FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 4

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CONTENTS

Last day: Presheaves and sheaves. Morphisms thereof. Sheafification.

Today: Understanding sheaves via stalks. Understanding sheaves via "sheaves on a nice base of a topology". Affine schemes $\operatorname{Spec} R$: the set.

Here's where we are. I introduced you to some of the notions of category theory. Our motivation is as follows. We will be creating some new mathematical objects, and we expect them to act like object we have seen before. We could try to nail down precisely what we mean, and what minimal set of things we have to check in order to verify that they act the way we expect. Fortunately, we don't have to — other people have done this before us, by defining key notions, like abelian categories, which behave like modules over a ring.

We then defined presheaves and sheaves. We have seen sheaves of sets and rings. We have also seen sheaves of abelian groups and of $\mathcal{O}_X$-modules, which form an abelian category. Let me contrast again presheaves and sheaves. Presheaves are simpler to define, and notions such as kernel and cokernel are straightforward, and are defined open set by open set. Sheaves are more complicated to define, and some notions such as cokernel require the notion of sheafification. But we like sheaves because they are in some sense geometric; you can get information about a sheaf locally. Today, I'd like to go over some of the things we talked about last day in more detail. I'm going to talk again about stalks, and how information about sheaves are contained in stalks.

First, a small comment I should have said earlier. Suppose we have an exact sequence of sheaves of abelian groups (or $\mathcal{O}_X$-modules) on X

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}.$$

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If $U \subset X$ is any open set, then

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. Translation: taking sections over U is a *left-exact functor*. Reason: the kernel sheaf of $\mathcal{G} \to \mathcal{H}$ is in fact the kernel presheaf (see the previous lectures). Note that $\mathcal{G}(U) \to \mathcal{H}(U)$ is not necessarily surjective (the functor is not exact); a counterexample is given by our old friend

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0.$$

(By now you should be able to guess what U to use.)

1. Stalks, and sheafification

1.1. *Important exercise.* Prove that a section of a sheaf is determined by its germs, i.e.

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$$

is injective. (Hint: you won't use the gluability axiom. So this is true for separated presheaves.)

Corollary. In particular, if a sheaf has all stalks 0, then it is the 0-sheaf.

1.2. Morphisms and stalks.

1.3. *Exercise.* Show that morphisms of presheaves (and sheaves) induce morphisms of stalks.

1.4. *Exercise.* Show that morphisms of sheaves are determined by morphisms of stalks. Hint # 1: you won't use the gluability axiom. So this is true of morphisms of separated presheaves. Hint # 2: study the following diagram.

(1)
$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \mathcal{G}_x \end{array}$$

1.5. *Exercise.* Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. (Hint: Use (1). Injectivity uses the previous exercise. Surjectivity will use gluability.)

1.6. *Exercise.* (a) Show that Exercise 1.1 is false for general presheaves. (Hint: take a 2-point space with the discrete topology, i.e. every subset is open.)
(b) Show that Exercise 1.4 is false for general presheaves. (Hint: a 2-point space suffices.)
(c) Show that Exercise 1.5 is false for general presheaves.

1.7. Description of sheafification. I described sheafification a bit quickly last time. I will do it again now.

Suppose $\mathcal{F}$ is a presheaf on a topological space X. We define $\mathcal{F}^{sh}$ as follows. Sections over $U \subset X$ are stalks at each point, with compatibility conditions (to each element of the stalk, there is a representative (g, U) with g restricting correctly to all stalks in U). More explicitly:

$$\mathcal{F}^{sh}(U) := \{(f_x \in \mathcal{F}_x)_{x \in U} : \forall x \in U, \exists U_X \text{ with } x \subset U_x \subset U, F^x \in \mathcal{F}(U_x) : F^x_y = f_y \forall y \in U_x\}.$$

(Those who want to worry about the empty set are welcome to.)

This is less confusing than it seems. $\mathcal{F}^{sh}(U)$ is clearly a sheaf: we have restriction maps; they commute; we have identity and gluability. It would be good to know that it satisfies the universal property of sheafification.

