FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 2

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CONTENTS

First, some bureaucratic details.

- We will move to **380-F** for Monday's class.
- Please sign up on this sign-up sheet. I'm going to use it to announce important things like room changes and problem sets.
- Problem sets will be due on Fridays, and I'll try to give them out at least a week in advance. The first set will be out by tomorrow, on the class website. I'll announce it by e-mail. The problems will all be from the notes, and almost all from the class.
- Jarod will be hosting problem sessions on Wednesdays from 5-6 pm, starting next week, at a location to be announced later. This is a great chance to ask him lots of questions, and to hear interesting questions from other people.

If you weren't here last day, you can see the notes on-line. The main warning is that this is going to be a hard class, and you should take it only if you really want to, and also that you should ask me lots of questions, both during class and out of class. And you should do lots of problems.

1. WHERE WE WERE

Last day, we begin by discussing some category theory. Keep in mind that our motivation in learning this is to formalize what we already know, so we can use it in new contexts. Today we should finish with category theory, and we may even begin to discuss sheaves.

The most important notion from last day was the fact that universal properties essentially determine things up to unique isomorphism.

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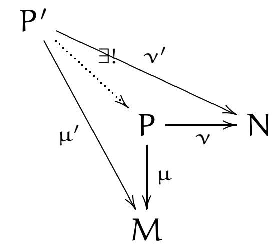
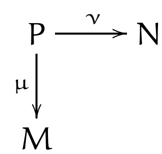
For example, in any category, the product of two objects M and N is an object P, along with maps $\mu : P \to M$ and $\nu : P \to N$, such that for *any other object* P' *with maps* $\mu' : P' \to M$ *and* $\nu' : P' \to N$, these maps must factor *uniquely* through P:

(1)

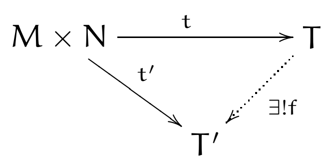


Thus a product is a *diagram*



and not just a set P, although the maps μ and ν are often left implicit.

Another good example of a universal property construction is the notion of a tensor product of A-modules. It is an A-module T along with an A-bilinear map $t : M \times N \to T$, such that any other such map factors through t: given any other $t' : M \times N \to T'$, there is a unique map $f : T \to T'$ such that $t' = f \circ t$.



I gave you the exercise of showing that $(T, t : M \times N \to T)$ (should it exist) is unique up to unique isomorphism. You should really do this, because I'm going to use universal property arguments a whole lot. If you know how to do one of these arguments, you'll know how to do them all.

I then briefly gave other examples: initial objects, final objects, and zero-objects (=initial+final).

2. Yoneda's lemma

2.1. Yoneda's Lemma.

Suppose A is an object of category $\mathcal{C}$. For any object $C \in \mathcal{C}$, we have a set of morphisms $\mathrm{Mor}(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

(2) $$\mathrm{Mor}(C, A) \to \mathrm{Mor}(B, A),$$

by composition: given a map from C to A, we get a map from B to A by precomposing with f. Hence this gives a contravariant functor $h^A : \mathcal{C} \to \textbf{Sets}$. Yoneda's Lemma states that the functor h^A determines A up to unique isomorphism. More precisely:

2.2. *Yoneda's lemma.* — Given two objects A and A', and bijections

$$i_C : \mathrm{Mor}(C, A) \to \mathrm{Mor}(C, A') \tag{3}$$

that commute with the maps (2), then the i_C must be induced from a unique isomorphism $A \to A'$.

2.A. IMPORTANT EXERCISE (THAT EVERYONE SHOULD DO ONCE IN THEIR LIFE). Prove this. (Hint: This sounds hard, but it really is not. This statement is so general that there are really only a couple of things that you could possibly try. For example, if you're hoping to find an isomorphism $A \to A'$, where will you find it? Well, you're looking for an element $\mathrm{Mor}(A, A')$. So just plug in $C = A$ to (3), and see where the identity goes. You'll quickly find the desired morphism; show that it is an isomorphism, then show that it is unique.)

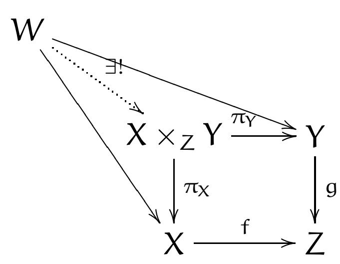
2.3. *Remark.* There is an analogous statement with the arrows reversed, where instead of maps into A, you think of maps *from* A.

