfaces of degree $d_1, \ldots, d_n$. (This will be the same calculation!) Find all possible cases where $\mathcal{K} \cong \mathcal{O}$. These are examples of *Calabi-Yau varieties* (or *Calabi-Yau manifolds* if $k = \mathbb{C}$), at least when they have dimension at least 2. If they have dimension precisely 2, they are called K3 surfaces.

8. The dualizing sheaf is an invertible sheaf on the smooth locus

(I didn't do this in class, but promised it in the notes. A simpler proof in the case where X is a curve is given in §9.)

We begin with some preliminaries.

(0) If $f : U \to U$ is the identity, and $\mathcal{F}$ is a quasicoherent sheaf on U, then $f'\mathcal{F} \cong \mathcal{F}$.

(i) The $'$ construction behaves well with respect to flat base change, as the pushforward does. In other words, if

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow g & & \downarrow e \\ Y' & \xrightarrow{f} & Y \end{array}$$

is a fiber diagram, where f (and hence h) is flat, and $\mathcal{F}$ is any quasicoherent sheaf on Y, then there is a canonical isomorphism $h^* e'\mathcal{F} \cong g' f^* \mathcal{F}$.

(ii) The $'$ construction behaves well with respect to disjoint unions of the source. In other words, if $f_i : X_i \to Y$ $(i = 1, 2)$ are two morphisms, $f : X_1 \cup X_2 \to Y$ is the induced morphism from the disjoint union, and $\mathcal{F}$ is a quasicoherent sheaf on Y, then $f'\mathcal{F}$ is $f_1'\mathcal{F}$ on X_1 and $f_2'\mathcal{F}$ on X_2. The reason again is that pushforward behaves well with respect to disjoint union.

Exercise. Prove both these facts, using abstract nonsense.

Given a smooth point $x \in X$, we can choose our projection so that $\pi : X \to \mathbb{P}^n$ is etale at that point. *Exercise.* Prove this. (Hint: We need only check isomorphisms of tangent spaces.)

So hence we need only check our desired result on the etale locus U for $X \to \mathbb{P}^n$. (This is an open set, as etaleness is an open condition.) Consider the base change.

$$\begin{array}{ccc} X \times_{\mathbb{P}^n_k} U & \xrightarrow{h} & X \\ \downarrow g & & \downarrow e \\ U & \xrightarrow{f} & \mathbb{P}^n_k. \end{array}$$

There is a section $U \to X \times_{\mathbb{P}^n} U$ of the vertical morphism on the left. *Exercise.* Show that it expresses U as a connected component of $X \times_{\mathbb{P}^n} U$. (Hint: Show that a section of an etale morphism always expresses the target as a component of the source as follows. Check that s is a homeomorphism onto its image. Use Nakayama's lemma.) The dualizing sheaf $\omega_{\mathbb{P}^n_k}$ is invertible, and hence $f^*\omega_{\mathbb{P}^n_k}$ is invertible on U. Hence $g^! f^* \omega_{\mathbb{P}^n_k}$ is invertible on $s(U)$ (by observation (0)). By observation (i) then, $h^* g^* \omega_{\mathbb{P}^n_k} \cong h^* \omega_X$ is an invertible sheaf.

We are now reduced to showing the following. Suppose $h : U \to X$ is an etale morphism. (In the etale topology, this is called an "etale open set", even though it isn't an open set in any reasonable sense.) Its image is an open subset of X (as etale morphisms are open maps). Suppose $\mathcal{F}$ is a coherent sheaf on X such that $h^*\mathcal{F}$ is an invertible sheaf on U. Then $\mathcal{F}$ is an invertible sheaf on the image of U.

(Experts will notice that this is a special case of *faithfully flat descent*.)

Exercise. Prove this. Hint: it suffices to check that the stalks of $\mathcal{F}$ are isomorphic to the stalks of the structure sheaf. Hence reduce the question to a map of local rings: suppose $(B, \mathfrak{n}) \to (A, \mathfrak{m})$ is etale, and N is a coherent B-module such that $M := N \otimes_B A$ is isomorphic to A. We wish to show that N is isomorphic to B. Use Nakayama's lemma to show that N has the same minimal number of generators (over B) as M (over A), by showing that $\dim_{B/\mathfrak{n}} N - \dim_{A/\mathfrak{m}} M$. Hence this number is 1, so $N \cong B/I$ for some ideal I. Then show that $I = 0$ — you'll use flatness here.

9. An easier proof that the dualizing sheaf of a smooth curve is invertible

Here is another proof that for curves, the dualizing sheaf is invertible. We'll show that it is torsion-free, and rank 1.