1.8. *Exercise.* The stalks of $\mathcal{F}^{sh}$ are the same as the stalks of $\mathcal{F}$. Reason: Use the concrete description of the stalks.

1.9. *Exercise.* For any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$.

We have a natural presheaf morphism $\mathcal{F} \to \mathcal{F}^{sh}$. This induces a natural morphism of stalks $\mathcal{F}_x \to \mathcal{F}^{sh}_x$ (Exercise 1.3). This is an isomorphism by remark a couple of paragraphs previous. Hence if $\mathcal{F}$ is a sheaf already, then $\mathcal{F} \to \mathcal{F}^{sh}$ is an isomorphism, by Exercise 1.5. If we knew that $\mathcal{F}^{sh}$ satisfied the universal property of sheafification, this would have been immediate by abstract nonsense, but we don't know that yet. In fact, we'll show that now. Suppose we have the solid arrows in

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^{sh} \\ & \searrow & \vdots \\ & & \mathcal{G}. \end{array}$$

We want to show that there exists a dashed arrow as in the diagram, making the diagram commute, and we want to show that it is unique. By 1.9, $\mathcal{F} \to \mathcal{G}$ induces a morphism $\mathcal{F}^{sh} \to \mathcal{G}^{sh} = \mathcal{G}$, so we have existence.

For uniqueness: as morphisms of sheaves are determined by morphisms of stalks (Exercise 1.4), and for any $x \in X$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_x & \xrightarrow{=} & \mathcal{F}^{sh}_x \\ & \searrow & \downarrow \\ & & \mathcal{G}_x, \end{array}$$

we are done. Thus $\mathcal{F} \to \mathcal{F}^{sh}$ is indeed the sheafification.

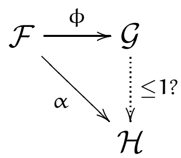
Four properties of morphisms of sheaves that you can check on stalks.

You can verify the following.

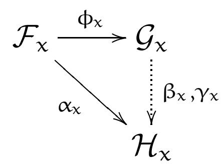
- A morphism of sheaves of sets is injective (monomorphism) if and only if it is injective on all stalks.
- Same with surjective (epimorphism).
- Same with isomorphic — we've already seen this.
- Suppose $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is a complex of sheaves of abelian groups (or $\mathcal{O}_X$-modules). Then it is exact if and only if it is on stalks.

I'll prove one of these, to show you how it works: surjectivity.

Suppose first that we have surjectivity on all stalks for a morphism $\phi : \mathcal{F} \to \mathcal{G}$. We want to check the definition of epimorphism. Suppose we have $\alpha : \mathcal{F} \to \mathcal{H}$, and $\beta, \gamma : \mathcal{G} \to \mathcal{H}$ such that $\alpha = \beta \circ \phi = \gamma \circ \phi$.

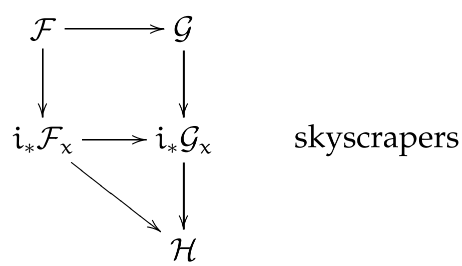


Then by taking stalks at x, we have



By surjectivity (epimorphism-ness) of the morphisms of stalks, $\beta_x = \gamma_x$. But as morphisms are determined by morphisms at stalks (Exercise 1.4), we must have $\beta = \gamma$.

Next assume that $\phi : \mathcal{F} \to \mathcal{G}$ is an epimorphism of sheaves, and $x \in X$. We will show that $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is a epimorphism for any given $x \in X$. Choose for $\mathcal{H}$ any skyscraper sheaf supported at x. (the stalk of a skyscraper sheaf at the skyscraper point is just the skyscraper set/group/ring). Then the maps α, β, γ factor through the stalk maps:



and then we are basically done.

