# MATH 121 PRACTICE FINAL EXAM

RAVI VAKIL

**Justify all answers!**

**1.** What is the degree of the minimal polynomial of $\sqrt{2} + \sqrt[3]{2}$ over $\mathbb{Q}$?

**2.** Suppose $x + y + z = 0$. Show that

$$\frac{x^5 + y^5 + z^5}{5} = \frac{x^2 + y^2 + z^2}{2} \times \frac{x^3 + y^3 + z^3}{3}$$

and

$$\frac{x^7 + y^7 + z^7}{7} = \frac{x^2 + y^2 + z^2}{2} \times \frac{x^5 + y^5 + z^5}{5}.$$

**3.** (a) For each square-free integer, $n$, describe which roots of unity lie in $\mathbb{Q}(\sqrt{n})$.

(b) As an application, solve the following problem in geometry: for which $m$ can a regular $m$-gon be found with vertices on lattice points $\{(x, y) : x, y \in \mathbb{Z}\} \subset \mathbb{R}^2$? How about a triangular lattice?

**4.** Show that $\mathbb{C}(x)/\mathbb{C}(x^3 + 1/x^3)$ is a Galois extension with Galois group $S_3$. (Hint: consider the group acting on $\mathbb{C}(x)$ by sending $x \mapsto x, 1/x, \omega x, \omega/x, \omega^2 x, \omega^2/x$, where $\omega$ is a cube root of $1$.) Give all intermediate extensions of $\mathbb{C}(x)/\mathbb{C}(x^3 + 1/x^3)$ (by describing them in the form $\mathbb{C}(f(x))$ for various $f(x)$). Hint: there are four of them.

**5.** Prove that $\sqrt[3]{2}$ does not lie in any cyclotomic field over $\mathbb{Q}$.

**6.** Let $E_1$ be the splitting field of $x^3 - 2$ and let $E_2$ be the splitting field of $x^3 - 5$ (both over $\mathbb{Q}$). What is the degree of $E_1 \cap E_2/\mathbb{Q}$? What is the degree of $E_1 E_2/\mathbb{Q}$?

---

*Date*: Saturday, March 17, 2007.