FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 15

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CONTENTS

Last day: quasicoherence is affine-local, (locally) free sheaves and vector bundles, invertible sheaves and line bundles, torsion-free sheaves, quasicoherent sheaves of ideals and closed subschemes.

Today: Quasicoherent sheaves form an abelian category; finite type and coherent sheaves; support; rank; quasicoherent sheaves of ideals and closed subschemes.

I'd like to start by restating some of the definitions and arguments from last day.

Suppose X is a scheme. Recall that $\mathcal{O}_X$-module $\mathcal{F}$ is a quasicoherent sheaf if one of two equivalent things is true.
(i) For every affine open $\operatorname{Spec} R$ and distinguished affine open $\operatorname{Spec} R_f$ thereof, the restriction map $\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R_f, \mathcal{F})$ factors as:

$$\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R, \mathcal{F})_f \cong \Gamma(\operatorname{Spec} R_f, \mathcal{F}).$$

(ii) For any affine open set $\operatorname{Spec} R$, $\mathcal{F}|_{\operatorname{Spec} R} \cong \tilde{M}$ for some R-module M.

I will use both definitions today.

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1. Quasicoherent sheaves form an abelian category

Last day, I showed that the quasicoherent sheaves on X form an abelian category, and in fact an abelian subcategory of $\mathcal{O}_X$-modules. I restated the argument in a better way today. I've moved this exposition back into the Class 14 notes.

2. Finiteness conditions on quasicoherent sheaves: finitely generated quasicoherent sheaves, and coherent sheaves

There are some natural finiteness conditions on an A-module M. I will tell you three. In the case when A is a Noetherian ring, which is the case that almost all of you will ever care about, they are all the same.

The first is the most naive: a module could be *finitely generated*. In other words, there is a surjection $A^p \to M \to 0$.

The second is reasonable too: it could be finitely presented. In other words, it could have a finite number of generators with a finite number of relations: there exists a *finite presentation*

$$A^q \to A^p \to M \to 0.$$

The third is frankly a bit surprising, and I'll justify it soon. We say that an A-module M is *coherent* if (i) it is finitely generated, and (ii) whenenver we have a map $A^p \to M$ (not necessarily surjective!), the kernel is finitely generated.

Clearly coherent implies finitely presented, which in turn implies finitely generated.

2.1. *Proposition. — If A is Noetherian, then these three definitions are the same.*

Preparatory facts. If R is any ring, not necessarily Noetherian, we say an R-module is Noetherian if it satisfies the ascending chain condition for submodules. **Exercise.** M Noetherian implies that any submodule of M is a finitely generated R-module. Hence for example if R is a Noetherian ring then finitely generated = Noetherian. **Exercise.** If $0 \to M' \to M \to M'' \to 0$ is exact, then M' and M'' are Noetherian if and only if M is Noetherian. (Hint: Given an ascending chain in M, we get two simultaneous ascending chains in M' and M''.) **Exercise.** A Noetherian as an A-module implies A^n is a Noetherian A-module. **Exercise.** If A is a Noetherian ring and M is a finitely generated A-module, then any submodule of M is finitely generated. (Hint: suppose $M' \hookrightarrow M$ and $A^n \twoheadrightarrow M$. Construct N with
$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, two heads] & A^n \arrow[d, two heads] \\ M' \arrow[r, hook] & M \end{tikzcd}$$
.)

Proof. Clearly both finitely presented and coherent imply finitely generated.

Suppose M is finitely generated. Then take any $A^p \xrightarrow{\alpha} M$. $\ker \alpha$ is a submodule of a finitely generated module over A, and is thus finitely generated. (Here's why submodules of finitely generated modules over Noetherian rings are also finitely generated: Show it is true for $M = A^n$ — this takes some inspiration. Then given $N \subset N$, consider $A^n \twoheadrightarrow M$, and take the submodule corresponding to N.) Thus we have shown coherence. By choosing a surjective $A^p \to M$, we get finite presentation. $\square$

Hence almost all of you can think of these three notions as the same thing.

2.2. *Lemma. — The coherent A-modules form an abelian subcategory of the category of A-modules.*

I will prove this in the case where A is Noetherian, but I'll include a series of short exercises in the notes that will show it in general.

Proof if A is Noetherian. Recall that we have four things to check (see our discussion earlier today). We quickly check that 0 is finitely generated (=coherent), and that if M and N are finitely generated, then $M \oplus N$ is finitely generated. Suppose now that $f : M \to N$ is a map of finitely generated modules. Then $\operatorname{coker} f$ is finitely generated (it is the image of N), and $\ker f$ is too (it is a submodule of a finitely generated module over a Noetherian ring). $\square$

Easy Exercise (only important for non-Noetherian people). Show A is coherent (as an A-module) if and only if the notion of finitely presented agrees with the notion of coherent.