2.4. *Remark: the full statement of Yoneda's Lemma.* It won't matter so much for us (so I didn't say it in class), but it is useful to know the full statement of Yoneda's Lemma. A covariant functor $F : \mathcal{A} \to \mathcal{B}$ is *faithful* if for all $A, A' \in \mathcal{A}$, the map $\mathrm{Mor}_{\mathcal{A}}(A, A') \to \mathrm{Mor}_{\mathcal{B}}(F(A), F(A'))$ is injective, and *full* if it is surjective. A functor that is full and faithful is *fully faithful*. A subcategory $i : \mathcal{A} \to \mathcal{B}$ is a *full subcategory* if i is full. If $\mathcal{C}$ is a category, consider the contravariant functor

$$h : \mathcal{C} \to \mathbf{Sets}^{\mathcal{C}}$$

where the category on the right is the "functor category" where the objects are contravariant functors $\mathcal{C} \to$ **Sets**. (What are the morphisms in this category? You will rediscover the notion of *natural transformation of functors*.) This functor h sends A to h^A. Yoneda's lemma states that this is a fully faithful functor, called the *Yoneda embedding*.

2.5. Example: Fibered products. (This notion of fibered product will be important for us later.) Suppose we have morphisms $X, Y \to Z$ (in *any* category). Then the *fibered product* is an object $X \times_Z Y$ along with morphisms to X and Y, where the two compositions $X \times_Z Y \to Z$ agree, such that given any other object W with maps to X and Y (whose compositions to Z agree), these maps factor through some unique $W \to X \times_Z Y$:



By a universal property argument, if it exists, it is unique up to unique isomorphism. (You should think this through until it is clear to you.) Thus the use of the phrase "the fibered product" (rather than "a fibered product") is reasonable, and we should reasonably be allowed to give it the name $X \times_Z Y$. We know what maps to it are: they are precisely maps to X and maps to Y that agree on maps to Z.

The right way to interpret this is first to think about what it means in the category of sets.

2.B. EXERCISE. Show that in **Sets**,

$$X \times_Z Y = \{(x \in X, y \in Y) : f(x) = g(y)\}.$$

More precisely, describe a natural isomorphism between the left and right sides. (This will help you build intuition for fibered products.)

2.C. EXERCISE. If X is a topological space, show that fibered products always exist in the category of open sets of X, by describing what a fibered product is. (Hint: it has a one-word description.)

2.D. EXERCISE. If Z is the final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, then "$X \times_Z Y = X \times Y$": "the" fibered product over Z is canonically isomorphic to "the" product. (This is an exercise about unwinding the definition.)

2.E. UNIMPORTANT EXERCISE. Show that in the category **Ab** of abelian groups, the kernel K of $f : A \to B$ can be interpreted as a fibered product:

$$\begin{array}{ccc} K & \longrightarrow & A \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & B \end{array}$$

We make a definition to set up an exercise.

2.6. Definition. A morphism $f : X \to Y$ is a **monomorphism** if any two morphisms $g_1, g_2 : Z \to X$ such that $f \circ g_1 = f \circ g_2$ must satisfy $g_1 = g_2$. This a generalization of of an injection of sets. In other words, there is a unique way of filling in the dotted arrow so that the following diagram commutes.

$$\begin{array}{ccc} Z & & \\ {\scriptstyle \leq 1}\,\vdots & \searrow & \\ X & \xrightarrow{f} & Y. \end{array}$$

Intuitively, it is the categorical version of an injective map, and indeed this notion generalizes the familiar notion of injective maps of sets.

2.7. *Remark.* The notion of an **epimorphism** is "dual" to this diagramatic definition, where all the arrows are reversed. This concept will not be central for us, although it is necessary for the definition of an abelian category. Intuitively, it is the categorical version of a surjective map.

2.F. EXERCISE. Prove a morphism is a monomorphism if and only if the natural morphism $X \to X \times_Y X$ is an isomorphism. (What is this natural morphism?!) We may then take this as the definition of monomorphism. (Monomorphisms aren't very central to future discussions, although they will come up again. This exercise is just good practice.)

2.G. EXERCISE. Suppose $X \to Y$ is a monomorphism, and $W, Z \to X$ are two morphisms. Show that $W \times_X Z$ and $W \times_Y Z$ are canonically isomorphic. We will use this later when talking about fibered products. (Hint: for any object V, give a natural bijection between maps from V to the first and maps from V to the second.)

2.H. EXERCISE. Given $X \to Y \to Z$, show that there is a natural morphism $X \times_Y X \to X \times_Z X$, assuming that both fibered products exist. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

2.I. UNIMPORTANT EXERCISE. Define *coproduct* in a category by reversing all the arrows in the definition of product. Show that coproduct for **Sets** is disjoint union.