2. Recovering sheaves from a "sheaf on a base"

Sheaves are natural things to want to think about, but hard to get one's hands on. We like the identity and gluability axioms, but they make proving things trickier than for presheaves. We've just talked about how we can understand sheaves using stalks. I now want to introduce a second way of getting a hold of sheaves, by introducing the notion of a *sheaf on a nice base*.

First, let me define the notion of a *base of a topology*. Suppose we have a topological space X, i.e. we know which subsets of X are open $\{U_i\}$. Then a base of a topology is a subcollection of the open sets $\{B_j\} \subset \{U_i\}$, such that each U_i is a union of the B_j. There is one example that you have seen early in your mathematical life. Suppose $X = \mathbb{R}^n$. Then the way the usual topology is often first defined is by defining *open balls* $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ *are open sets*, and declaring that any union of balls is open. So the balls form a base of the usual topology. Equivalently, we often say that they *generate* the usual topology. As an application of how we use them, if you want to check continuity of some map $f : X \to \mathbb{R}^n$ for example, you need only think about the pullback of balls on $\mathbb{R}^n$.

There is a slightly nicer notion I want to use. A base is particularly pleasant if the intersection of any two elements is also an element of the base. (Does this have a name?) I will call this a *nice base*. For example if $X = \mathbb{R}^n$, then a base would be *convex open sets*. Certainly the intersection of two convex open sets is another convex open set. Also, this certainly forms a base, because it includes the balls.

Now suppose we have a sheaf $\mathcal{F}$ on X, and a nice base $\{B_i\}$ on X. Then consider the information $(\{\mathcal{F}(B_i)\}, \{\phi_{ij} : \mathcal{F}(B_i) \to \mathcal{F}(B_j)\}$, which is a subset of the information contained in the sheaf — we are only paying attention to the information involving elements of the base, not all open sets.

Observation. We can recover the entire sheaf from this information. Proof:

$$\mathcal{F}(U) = \{(f_i \in \mathcal{F}(B_i))_{B_i \subset U} : \phi_{ij}(f_i) = f_j\}.$$

The map from the left side to the right side is clear. We get a map from the right side to the left side as follows. By gluability, each element gives at least one element of the left side. By identity, it gives a unique element.

Conclusion: we can recover a sheaf from less information. This even suggests a notion, that of a *sheaf on a nice base*.

A sheaf of sets (rings etc.) on a nice base $\{B_i\}$ is the following. For each B_i in the base, we have a set $\mathcal{F}(B_i)$. If $B_i \subset B_j$, we have maps $\mathrm{res}_{ji} : \mathcal{F}(B_j) \to \mathcal{F}(B_i)$. (Everywhere things called B are assumed to be in the base.) If $B_i \subset B_j \subset B_k$, then $\mathrm{res}_{B_k,B_i} = \mathrm{res}_{B_j,B_i} \circ \mathrm{res}_{B_k,B_j}$. For the pedants, $\mathcal{F}(\emptyset)$ is a one-element set (a final object). So far we have defined a *presheaf on a nice base*.

We also have base identity: If $B = \cup B_i$, then if $f, g \in \mathcal{F}(B)$ such that $\mathrm{res}_{B,B_i} f = \mathrm{res}_{B,B_i} g$ for all i, then $f = g$.

And base gluability: If $B = \cup B_i$, and we have $f_i \in \mathcal{F}(B_i)$ such that f_i agrees with f_j on basic open set $B_i \cap B_j$ (i.e. $\mathrm{res}_{B_i,B_i \cap B_j} f_i = \mathrm{res}_{B_j,B_i \cap B_j} f_j$) then there exist $f \in \mathcal{F}(B)$ such that $\mathrm{res}_{B,B_i} = f_i$ for all i.

2.1. *Theorem. — Suppose we have data* $F(U_i)$, ϕ_{ij}, *satisfying "base presheaf", "base identity" and "base gluability". Then (if the base is nice) this uniquely determines a sheaf of sets (or rings, etc.)* $\mathcal{F}$, *extending this.*

This argument will later get trumped by one given in Class 13.