I want to say a few words on the notion of coherence. There is a good reason for this definition — because of this lemma. There are two sorts of people who should care. Complex geometers should care. They consider complex-analytic spaces with the classical topology. One can define the notion of coherent $\mathcal{O}_X$-module in a way analogous to this. You can then show that the structure sheaf is coherent, and this is very hard. (It is called Oka's theorem, and takes a lot of work to prove.) I believe the notion of coherence may have come originally from complex geometry.

The second sort of people who should care are the sort of arithmetic people who sometimes are forced to consider non-Noetherian rings. (For example, for people who know what they are, the ring of adeles is non-Noetherian.)

Warning: it is common in the later literature to define coherent as finitely generated. It's possible that Hartshorne does this. Please don't do this, as it will only cause confusion. (In fact, if you google the notion of coherent sheaf, you'll get this faulty definition repeatedly.) I will try to be scrupulous about this. Besides doing this for the reason of honesty, it will also help you see what hypotheses are actually necessary to prove things — and that always helps me remember what the proofs are.

2.3. Exercise. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module.

Exercise. If $(f_1, \dots, f_n) = A$, and M_{f_i} is finitely generated (resp. coherent) A_{f_i}-module for all i, then M is a finitely generated (resp. coherent) A-module.

I'm not sure if that exercise is even true for finitely presented. That's one of several reasons why I think that "finitely presented" is a worse notion than coherence.

Definition. A quasicoherent sheaf $\mathcal{F}$ is *finite type* (resp. *coherent*) if for every affine open $\operatorname{Spec} R$, $\Gamma(\operatorname{Spec} R, \mathcal{F})$ is a finitely generated (resp. coherent) R-module.

Thanks to the affine communication lemma, and the two previous exercises, it suffices to check this on the opens in a single affine cover.

3. Coherent modules over non-Noetherian rings

Here are some notes on coherent modules over a general ring. Read this only if you really want to! I did not discuss this in class, but promised it in the notes.

Suppose A is a ring. We say an A-module M is *finitely generated* if there is a surjection $A^n \to M \to 0$. We say it is *finitely presented* if there is a presentation $A^m \to A^n \to M \to 0$. We say M is *coherent* if (i) M is finitely generated, and (ii) every map $A^n \to M$ has a finitely generated kernel. The reason we like this third definition is that coherent modules form an abelian category.

Here are some quite accessible problems working out why these notions behave well.

1. Show that coherent implies finitely presented implies finitely generated.

2. Show that 0 is coherent.

Suppose for problems 3–9 that

$$0 \to M \to N \to P \to 0 \tag{1}$$

is an exact sequence of A-modules.

Hint $\star$**.** Here is a *hint* which applies to several of the problems: try to write

$$\begin{tikzcd}
0 \arrow[r] & A^p \arrow[r] \arrow[d] & A^{p+q} \arrow[r] \arrow[d] & A^q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

and possibly use the snake lemma.

3. Show that N finitely generated implies P finitely generated. (You will only need right-exactness of (1).)

4. Show that M, P finitely generated implies N finitely generated. (Possible hint: $\star$.) (You will only need right-exactness of (1).)

5. Show that N, P finitely generated need not imply M finitely generated. (Hint: if I is an ideal, we have $0 \to I \to A \to A/I \to 0$.)

6. Show that N coherent, M finitely generated implies M coherent. (You will only need left-exactness of (1).)

7. Show that N, P coherent implies M coherent. Hint for (i):

$$\begin{tikzcd}
 & A^q \arrow[dr] \arrow[dd] & & & \\
 & & A^p \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[dr] & 0 \\
 & 0 & 0 & & 0
\end{tikzcd}$$

(You will only need left-exactness of (1).)

8. Show that M finitely generated and N coherent implies P coherent. (Hint for (ii): $\star$.)

9. Show that M, P coherent implies N coherent. (Hint: $\star$.) We don't need exactness on the left for this.

At this point, we have shown that if two of (1) are coherent, the third is as well.

10. Show that a finite direct sum of coherent modules is coherent.

11. Suppose M is finitely generated, N coherent. Then if $\phi : M \to N$ is any map, then show that $\operatorname{Im} \phi$ is coherent.

12. Show that the kernel and cokernel of maps of coherent modules are coherent.

At this point, we have verified that coherent A-modules form an abelian subcategory of the category of A-modules. (Things you have to check: 0 should be in this set; it should be closed under finite sums; and it should be closed under taking kernels and cokernels.)