2.J. EXERCISE. Suppose $C \to A, B$ are two ring morphisms, so in particular A and B are C-modules. Define a ring structure $A \otimes_C B$ with multiplication given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$. There is a natural morphism $A \to A \otimes_C B$ given by $a \mapsto (a, 1)$. (Warning: This is not necessarily an inclusion.) Similarly, there is a natural morphism $B \to A \otimes_C B$. Show that this gives a coproduct on rings, i.e. that

$$\begin{array}{ccc} A \otimes_C B & \longleftarrow & B \\ \uparrow & & \uparrow \\ A & \longleftarrow & C \end{array}$$

satisfies the universal property of coproduct.

2.K. IMPORTANT EXERCISE FOR LATER. We continue the notation of the previous exercise. Let I be an ideal of A. Let I^e be the extension of I to $A \otimes_C B$. (These are the elements $\sum_j i_j \otimes b_j$ where $i_j \in I$, $b_j \in B$.) Show that there is a natural isomorphism

$$(A/I) \otimes_C B \cong (A \otimes_C B)/I^e.$$

(Hint: consider $I \to A \to A/I \to 0$, and use the right exactness of $\otimes_C B$.)

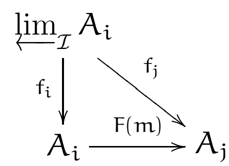
Hence the natural morphism $B \to B \otimes_C (A/I)$ is a surjection. As an application, we can compute tensor products of finitely generated k algebras over k. For example, we have a canonical isomorphism

$$k[x_1, x_2]/(x_1^2 - x_2) \otimes_k k[y_1, y_2]/(y_1^3 + y_2^3) \cong k[x_1, x_2, y_1, y_2]/(x_1^2 - x_2, y_1^3 + y_2^3).$$

Limits and colimits provide two important examples defined by universal properties. They generalize a number of familiar constructions. I'll give the definition first, and then show you why it is familiar. (For example, we'll see that the p-adics are a limit, and fractions are a colimit.)

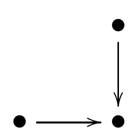
3.1. Limits. We say that a category is an *index category* (a technical condition intended only for experts) the objects form a set. An example is a partially ordered set (in which there in particular there is only one morphism between objects), and indeed all of our examples will be partially ordered sets. Suppose $\mathcal{I}$ is any index category (such as a partially ordered set), and $\mathcal{C}$ is any category. Then a functor $F : \mathcal{I} \to \mathcal{C}$ (i.e. with an object $A_i \in \mathcal{C}$ for each element $i \in \mathcal{I}$, and appropriate commuting morphisms dictated by $\mathcal{I}$) is said to be a *diagram indexed by* $\mathcal{I}$. Commuting squares can be interpreted in this way.

Then the *limit* is an object $\varprojlim_{\mathcal{I}} A_i$ of $\mathcal{C}$ along with morphisms $f_i : \varprojlim_{\mathcal{I}} A_i$ such that if $m : i \to j$ is a morphism in $\mathcal{I}$, then



commutes, and this object and maps to each A_i is universal (final) respect to this property. (The limit is sometimes called the *inverse limit* or *projective limit*.) By the usual universal property argument, if the limit exists, it is unique up to unique isomorphism.

3.2. *Examples: products.* For example, if $\mathcal{I}$ is the partially ordered set



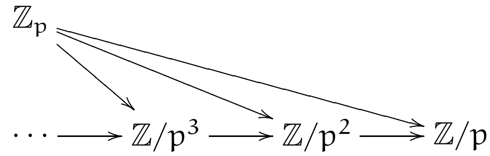
we obtain the fibered product.

If $\mathcal{I}$ is

we obtain the product.

If $\mathcal{I}$ is a set (i.e. the only morphisms are the identity maps), then the limit is called the *product* of the A_i, and is denoted $\prod_i A_i$. The special case where $\mathcal{I}$ has two elements is the example of the previous paragraph.

3.3. *Example: the* p*-adics.* The p-adic numbers, $\mathbb{Z}_p$, are often described informally (and somewhat unnaturally) as being of the form $\mathbb{Z}_p = ? + ?p + ?p^2 + ?p^3 + \cdots$. They are an example of a limit in the category of rings:



Limits do not always exist. For example, there is no limit of $\cdots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$ in the category of finite rings.

However, you can often easily check that limits exist if the elements of your category can be described as sets with additional structure, and arbitrary products exist (respecting the set structure).