First, here is why it is rank 1 at the generic point. We have observed that $f^!$ behaves well with respect to flat base change. Suppose L/K is a finite extension of degree n. Then $\mathrm{Hom}_K(L, K)$ is an L-module. What is its rank? As a K-module, it has rank n. Hence as an L-module it has rank 1. Applying this to $C \to \mathbb{P}^1$ at the generic point ($L = FF(C)$, $K = FF(\mathbb{P}^1)$) gives us the desired result. (Side remark: its structure as an L-module is a little mysterious. You can see that some sort of duality is relevant here. Illuminating this module's structure involves the norm map.)

Conclusion: the dualizing sheaf is rank 1 at the generic point.

Here is why it is torsion free. Let ω_t be the torsion part of ω, and ω_{nt} be the torsion-free part, so we have an exact sequence

$$0 \to \omega_t \to \omega \to \omega_{nt} \to 0.$$

9.1. *Exercise.* Show that this splits: $\omega = \omega_t \oplus \omega_{nt}$. (Hint: It suffices to find a splitting map $\omega \to \omega_t$. As ω_t is supported at a finite set of points, it suffices to find this map in a neighborhood of one of the points in the support. Restrict to a small enough affine open set where ω_{nt} is free. Then on this there is a splitting $\omega_{nt} \to \omega$, from which on that open set we have a splitting $\omega \to \omega_t$.)

Notice that ω_{nt} is rank 1 and torsion-free, hence an invertible sheaf. By Serre duality, for any invertible sheaf $\mathcal{L}$, $h^0(\mathcal{L}) = h^1(\omega_{nt} \otimes \mathcal{L}^\vee)$ and $h^1(\mathcal{L}) = h^0(\omega_{nt} \otimes \mathcal{L}^\vee) + h^0(\omega_t \otimes \mathcal{L})$. Substitute $\mathcal{L} = \mathcal{O}_X$ in the first of these equations and $\mathcal{L} = \omega_X$ in the second, to obtain that $h^0(X, \omega_t) = 0$. But the only skyscraper sheaf with no sections is the 0 sheaf, hence $\omega_t = 0$.

10. THE SHEAF OF DIFFERENTIALS IS DUALIZING FOR A SMOOTH PROJECTIVE CURVE

One can show that the determinant of the sheaf of differentials is the dualizing sheaf using Ext groups, but this involves developing some more machinery, without proof. Instead, I'd like to prove it directly for curves, using what we already have proved. (Note again that our proof of Serre duality for curves was rigorous — the cup product was already well-defined for dimension 1 schemes.)

I'll do this in a sequence of exercises.

Suppose C is an geometrically irreducible, smooth projective k-curve.

We wish to show that $\Omega_C \cong \omega_C$. Both are invertible sheaves. (Proofs that ω_C is invertible were given in §8 and §9.)

Define the genus of a curve as $g = h^1(C, \mathcal{O}_C)$. By Serre duality, this is $h^0(C, \omega_C)$. Also, $h^0(C, \mathcal{O}_C) = h^1(C, \omega_C) = 1$.

Suppose we knew that $h^0(C, \Omega_C) = h^0(C, \omega_C)$, and $h^1(C, \Omega_C) = h^1(\omega_C)$ $(= 1)$. Then $\deg \Omega_C = \deg \omega_C$. Also, by Serre duality $h^0(C, \Omega_C^\vee \otimes \omega_C) = h^1(\Omega_C) = 1$. Thus $\Omega_C^\vee \otimes \omega_C$ is a degree 0 invertible sheaf with a nonzero section. We have seen that this implies that the sheaf is trivial, so $\Omega_C \cong \omega_C$.

Thus it suffices to prove that $h^1(C, \Omega_C) = 1$, and $h^0(C, \Omega_C) = h^0(C, \omega_C)$. By Serre duality, we can restate the latter equality without reference to ω: $h^0(C, \Omega) = h^1(C, \mathcal{O}_C)$. Note that we can assume $k = \overline{k}$: all three cohomology group dimensions $h^i(C, \Omega_C)$, $h^0(C, \mathcal{O}_C)$ are preserved by field extension (shown earlier).

Until this point, the argument is slick and direct. What remains is reasonably pleasant, but circuitous. Can you think of a faster way to proceed, for example using branched covers of $\mathbb{P}^1$?

10.1. *Exercise.* Show that C can be expressed as a plane curve with only nodes as singularities. (Hint: embed C in a large projective space, and take a general projection. The Kleiman-Bertini theorem, or at least its method of proof, will be handy.)

Let the degree of this plane curve be d, and the number of nodes be δ. We then blow up $\mathbb{P}^2$ at the nodes (let $S = \mathrm{Bl}\,\mathbb{P}^2$), obtaining a closed immersion $C \hookrightarrow S$. Let H be the divisor class that is the pullback of the line ($\mathcal{O}(1)$) on $\mathbb{P}^2$. Let $E_1, \ldots, E_\delta$ be the classes of the exceptional divisors.

10.2. *Exercise.* Show that the class of C on $\mathbb{P}^2$ is $dH - 2\sum E_i$. (Reason: the total transform has class dH. Each exceptional divisor appears in the total transform with multiplicity two.)