Proof. Step 1: define the sections over an arbitrary U. For $U \neq \emptyset$, define

$$\mathcal{F}(U) = \{f_i \in F(B_i) \text{ for all } B_i \subset U : \operatorname{res}_{B_i, B_i \cap B_j} f_i = \operatorname{res}_{B_j, B_i \cap B_j} f_j \text{ in } F(B_i \cap B_j)\}$$

where if the set is empty, then we use the final object in our category; this is the only place where we needed to determine our category in advance. We get $\operatorname{res}_{U,V}$ in the obvious way. We get a presheaf.

$\mathcal{F}(B_i) = F(B_i)$ and $\operatorname{res}_{B_i, B_j}$ is as expected; both are clear.

Step 2: check the identity axiom. Take $f, g \in \mathcal{F}(U)$ restricting to $f_i \in \mathcal{F}(U_i)$. Then f, g agree on any base element contained in some U_i. We'll show that for each $B_j \subset U$, they agree. Take a cover of B_j by base elements each contained entirely in some U_i. The intersection of any two is also contained some U_i; they agree there too. Hence by "base identity" we get identity.

Step 3: check the gluability axiom. Suppose we have some $f_i \in \mathcal{F}(U_i)$ that agree on overlaps. Take any $B_j \subset \cup U_i$. Take a cover by basic opens that each lie in some U_i. Then they agree on overlaps. By "base gluability", we get a section over B_j. (Unique by "base identity".) Any two of the f_j's agree on the overlap. □

2.2. *Remark.* In practice, to find a section of such a sheaf over some open set U we may choose a smaller (finite if possible) cover of U.

Eventually, we will define a sheaf on a base in general, not just on a nice base. Experts may want to ponder the definition, and how to prove the above theorem in that case.

2.3. *Important Exercise.* (a) Verify that a morphism of sheaves is determined by a morphism on the base. (b) Show that a "morphism of sheaves on the base" (i.e. such that the diagram

$$\begin{array}{ccc} \Gamma(B_i, \mathcal{F}) & \longrightarrow & \Gamma(B_i, \mathcal{G}) \\ \downarrow & & \downarrow \\ \Gamma(B_j, \mathcal{F}) & \longrightarrow & \Gamma(B_j, \mathcal{G}) \end{array}$$

commutes) gives a morphism of sheaves.

2.4. *Remark.* Suppose you have a presheaf you want to sheafify, and when restricted to a base it is already a sheaf. Then the sheafification is obtained by taking this process.

Example: Let $X = \mathbb{C}$, and consider the sequence

$$1 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1.$$

Let's check that it is exact, using our new knowledge. We instead work on the nice base of convex open sets. then on these open sets, this is indeed exact. The key fact here is that on any convex open set B, every element of $\mathcal{O}_X^*(B)$ has a logarithm, so we have surjectivity here.

3. Toward schemes

We're now ready to define schemes! Here is where we are going. After some more motivation for what kind of objects affine schemes are, I'll define affine schemes, which are like balls in the analytic topology. We'll generalize in three transverse directions. I'll define schemes in general, including projective schemes. I'll define morphisms between schemes. And I'll define sheaves on schemes. These notions will take up the rest of the quarter.

We will define schemes as a *topological space* along with a *sheaf of "algebraic functions"* (that we'll call the *structure sheaf*). Thus our construction will have three steps: we'll describe the *set*, then the *topology*, and then the *sheaf*.

We will try to draw pictures throughout; geometric intuition can guide algebra (and vice versa). Pictures develop geometric intuition. We learn to draw them; the algebra tells how to think about them geometrically. So these comments are saying: "this is a good way to think". Eventually the picture tells you some algebra.

4. Motivating examples

As motivation for why this is a good foundation for a kind of "space", we'll reinterpret differentiable manifolds in this way. We will feel free to be informal in this section.