13. Suppose M and N are coherent submodules of the coherent module P. Show that $M + N$ and $M \cap N$ are coherent. (Hint: consider the right map $M \oplus N \to P$.)

14. Show that if A is coherent (as an A-module) then finitely presented modules are coherent. (Of course, if finitely presented modules are coherent, then A is coherent, as A is finitely presented!)

15. If M is finitely presented and N is coherent, show that $\operatorname{Hom}(M, N)$ is coherent. (Hint: Hom is left-exact in its first entry.)

16. If M is finitely presented, and N is coherent, show that $M \otimes N$ is coherent.

17. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. Hint: localization is exact. (This problem appears earlier as well, as Exercise 2.3.)

18. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is finitely generated for all i, then M is too. (Hint: Say M_{f_i} is generated by $m_{ij} \in M$ as an A_{f_i}-module. Show that the m_{ij} generate M. To check surjectivity $\oplus_{i,j} A \to M$, it suffices to check "on $D(f_i)$" for all i.)

19. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is coherent for all i, then M is too. (Hint from Rob Easton: if $\phi : A^2 \to M$, then $(\ker \phi)_{f_i} = \ker(\phi_{f_i})$, which is finitely generated for all i. Then apply the previous exercise.)

20. Show that the ring $A := k[x_1, x_2, \dots]$ is not coherent over itself. (Hint: consider $A \to A$ with $x_1, x_2, \dots \mapsto 0$.) Thus we have an example of a finitely presented module that is not coherent; a surjection of finitely presented modules whose kernel is not even finitely generated; hence an example showing that finitely presented modules don't form an abelian category.

4. Support of a sheaf

Suppose $\mathcal{F}$ is a sheaf of abelian groups (resp. sheaf of $\mathcal{O}_X$-modules) on a topological space X (resp. ringed space $(X, \mathcal{O}_X)$). Define the *support* of a section s of $\mathcal{F}$ to be

$$\operatorname{Supp} s = \{p \in X : s_p \neq 0 \text{ in } \mathcal{F}_p\}.$$

I think of this as saying where s "lives". Define the *support* of $\mathcal{F}$ as

$$\operatorname{Supp} \mathcal{F} = \{p \in X : \mathcal{F}_p \neq 0\}.$$

It is the union of "all the supports of sections on various open sets". I think of this as saying where $\mathcal{F}$ "lives".

4.1. *Exercise.* The support of a finite type quasicoherent sheaf on a scheme is a closed subset. (Hint: Reduce to an affine open set. Choose a finite set of generators of the corresponding module.) Show that the support of a quasicoherent sheaf need not be closed. (Hint: If $A = \mathbb{C}[t]$, then $\mathbb{C}[t]/(t-a)$ is an A-module supported at a. Consider $\oplus_{a \in \mathbb{C}} \mathbb{C}[t]/(t-a)$.)

5. Rank of a finite type sheaf at a point

The *rank* $\mathcal{F}$ of a finite type sheaf at a point p is $\dim \mathcal{F}_p / \mathfrak{m}\mathcal{F}_p$ where $\mathfrak{m}$ is the maximal ideal corresponding to p. More explicitly, on any affine set $\operatorname{Spec} A$ where $p = [\mathfrak{p}]$ and $\mathcal{F}(\operatorname{Spec} A) = M$, then the rank is $\dim_{A/\mathfrak{p}} M_{\mathfrak{p}} / \mathfrak{p} M_{\mathfrak{p}}$. By Nakayama's lemma, this is the minimal number of generators of $M_{\mathfrak{p}}$ as an $A_{\mathfrak{p}}$-module.

Note that this definition is consistent with the notion of rank of a locally free sheaf. In that case, the rank is a (locally) constant function of the point. The converse is sometimes true, as is shown in Exercise 5.2 below.

If $\mathcal{F}$ is quasicoherent (not necessarily finite type), then $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$ can be interpreted as the fiber of the sheaf at the point. A section of $\mathcal{F}$ over an open set containing p can be said to take on a value at that point, which is an element of $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$.