3.A. EXERCISE. Show that in the category **Sets**,

$$\{(a_i)_{i \in I} \in \prod_i A_i : F(m)(a_i) = a_j \text{ for all } [m : i \to j] \in \operatorname{Mor}(\mathcal{I})\},$$

along with the projection maps to each A_i, is the limit $\varprojlim_{\mathcal{I}} A_i$.

This clearly also works in the category $\mathbf{Mod}_A$ of A-modules, and its specializations such as $\mathbf{Vec}_k$ and $\mathbf{Ab}$.

From this point of view, $2 + 3p + 2p^2 + \cdots \in \mathbb{Z}_p$ can be understood as the sequence $(2, 2 + 3p, 2 + 3p + 2p^2, \dots)$.

3.4. Colimits. More immediately relevant for us will be the dual of the notion of inverse limit. We just flip all the arrows in that definition, and get the notion of a *direct limit*. Again, if it exists, it is unique up to unique isomorphism. (The colimit is sometimes called the direct limit or injective limit.)

A limit maps *to* all the objects in the big commutative diagram indexed by $\mathcal{I}$. A colimit has a map *from* all the objects.

Even though we have just flipped the arrows, somehow colimits behave quite differently from limits.

3.5. *Example.* The ring $5^{-\infty}\mathbb{Z}$ of rational numbers whose denominators are powers of 5 is a colimit $\varinjlim 5^{-i}\mathbb{Z}$. More precisely, $5^{\infty}\mathbb{Z}$ is the colimit of

$$\mathbb{Z} \longrightarrow 5^{-1}\mathbb{Z} \longrightarrow 5^{-2}\mathbb{Z} \longrightarrow \cdots$$

The colimit over an index *set* I is called the *coproduct*, denoted $\coprod_i A_i$, and is the dual notion to the product.

3.B. EXERCISE. (a) Interpret the statement "$\mathbb{Q} = \varinjlim \frac{1}{n}\mathbb{Z}$". (b) Interpret the union of some subsets of a given set as a colimit. (Dually, the intersection can be interpreted as a limit.)

Colimits always exist in the category of sets:

3.C. EXERCISE. Consider the set $\{(i \in \mathcal{I}, a_i \in A_i)\}$ modulo the equivalence generated by: if $m : i \to j$ is an arrow in $\mathcal{I}$, then $(i, a_i) \sim (j, F(m)(a_i))$. Show that this set, along with the obvious maps from each A_i, is the colimit.

Thus in Example 3.5, each element of the direct limit is an element of something upstairs, but you can't say in advance what it is an element of. For example, 17/125 is an element of the $5^{-3}\mathbb{Z}$ (or $5^{-4}\mathbb{Z}$, or later ones), but not $5^{-2}\mathbb{Z}$.

3.6. *Example: colimits of A-modules.* A variant of this construction works in a number of categories that can be interpreted as sets with additional structure (such as abelian groups, A-modules, groups, etc.). While in the case of sets, the direct limit is a quotient object of the direct sum (= disjoint union) of the A_i, in the case of A-modules (for example), the direct limit is a quotient object of the direct sum of rings. thus the direct limit is $\oplus A_i$ modulo $a_j - F(m)(a_i)$ for every $m : i \to j$ in $\mathcal{I}$.

3.D. EXERCISE. Verify that the A-module described above is indeed the colimit.

3.7. Summary. One useful thing to informally keep in mind is the following. In a category where the objects are "set-like", an element of a colimit can be thought of ("has a representative that is") an element of a single object in the diagram. And an element of a limit can be thought of as an element in each object in the diagram, that are "compatible". Even though the definitions of limit and colimit are the same, just with arrows reversed, these interpretations are quite different.

4. ADJOINTS

Here is another example of a construction closely related to universal properties. We now define adjoint functors. Two *covariant* functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are *adjoint* if there is a natural bijection for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$\tau_{AB} : \mathrm{Mor}_{\mathcal{B}}(F(A), B) \to \mathrm{Mor}_{\mathcal{A}}(A, G(B)).$$

In this instance, let me make precise what "natural" means. For all $f : A \to A'$ in $\mathcal{A}$, we require

(4)
$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{B}}(F(A'), B) & \xrightarrow{Ff^*} & \mathrm{Mor}_{\mathcal{B}}(F(A), B) \\ \downarrow \tau & & \downarrow \tau \\ \mathrm{Mor}_{\mathcal{A}}(A', G(B)) & \xrightarrow{f^*} & \mathrm{Mor}_{\mathcal{A}}(A, G(B)) \end{array}$$

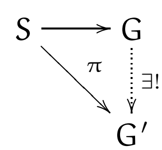
to commute, and for all $g : B \to B'$ in $\mathcal{B}$ we want a similar commutative diagram to commute. (Here f^* is the map induced by $f : A \to A'$, and Ff^* is the map induced by $Ff : L(A) \to L(A')$.)

