MATH 210 PROBLEM SET 5

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This problem set is due on Friday, March 2 at Jarod Alper's office door.

1. Prove the following fact used in our proof of the primitive element theorem. Suppose F is an infinite field, and V is a finite-dimensional vector space over F. Suppose $V_1, \dots, V_n$ are proper subspaces of V (i.e. $V_i \neq V$). Show that $\cup V_i \neq V$. Show that the statement is false without the hypothesis that F is infinite.

2. Determine the Galois closure of the extension $\mathbb{Q}\left(\sqrt{1+\sqrt{2}}\right)/\mathbb{Q}$. What is its degree? (Dummit and Foote §14.4 problem 1)

3. *Kummer generators for cyclic extensions.* (Feel free to assume n is prime.) Let F be a field of characteristic not dividing n containing the nth root of unity and let K be a cyclic extension of degree d dividing n. Then $K = F(\sqrt[n]{a})$ for some nonzero $a \in F$. Let σ be a generator for the cyclic group $\mathrm{Gal}(K/F)$.

(a) Show that $\sigma(\sqrt[n]{a}) = \zeta \sqrt[n]{a}$ for some primitive dth root of unity ζ.

(b) Suppose $K = F(\sqrt[n]{a}) = F(\sqrt[n]{b})$. Use (a) to show that

$$\frac{\sigma(\sqrt[n]{a})}{\sqrt[n]{a}} = \left(\frac{\sigma(\sqrt[n]{b})}{\sqrt[n]{b}}\right)^i$$

for some integer i relatively prime to d. Conclude that σ fixes the element $\frac{\sqrt[n]{a}}{(\sqrt[n]{b})^i}$, so this is an element of F.

(c) Prove that $K = F(\sqrt[n]{a}) = F(\sqrt[n]{b})$ if and only if $a = b^i c^n$ and $b = a^j d^n$ for some $c, d \in$, i.e., if and only if a and b generate the same subgroup of $F^\times$ modulo nth powers. (Dummit and Foote §14.7, problem 7 — note as a special case that this classifies degree 2 extensions)

4. Prove that if R is Noetherian, then so is the ring $R[[x]]$ of formal power series in the variable x with coefficients from R. Hint: mimic the proof of the Hilbert basis theorem. (Dummit and Foote §15.1, problem 4)

Date: Friday, February 23, 2007.