5.1. *Exercise.*

(a) If $m_1, \ldots, m_n$ are generators at P, they are generators in an open neighborhood of P. (Hint: Consider $\operatorname{coker} A^n \xrightarrow{(f_1,\ldots,f_n)} M$ and Exercise 4.1.)
(b) Show that at any point, $\operatorname{rank}(\mathcal{F} \oplus \mathcal{G}) = \operatorname{rank}(\mathcal{F}) + \operatorname{rank}(\mathcal{G})$ and $\operatorname{rank}(\mathcal{F} \otimes \mathcal{G}) = \operatorname{rank} \mathcal{F} \operatorname{rank} \mathcal{G}$ at any point. (Hint: Show that direct sums and tensor products commute with ring quotients and localizations, i.e. $(M \oplus N) \otimes_R (R/I) \cong M/IM \oplus N/IN$, $(M \otimes_R N) \otimes_R (R/I) \cong (M \otimes_R R/I) \otimes_{R/I} (N \otimes_R R/I) \cong M/IM \otimes_{R/I} N/IM$, etc.) Thanks to Jack Hall for improving this problem.
(c) Show that rank is an upper semicontinuous function on X. (Hint: Generators at P are generators nearby.)

5.2. *Important Exercise.* If X is reduced, $\mathcal{F}$ is coherent, and the rank is constant, show that $\mathcal{F}$ is locally free. (Hint: choose a point $p \in X$, and choose generators of the stalk $\mathcal{F}_p$. Let U be an open set where the generators are sections, so we have a map $\phi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$. The cokernel and kernel of ϕ are supported on closed sets by Exercise 4.1. Show that these closed subsets don't include p. Make sure you use the reduced hypothesis!) Thus coherent sheaves are locally free on a dense open set. Show that this can be false if X is not reduced. (Hint: $\operatorname{Spec} k[x]/x^2$, $M = k$.)

You can use the notion of rank to help visualize finite type sheaves, or even quasicoherent sheaves. (We discussed first finite type sheaves on reduced schemes. We then generalized to quasicoherent sheaves, and to nonreduced schemes.)

5.3. Exercise: Geometric Nakayama. Suppose X is a scheme, and $\mathcal{F}$ is a finite type quasicoherent sheaf. Show that if $\mathcal{F}_x \otimes k(x) = 0$, then there exists V such that $\mathcal{F}|_V = 0$. Better: if I have a set that generates the fiber, it generates the stalk.

5.4. *Less important Exercise.* Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. Then $\mathcal{F}$ and $\mathcal{G}$ are both invertible (Hint: Nakayama.) This is the reason for the adjective "invertible" these sheaves are the invertible elements of the "monoid of finite type sheaves".

6. Quasicoherent sheaves of ideals, and closed subschemes

This section is important, and short only because we have built up some machinery.

We now define closed subschemes, and what it means for some functions on a scheme to "cut out" another scheme. The intuition we want to make precise is that a closed subscheme of X is something that on each affine looks like $\operatorname{Spec} R/I$ "$\hookrightarrow$" $\operatorname{Spec} R$.

Suppose $\mathcal{I} \hookrightarrow \mathcal{O}_X$ is a *quasicoherent sheaf of ideals*. (Quasicoherent sheaves of ideals are, not suprisingly, sheaves of ideals that are quasicoherent.) Not all sheaves of ideals are quasicoherent.

6.1. Exercise. *(A non-quasicoherent sheaf of ideals)* Let $X = \operatorname{Spec} k[x]_{(x)}$, the germ of the affine line at the origin, which has two points, the closed point and the generic point η. Define $\mathcal{I}(X) = \{0\} \subset \mathcal{O}_X(X) = k[x]_{(x)}$, and $\mathcal{I}(\eta) = k(x) = \mathcal{O}_X(\eta)$. Show that $\mathcal{I}$ is not a quasicoherent sheaf of ideals.

The cokernel of $\mathcal{I} \to \mathcal{O}_X$ is also quasicoherent, so we have an exact sequence of quasicoherent sheaves

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0. \tag{2}$$

(This exact sequence will come up repeatedly. We could call it the *closed subscheme exact sequence*.) Now $\mathcal{O}_X/\mathcal{I}$ is finite tyupe (as over any affine open set, the corresponding module is generated by a single element), so $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ is a closed subset. Also, $\mathcal{O}_X/\mathcal{I}$ is a sheaf of rings. Thus we have a topological space $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ with a sheaf of rings. I claim this is a scheme. To see this, we look over an affine open set $\operatorname{Spec} R$. Here $\Gamma(\operatorname{Spec} R, \mathcal{I})$ is an ideal I of R. Then $\Gamma(\operatorname{Spec} R, \mathcal{O}_X/\mathcal{I}) = R/I$ (because quotients behave well on affine open sets).