Usual definition of differentiable manifold: atlas, and gluing functions. (There is also a Hausdorff axiom, which I'm going to neglect for now.)

A fancier definition is as follows: as a topological space, with a sheaf of differentiable functions. (Some observations: Functions are determined by values at points. This is an obvious statement, but won't be true for schemes in general. Note: Stalks are local rings $(\mathcal{O}_x, \mathfrak{m}_x)$; the residue map is "value at a point" $0 \to \mathfrak{m}_x \to \mathcal{O}_x \to \mathbb{R} \to 0$, as I described in an earlier class, probably class 1 or class 2.)

There is an interesting fact that I'd like to mention now, but that you're not quite ready for. So don't write this down, but hopefully let some of it subconsciously sink into your head. The tangent space at a point x can be naturally identified with $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. Let's make this a bit explicit. Every function vanishing at p canonically gives a functional on the tangent space to X at p. If $X = \mathbb{R}^2$, the function $\sin x - y + y^2$ gives the functional $x - y$.

Morphisms $X \to Y$: these are certain continuous maps — but which ones? We can pull back functions along continuous maps. Differentiable functions pull back to differentiable functions. We haven't defined the inverse image of sheaves yet — if you're curious, that will be in the second problem set — but if we had, we would have a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. (I don't want to call it "pullback" because that word is used for a slightly different concept.) Inverse image is left-adjoint to pushforward, which we *have* seen, so we get map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Interesting question: which continuous maps are differentiable? Answer: Precisely those for which the induced map of functions sends differentiable functions to differentiable functions. (Check on local patches.)

4.1. *Unimportant Exercise.* Show that a morphism of differentiable manifolds $f : X \to Y$ with $f(p) = q$ induces a morphism of stalks $f^\# : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$. Show that $f^\#(\mathfrak{m}_{Y,q}) \subset \mathfrak{m}_{X,p}$. (In other words, if you pull back a function that vanishes at q, you get a function that vanishes at p.)

More for experts: Notice that this induces a map on tangent spaces $(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2)^* \to (\mathfrak{m}_{Y,q}/\mathfrak{m}_{Y,q}^2)^*$. This is the tangent map you would geometrically expect. Interesting fact: the cotangent space, and cotangent map, is somehow more algebraically natural, despite the fact that tangent spaces, and tangent maps, are more geometrically natural. Rhetorical questions: How to check for submersion ("smooth morphism")? How to check for inclusion, but not just set-theoretically? Answer: differential information.

[Then we have a normal exact sequence.

Vector bundle can be rewritten in terms of sheaves; explain how.]

Side Remark. Manifolds are covered by disks that are all isomorphic. Schemes will not have isomorphic open sets, even varieties won't. An example will be given later.

5. Affine schemes I: the underlying set

For any ring R, we are going to define something called $\operatorname{Spec} R$. First I'll define it as a set, then I'll tell you its topology, and finally I'll give you a sheaf of rings on it, which I'll call the sheaf of functions. Such an object is called an affine scheme. In the future, $\operatorname{Spec} R$ will denote the set, the topology, and the structure sheaf, and I might use the notation $\operatorname{sp}(\operatorname{Spec} R)$ to mean the underlying topological space. But for now, as there is no possibility of confusion, $\operatorname{Spec} R$ will just be the set.

The set $\operatorname{sp}(\operatorname{Spec} R)$ is the set of prime ideals of R.

Let's do some examples. Along with the examples, I'll say a few things that aren't yet rigorously defined. But I hope they will motivate the topological space we'll eventually define, and also the structure sheaf.

