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## Problem 6

We can compute the roots of $x^3 - 3x + 1$ in $\mathbb{C}$ as follows. Suppose the solution of $x^3 - 3x + 1$ has the form $x = u + u^{-1}$ for some $u \in \mathbb{C}$. Then

$$\begin{aligned} 0 &= x^3 - 3x + 1 = (u + u^{-1})^3 - 3(u + u^{-1}) + 1 \\ &= (u^3 + 3u + 3u^{-1} + u^{-3}) - 3(u + u^{-1}) + 1 \\ &= u^3 + u^{-3} + 1. \end{aligned}$$

Hence $u$ is a solution to $u^6 + u^3 + 1 = 0$, or equivalently, $v = u^3$ is a solution to $v^2 + v + 1 = 0$. Recall $v^3 - 1 = (v - 1)(v^2 + v + 1)$, so $u^3 = v = e^{\pm i2\pi/3}$ (one could also compute $v$ using the quadratic equation). Therefore

$$u = e^{\pm i2\pi/9},\ u = e^{\pm i8\pi/9},\ \text{or } u = e^{\pm 14\pi/9} = e^{\pm(-4i\pi/9)}.$$

Hence

$$x = e^{i2\pi/9} + e^{-i2\pi/9},\ u = e^{i8\pi/9} + e^{-i8\pi/9},\ \text{or } u = e^{-i4\pi/9} + e^{i4\pi/9}.$$

Let $\alpha = e^{i2\pi/9} + e^{-i2\pi/9}$. Then

$$\alpha^2 = e^{i4\pi/9} + 2 + e^{-i4\pi/9} = (e^{i4\pi/9} + e^{-i4\pi/9}) + 2,$$

so $\alpha^2 - 2$ is also a root of $x^3 - 3x + 1$. Since zero is the sum of the roots of $x^3 - 3x + 1$,

$$e^{-i4\pi/9} + e^{i4\pi/9} = -(e^{i2\pi/9} + e^{-i2\pi/9} + e^{i8\pi/9} + e^{-i8\pi/9}) = -(\alpha + \alpha^2 - 2) = 2 - \alpha - \alpha^2.$$

Therefore the roots of $x^3 - 3x + 1$ are $\alpha = e^{i2\pi/9} + e^{-i2\pi/9}$, $\alpha^2 - 2$, and $2 - \alpha - \alpha^2$, so $\mathbb{Q}(\alpha)$ is the splitting field of $x^3 - 3x + 1$. We know that $x^3 - 3x + 1$ is irreducible since $x^3 - 3x + 1$ is a cubic and thus $x^3 - 3x + 1$ is reducible if and only if it has as rational root. But by the Rational Root Test, the only possible rational roots of $x^3 - 3x + 1$ are $-1$ and 1. Since $(-1)^3 - 3(-1) + 1 = 3 \neq 0$ and $(1)^3 - 3(1) + 1 = -1 \neq 0$, $x^3 - 3x + 1$ is irreducible over $\mathbb{Q}$. Since $x^3 - 3x + 1$ is the minimal polynomial of $\alpha$, the splitting field of $x^3 - 3x + 1$ has degree $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$ over $\mathbb{Q}$.

In light of this calculation, we could show abstractly that the splitting field of $x^3 - 3x + 1$ has degree 3 over $\mathbb{Q}$ without regarding the splitting field is a subset of $\mathbb{C}$. Let $\alpha$ be a root of $x^3 - 3x + 1 \in \mathbb{Q}[x]$, i.e. let $\alpha$ correspond to $x$ in $\mathbb{Q}[x]/(x^3 - 3x + 1)$. We claim that $\alpha^2 - 2$ and $2 - \alpha - \alpha^2$ are also solutions to $x^3 - 3x + 1 = 0$. We have

$$\begin{aligned} (\alpha^2 - 2)^3 - 3(\alpha^2 - 2) + 1 &= (\alpha^6 - 6\alpha^4 + 12\alpha^2 - 8) - 3(\alpha^2 - 2) + 1 \\ &= \alpha^6 - 6\alpha^4 + 9\alpha^2 - 1 \\ &= (3\alpha - 1)^2 - 6(3\alpha - 1)\alpha + 9\alpha^2 - 1 \\ &= (9\alpha^2 - 6\alpha + 1) - 6(3\alpha^2 - \alpha) + 9\alpha^2 - 1 \\ &= 0, \end{aligned}$$

so $\alpha^2 - 2$ is a root of $x^3 - 3x + 1$. Let $\beta$ be the third root of $x^3 - 3x + 1$ in the splitting field of $x^3 - 3x + 1$ over $\mathbb{Q}(\alpha)$. Then

$$x^3 - 3x + 1 = (x - \alpha)(x - \alpha^2 + 2)(x - \beta),$$

so $0 = \alpha + (\alpha^2 - 2) + \beta \Rightarrow \beta = 2 - \alpha - \alpha^2 \in \mathbb{Q}(\alpha)$. Therefore $\mathbb{Q}(\alpha)$ is the splitting field of $x^3 - 3x + 1$. Since $x^3 - 3x + 1$ is the minimal polynomial of $\alpha$, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. Hence the splitting field of $x^3 - 3x + 1$ has degree $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$ over $\mathbb{Q}$.