I claim that on this open set, $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ is the closed subset $V(I)$, which I can identify with the topological space $\operatorname{Spec} R/I$. Reason: $[\mathfrak{p}] \in \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ if and only if $(R/I)_{\mathfrak{p}} \neq 0$ if and only if $\mathfrak{p}$ contains I if and only if $[\mathfrak{p}] \in \operatorname{Spec} R/I$.

(Remark for experts: when you have a sheaf supported in a closed subset, you can interpret it as a sheaf *on* that closed subset. More precisely, suppose X is a topological space, $i : Z \hookrightarrow X$ is an inclusion of a closed subset, and $\mathcal{F}$ is a sheaf on X with $\operatorname{Supp} \mathcal{F} \subset Z$. Then we have a natural map $\mathcal{F} \to i_* i^{-1} \mathcal{F}$ (corresponding to the map $i^{-1}\mathcal{F} \to i^{-1}\mathcal{F}$, using the adjointness of i^{-1} and i_*). You can check that this is an isomorphism on stalks, and hence an isomorphism, so $\mathcal{F}$ can be interpreted as the pushforward of a sheaf on the closed subset. Thanks to Jarod and Joe for this comment.)

I next claim that on the distinguished open set $D(f)$ of $\operatorname{Spec} R$, the sections of $\mathcal{O}_X/\mathcal{I}$ are precisely $(R/I)_f \cong R_f/I_f$. (Reason that $(R/I)_f \cong R_f/I_f$: take the exact sequence $0 \to I \to R \to R/I \to 0$ and tensor with R_f, which preserves exactness.) Reason: On $\operatorname{Spec} R$, the sections of $\mathcal{O}_X/\mathcal{I}$ are R/I, and $\mathcal{O}_X/\mathcal{I}$ is quasicoherent, hence the sections over $D(f)$ are $(R/I)_f$.

That's it!

We say that a *closed subscheme* of X is anything arising in this way from a quasicoherent sheaf of ideals. In other words, there is a tautological correspondence between quasicoherent sheaves of ideals and closed subschemes.

Important remark. Note that closed subschemes of affine schemes are affine. (This is tautological using our definition, but trickier using other definitions.)

Exercise. Suppose $\mathcal{F}$ is a locally free sheaf on a scheme X, and s is a section of $\mathcal{F}$. Describe how $s = 0$ "cuts out" a closed subscheme. (A picture is very useful here!)

6.2. Reduction of a scheme. The *reduction* of a scheme is the "reduced version" of the scheme. If R is a ring, then the nilradical behaves well with respect to localization with respect to an element of the ring: $\mathfrak{N}(R)_f$ is naturally isomorphic to $\mathfrak{N}(R_f)$ (check this!). Thus on any scheme, we have an ideal sheaf of nilpotents, and the corresponding closed subscheme is called the *reduction* of X, and is denoted X^{red}. We will soon see that X^{red} satisfies a universal property; we will need the notion of a morphism of schemes to say what this universal property is.

6.3. *Unimportant exercise.*

(a) X^{red} has the same underlying topological space as X: there is a natural homeomorphism of the underlying topological spaces $X^{\text{red}} \cong X$. Picture: taking the reduction may be interpreted as shearing off the fuzz on the space.
(b) Give an example to show that it is *not* true that $\Gamma(X^{\text{red}}, \mathcal{O}_{X^{\text{red}}}) = \Gamma(X, \mathcal{O}_X)/\sqrt{\Gamma(X, \mathcal{O}_X)}$. (Hint: $\coprod_{n>0} \operatorname{Spec} k[t]/(t^n)$ with global section $(t, t, t, \dots)$.) Motivation for this exercise: this *is* true on each affine open set.

By Exercise 4.1, we have that the reduced locus of a locally Noetherian scheme is open. More precisely: Let $\mathcal{I}$ be the ideal sheaf of X^{red}, so on X we have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_{X^{\text{red}}} \to 0$$

of quasicoherent sheaves on X. Then $\mathcal{I}$ is coherent as X is locally Noetherian. Hence the support of I is closed. The complement of the support of I is the **reduced locus**. Geometrically, this says that "the fuzz is on a closed subset". (A picture is really useful here!)

6.4. Important exericse (the reduced subscheme induced by a closed subset). Suppose X is a scheme, and K is a closed subset of X. Show that the following construction determines a closed subscheme Y: on any affine open subset $\operatorname{Spec} R$ of X, consider the ideal $I(K \cap \operatorname{Spec} R)$. This is called the *reduced subscheme induced by* K. Show that Y is reduced.

7. Discussion of future topics

I then discussed the notion of when a sheaf is *generated by global sections*, and gave a preview of quasicoherent sheaves on projective A-schemes. These ideas will appear in the notes for class 16.

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