FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 21

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This set is due Thursday June 8. You can hand it in to Rob Easton, for example via his mailbox. It covers (roughly) classes 49 and 50.

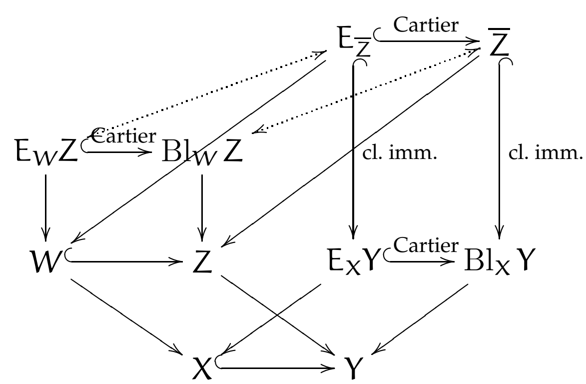
Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. Suppose X is an open subscheme of Y, cut out by a finite type sheaf of ideals. If U is an open subset of Y, show that $\mathrm{Bl}_{U \cap X} U \cong \beta^{-1}(U)$, where $\beta : \mathrm{Bl}_X Y \to Y$ is the blow-up. (Hint: show $\beta^{-1}(U)$ satisfies the universal property!)

2. *(The blow up can be computed locally.)* Show that if Y_α is an open cover of Y (as α runs over some index set), and the blow-up of Y_α along $X \cap Y_\alpha$ exists, then the blow-up of Y along X exists.

3. *(The blow-up preserves irreducibility and reducedness.)* Show that if Y is irreducible, and X doesn't contain the generic point of Y, then $\mathrm{Bl}_X Y$ is irreducible. Show that if Y is reduced, then $\mathrm{Bl}_X Y$ is reduced.

4+. Prove the blow-up closure lemma (see the class notes). Hint: obviously, construct maps in both directions, using the universal property. The following diagram may or may not help.



5. If Y and Z are closed subschemes of a given scheme X, show that $\mathrm{Bl}_Y Y \cup Z \cong \mathrm{Bl}_{Y \cap Z} Z$. (In particular, if you blow up a scheme along an irreducible component, the irreducible component is blown out of existence.)

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6. Consider the curve $y^2 = x^3 + x^2$ inside the plane $\mathbb{A}^2_k$. Blow up the origin, and compute the total and proper transform of the curve. (By the blow-up closure lemma, the latter is the blow-up of the nodal curve at the origin.) Check that the proper transform is nonsingular. (All but the last sentence were done in class.)

7. Describe both the total and proper transform of the curve C given by $y = x^2 - x$ in $\mathrm{Bl}_{(0,0)} \mathbb{A}^2$. Verify that the proper transform of C is isomorphic to C. Interpret the intersection of the proper transform of C with the exceptional divisor E as the slope of C at the origin.

8. *(blowing up a cuspidal plane curve)* Describe the proper transform of the cuspidal curve C' given by $y^2 = x^3$ in the plane $\mathbb{A}^2_k$. Show that it is nonsingular. Show that the proper transform of C meets the exceptional divisor E at one point, and is tangent to E there.

9. (a) Desingularize the tacnode $y^2 = x^4$ by blowing up the plane at the origin (and taking the proper transform), and then blowing up the resulting surface once more.
(b) Desingularize $y^8 - x^5 = 0$ in the same way. How many blow-ups do you need?
(c) Do (a) instead in one step by blowing up (y, x^2).

10. Blowing up something nonreduced in nonsingular can give you something singular, as shown in this example. Describe the blow up of the ideal (x, y^2) in $\mathbb{A}^2_k$. What singularity do you get? (Hint: it appears in a nearby exercise.)

11. Blow up the cone point $z^2 = x^2 + y^2$ at the origin. Show that the resulting surface is nonsingular. Show that the exceptional divisor is isomorphic to $\mathbb{P}^1$.

12+. If $X \hookrightarrow \mathbb{P}^n$ is a projective scheme, show that the exceptional divisor of the blow up the affine cone over X at the origin is isomorphic to X, and that its normal bundle is $\mathcal{O}_X(-1)$. (In the case $X = \mathbb{P}^1$, we recover the blow-up of the plane at a point. In particular, we again recover the important fact that the normal bundle to the exceptional divisor is $\mathcal{O}(-1)$.)

13. Show that the multiplicity of the exceptional divisor in the total transform of a subscheme of $\mathbb{A}^n$ when you blow up the origin is the lowest degree that appears in a defining equation of the subscheme. (For example, in the case of the nodal and cuspidal curves above, Example **??** and Exercise respectively, the exceptional divisor appears with multiplicity 2.) This is called the *multiplicity* of the singularity.

14. Suppose Y is the cone $x^2 + y^2 = z^2$, and X is the ruling of the cone $x = 0, y = z$. Show that $\mathrm{Bl}_X Y$ is nonsingular. (In this case we are blowing up a codimension 1 locus that is not a Cartier divisor. Note that it *is* Cartier away from the cone point, so you should expect your answer to be an isomorphism away from the cone point.)

15+. *(blow-ups resolve base loci of rational maps to projective space)* Suppose we have a scheme Y, an invertible sheaf $\mathcal{L}$, and a number of sections $s_0, \ldots, s_n$ of $\mathcal{L}$. Then away from the closed subscheme X cut out by $s_0 = \cdots = s_n = 0$, these sections give a morphism to $\mathbb{P}^n$. Show that this morphism extends to a morphism $\mathrm{Bl}_X Y \to \mathbb{P}^n$, where this morphism corresponds to the invertible sheaf $(\pi^* \mathcal{L})(-E_X Y)$, where $\pi : \mathrm{Bl}_X Y \to Y$ is the blow-up morphism. In other words, "blowing up the base scheme resolves this rational map". (Hint: it suffices to consider an affine open subset of Y where $\mathcal{L}$ is trivial.)

16. Blow up (xy, z) in $\mathbb{A}^3$, and verify that the exceptional divisor is indeed the projectivized normal bundle.

17. Suppose X is an irreducible nonsingular subvariety of a nonsingular variety Y, of codimension at least 2. Describe a natural isomorphism $\operatorname{Pic} \operatorname{Bl}_X Y \cong \operatorname{Pic} Y \oplus \mathbb{Z}$. (Hint: compare divisors on $\operatorname{Bl}_X Y$ and Y. Show that the exceptional divisor $E_X Y$ gives a non-torsion element of $\operatorname{Pic}(\operatorname{Bl}_X Y)$ by describing a $\mathbb{P}^1$ on $\operatorname{Bl}_X Y$ which has intersection number -1 with $E_X Y$.)

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