4.A. EXERCISE. Write down what this diagram should be. (Hint: do it by extending diagram (4) above.)

You've actually seen this before, in linear algebra, when you have seen adjoint matrices. Here is another example.

4.B. EXERCISE. Suppose M, N, and P are A-modules. Describe a natural bijection $\mathrm{Mor}_A(M \otimes_A N, P) = \mathrm{Mor}_A(M, \mathrm{Mor}_A(N, P))$. (Hint: try to use the universal property.) If you wanted, you could check that $\cdot \otimes_A N$ and $\mathrm{Mor}_A(N, \cdot)$ are adjoint functors. (Checking adjointness is never any fun!)

4.1. Example: groupification. Here is another motivating example: getting an abelian group from an abelian semigroup. An abelian semigroup is just like a group, except you don't require an inverse. One example is the non-negative integers $0, 1, 2, \ldots$ under addition. Another is the positive integers under multiplication $1, 2, \ldots$. From an abelian semigroup, you can create an abelian group, and this could be called groupification. Here is a formalization of that notion. If S is a semigroup, then its groupification is a map of semigroups $\pi : S \to G$ such that G is a group, and any other map of semigroups from S to a *group* G' factors *uniquely* through G.



4.C. EXERCISE. Define groupification H from the category of abelian semigroups to the category of abelian groups. (One possibility of a construction: given an abelian semigroup S, the elements of its groupification $H(S)$ are (a, b), which you may think of as $a - b$, with the equivalence that $(a, b) \sim (c, d)$ if $a + d = b + c$. Describe addition in this group, and show that it satisfies the properties of an abelian group. Describe the semigroup map $S \to H(S)$.) Let F be the forgetful morphism from the category of abelian groups **Ab** to the category of abelian semigroups. Show that H is left-adjoint to F.

(Here is the general idea for experts: We have a full subcategory of a category. We want to "project" from the category to the subcategory. We have $\mathrm{Mor}_{\text{category}}(S, H) = \mathrm{Mor}_{\text{subcategory}}(G, H)$ automatically; thus we are describing the left adjoint to the forgetful functor. How the argument worked: we constructed something which was in the small category, which automatically satisfies the universal property.)

4.D. EXERCISE. Show that if a semigroup is *already* a group then groupification is the identity morphism, by the universal property.

4.E. EXERCISE. The purpose of this exercise is to give you some practice with "adjoints of forgetful functors", the means by which we get groups from semigroups, and sheaves from presheaves. Suppose A is a ring, and S is a multiplicative subset. Then $S^{-1}A$-modules are a fully faithful subcategory of the category of A-modules (meaning: the objects of the first category are a subset of the objects of the second; and the morphisms between any two objects of the second that are secretly objects of the first are just the morphisms from the first). Then $M \to S^{-1}M$ satisfies a universal property. Figure out what the universal property is, and check that it holds. In other words, describe the universal property enjoyed by $M \to S^{-1}M$, and prove that it holds.

(Here is the larger story. Let $S^{-1}A$-Mod be the category of $S^{-1}A$-modules, and A-Mod be the category of A-modules. Every $S^{-1}A$-module is an A-module, and this is an injective map, so we have a (covariant) forgetful functor $F : S^{-1}A\text{-Mod} \to A\text{-Mod}$. In fact this is a fully faithful functor: it is injective on objects, and the morphisms between any two $S^{-1}A$-modules *as A-modules* are just the same when they are considered as $S^{-1}A$-modules. Then there is a functor $G : A\text{-Mod} \to S^{-1}A\text{-Mod}$, which might reasonably be called "localization with respect to S", which is left-adjoint to the forgetful functor. Translation: If M is an A-module, and N is an $S^{-1}A$-module, then $\text{Mor}(GM, N)$ (morphisms as $S^{-1}A$-modules, which is incidentally the same as morphisms as A-modules) are in natural bijection with $\text{Mor}(M, FN)$ (morphisms as A-modules).)

4.2. Useful comment for experts. Here is one last useful comment intended only for people who have seen adjoints before. If (F, G) is an adjoint pair of functors, then F preserves all colimits, and G preserves all limits.

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