10.3. *Exercise.* Use long exact sequences to show that $h^1(C, \mathcal{O}_C) = \binom{d-1}{2} - \delta$. (Hint: Compute $\chi(C, \mathcal{O}_C)$ instead. One possibility is to compute $\chi(C', \mathcal{O}_{C'})$ where C' is the image of C in $\mathbb{P}^2$, and use the Leray spectral sequence for $C \to C'$. Another possibility is to work on S directly.)

10.4. *Exercise.* Show that $\Omega_C = \mathcal{K}_S(C)|_C$. Show that this is

$$(-3H + \sum E_i) + (dH - \sum 2E_i).$$

Show that this has degree $2g - 2$ where $g = h^1(\mathcal{O}_C)$. (Possible hint: use long exact sequences.)

10.5. *Exercise.* Show that $h^0(\Omega_C) > 2g - 2 - g + 1 = g - 1$ from

$$0 \to H^0(S, \mathcal{K}_S) \to H^0(S, \mathcal{K}_S(C)) \to H^0(C, \Omega_C).$$

10.6. *Exercise.* Show that $\Omega_C \cong \omega_C$.

11. Ext groups, and adjunction

Let me now introduce Ext groups and their properties, without proof. Suppose i is a non-negative integer. Given two quasicoherent sheaves, $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ is a quasicoherent sheaf. $\mathrm{Ext}^0(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. Then there are long exact sequences in both arguments. In other words, if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence, then there is a long exact sequence starting

$$0 \to \mathrm{Ext}^0(\mathcal{F}'', \mathcal{G}) \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^0(\mathcal{F}', \mathcal{G}) \to \mathrm{Ext}^1(\mathcal{F}'', \mathcal{G}) \to \cdots,$$

and if

$$0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$$

is a short exact sequence, then there is a long exact sequence starting

$$0 \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}') \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}'') \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}') \to \cdots.$$

Also, if $\mathcal{F}$ is locally free, there is a canonical isomorphism $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \cong H^i(X, \mathcal{G} \otimes \mathcal{F}^\vee)$.

For any quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is a natural map $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \times H^j(X, \mathcal{F}) \to H^{i+j}(\mathcal{G})$.

For any coherent sheaf on X, there is a natural morphism ("cup product") $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \times H^j(X, \mathcal{F}) \to H^{i+j}(X, \mathcal{G})$.

11.1. *Exercise.* Suppose X is Cohen-Macaulay, and finite type and projective over k (so Serre duality holds). Via this morphism, show that

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k$$

is a perfect pairing. Feel free to assume whatever nice properties of Ext-groups you need (as we haven't proven any of them anyway).

Hence Serre duality yields a natural extension to coherent sheaves. This is sometimes called Serre duality as well. This more general statement is handy to prove the adjunction formula.

11.2. *Adjunction formula.* — If X is a Serre duality space (i.e. a space where Serre duality holds), and D is an effective Cartier divisor, then $\omega_D = (\omega_X \otimes \mathcal{O}(D))|_D$.

We've seen that if X and D were smooth, and we knew that $\omega_X \cong \det \Omega_X$ and $\omega_D \cong \det \Omega_D$, we would be able to prove this easily (Exercise 7.2).

But we get more. For example, complete intersections in projective space have invertible dualizing sheaves, no matter how singular or how nonreduced. Indeed, complete intersections in *any* smooth projective k-scheme have invertible dualizing sheaves.

A projective k-schemes with invertible dualizing sheaf is so nice that it has a name: it is said to be *Gorenstein*. (Gorenstein has a more general definition, that also involves a dualizing sheaf. It is a local definition, like nonsingularity and Cohen-Macaulayness.)

11.3. *Exercise.* Prove the adjunction formula. (Hint: Consider $0 \to \omega_X \to \omega_X(D) \to \omega_X(D)|_D \to 0$. Apply $\mathrm{Hom}_X(\mathcal{F}, \cdot)$ to this, and take the long exact sequence in Ext-groups.) As before, feel free to assume whatever facts about Ext groups you need.

The following exercise is a bit distasteful, but potentially handy. Most likely you should skip it, and just show that $\omega_X \cong \det \Omega_X$ using the theory of Ext groups.

11.4. *Exercise.* We make a (temporary) definition inductively by definition. A k-variety is "nice" if it is smooth, and (i) it has dimension 0 or 1, or (ii) for any nontrivial invertible sheaf $\mathcal{L}$ on X, there is a nice divisor D such that $\mathcal{L}|_D \neq 0$. Show that for any nice k-variety, $\omega_X \cong \det \Omega_X$. (Hint: use the adjunction formula, and the fact that we know the result for curves.)

11.5. *Remark.* You may wonder if ω_X is always an invertible sheaf. In fact it isn't, for example if $X = \operatorname{Spec} k[x,y]/(x,y)^2$. I think I can give you a neat and short explanation of this fact. If you are curious, just ask.

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