Example 1. $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$. "The affine line", "the affine line over $\mathbb{C}$". What are the prime ideals? 0. $(x - a)$ where $a \in \mathbb{C}$. There are no others. Proof: $\mathbb{C}[x]$ is a Unique Factorization Domain. Suppose $\mathfrak{p}$ were prime. If $\mathfrak{p} \neq 0$, then suppose $f(x) \in \mathfrak{p}$ is an element of smallest degree. If $f(x)$ is not linear, then factor $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have positive degree. Then $g(x)$ or $h(x) \in \mathfrak{p}$, contradicting the minimality of the degree of f. Hence there is a linear element $(x - a)$ of $\mathfrak{p}$. Then I claim that $\mathfrak{p} = (x - a)$. Suppose $f(x) \in \mathfrak{p}$. Then the division algorithm would give $f(x) = g(x)(x - a) + m$ where $m \in \mathbb{C}$. Then $m = f(x) - g(x)(x - a) \in \mathfrak{p}$. If $m \neq 0$, then $1 \in \mathfrak{p}$, giving a contradiction: prime ideals can't contain 1.

Thus we have a picture of $\operatorname{Spec} \mathbb{C}[x]$. There is one point for each complex number, plus one extra point. Where is this point? How do we think of it? We'll soon see; but it is a special kind of point. Because (0) is contained in all of these primes, I'm going to somehow identify it with this line passing through all the other points. Here is one way to think of it. You can ask me: is it on the line? I'd answer yes. You'd say: is it here? I'd answer no. This is kind of zen.

To give you an idea of this space, let me make some statements that are currently undefined. The functions on $\mathbb{A}^1_{\mathbb{C}}$ are the polynomials. So $f(x) = x^2 - 3x + 1$ is a function. What is its value at $(x - 1)$ = "1"? Plug in 1! Or evaluate mod $x - 1$ — same thing by division algorithm! (What is its value at (0)? We'll see later. In general, values at maximal ideals are immediate, and we'll have to think a bit more when primes aren't maximal.)

Here is a "rational function": $(x - 3)^3/(x - 2)$. This function is defined everywhere but $x = 2$; it is an element of the structure sheaf on the open set $\mathbb{A}^1_{\mathbb{C}} - \{2\}$. It has a triple zero at 3, and a single pole at 2.

Example 2. Let k be an algebraically closed field. $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. The same thing works, without change.

Example 3. $\operatorname{Spec} \mathbb{Z}$. One amazing fact is that from our perspective, this will look a lot like the affine line. Another unique factorization domain. Prime ideals: (0), (p) where p is prime. (Do this if you don't know it!!) Hence we have a picture of this Spec (omitted from notes).

Fun facts: 100 is a function on this space. It's value at (3) is "$1 \pmod 3$". It's value at (2) is "$0 \pmod 2$", and in fact it has a double 0. We will have to think a little bit to make sense of its value at (0).

27/4 is a rational function on $\operatorname{Spec} \mathbb{Z}$. It has a double pole at (2), a triple zero at (3). What is its value at (5)? Answer

$$27 \times 4^{-1} \equiv 2 \times (-1) \equiv 3 \pmod 5.$$

Example 4: stupid examples. $\operatorname{Spec} k$ where k is any field is boring: only one point. $\operatorname{Spec} 0$, where 0 is the zero-ring: the empty set, as 0 has no prime ideals.

Exercise. Describe the set $\operatorname{Spec} k[x]/x^2$. The ring $k[x]/x^2$ is called the *ring of dual numbers* (over k).

Example 5: $\mathbb{R}[x]$. The primes are (0), $(x-a)$ where $a \in \mathbb{R}$, and (x^2+ax+b) where x^2+ax+b is an irreducible quadratic **(exercise)**. The latter two are maximal ideals, i.e. their quotients are fields. Example: $\mathbb{R}[x]/(x-3) \cong \mathbb{R}$, $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$. So things are a bit more complicated: we have points corresponding to real numbers, and points corresponding to *conjugate pairs* of complex numbers. So consider the "function" x^3-1 at the point $(x-2)$. We get 7. How about at (x^2+1)? We get

$$x^3 - 1 \equiv x - 1 \pmod{x^2+1}.$$

This is profitably thought of as $i-1$.

One moral of this example is that we can work over a non-algebraically closed field if we wish. (i) It is more complicated, (ii) but we can recover much of the information we wanted.

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