FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 9 AND 10

RAVI VAKIL

CONTENTS

This week, we will define some useful properties of schemes.

1. TOPOLOGICAL PROPERTIES: IRREDUCIBILITY, CONNECTEDNESS, QUASICOMPACTNESS

We will start with some topological properties. The definitions of *irreducible*, *closed point*, *specialization*, *generalization*, *generic point*, *connected component*, and *irreducible component* were given earlier. You should have pictures in your mind of each of these notions.

An earlier exercise showed that $\mathbb{A}^n$ is irreducible (it was easy). This argument "behaves well under gluing", yielding:

1.A. EXERCISE. Show that $\mathbb{P}^n_k$ is irreducible.

1.B. EXERCISE. An earlier exercise showed that there is a bijection between irreducible closed subsets and points. Show that this is true of schemes as well.

1.C. EXERCISE. Prove that if X is a scheme that has a finite cover $X = \cup_{i=1}^n \operatorname{Spec} A_i$ where A_i is Noetherian, then X is a Noetherian topological space. (We will soon call such a scheme a *Noetherian scheme*, §3.5.)

Thus $\mathbb{P}^n_k$ and $\mathbb{P}^n_{\mathbb{Z}}$ are Noetherian topological spaces: we built them by gluing together a finite number of Spec's of Noetherian rings.

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1.1. Definition. A topological space X is **connected** if it cannot be written as the disjoint union of two non-empty open sets.

1.D. EXERCISE. Show that an irreducible topological space is connected.

1.E. EXERCISE. Give (with proof!) an example of a scheme that is connected but reducible. (Possible hint: a picture may help. The symbol "$\times$" has two "pieces" yet is connected.)

1.F. EXERCISE. If $A = \prod A_1 \times A_2 \times \cdots \times A_n$, describe an isomorphism $\operatorname{Spec} A = \operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2 \coprod \cdots \coprod \operatorname{Spec} A_n$. Show that each $\operatorname{Spec} A_i$ is a distinguished open subset $D(f_i)$ of $\operatorname{Spec} A$. (Hint: let $f_i = (0, \cdots, 0, 1, 0, \cdots 0)$ where the 1 is in the ith component.) In other words, $\coprod_{i=1}^n \operatorname{Spec} A_i = \operatorname{Spec} \prod_{i=1}^n A_i$.

1.2. *Fun but irrelevant remark.* As affine schemes are quasicompact, $\coprod_{i=1}^\infty \operatorname{Spec} A_i$ cannot be isomorphic to $\operatorname{Spec} \prod_{i=1}^\infty A_i$. This lack of isomorphism has an entertaining consequence. Suppose the A_i are isomorphic to the field k. Then we certainly have an inclusion as sets

$$\coprod_{i=1}^{\infty} \operatorname{Spec} A_i \hookrightarrow \operatorname{Spec} \prod_{i=1}^{\infty} A_i$$

— there is a maximal ideal of $\operatorname{Spec} \prod A_i$ corresponding to each i (precisely those elements 0 in the ith component.) But there are other maximal ideals of $\prod A_i$. Hint: describe a proper ideal not contained in any of these maximal ideal. (One idea: consider elements $\prod a_i$ that are "eventually zero", i.e. $a_i = 0$ for $i \gg 0$.) This leads to the notion of *ultrafilters*, which are very useful, but irrelevant to our current discussion.

As long as we are on the topic of quasicompactness...

1.3. Definition. A scheme is **quasicompact** if its underlying topological space is quasicompact. This seems like a strong condition, but because Zariski-open sets are so large, almost any scheme naturally coming up in nature will be quasicompact.

1.G. EASY EXERCISE. Show that a scheme X is quasicompact if and only if it can be written as a finite union of affine schemes (Hence $\mathbb{P}_k^n$ is quasicompact.)

1.H. EXERCISE: QUASICOMPACT SCHEMES HAVE CLOSED POINTS. Show that if X is a nonempty quasicompact scheme, then it has a closed point. (Warning: there exist nonempty schemes with no closed points, so your argument had better use the quasicompactness hypothesis! We will see that in good situations, the closed points are dense, Exercise 3.H.)

1.4. Quasiseparatedness.

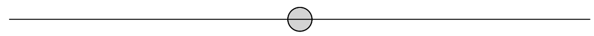

FIGURE 1. A picture of the scheme $\operatorname{Spec} k[x,y]/(xy,y^2)$

Quasiseparatedness is a weird notion that comes in handy for certain kinds of people. Most people, however, can ignore this notion. A scheme is **quasiseparated** if the intersection of any two quasicompact sets is quasicompact, or equivalently, if the intersection of any two affine open subsets is a finite union of affine open subsets.

1.I. SHORT EXERCISE. Prove this equivalence.

We will see later that this will be a useful hypothesis in theorems (in conjunction with quasicompactness), and that various interesting kinds of schemes (affine, locally Noetherian, separated, see Exercise 1.J, Exercise 3.B, and an exercise next quarter resp.) are quasiseparated, and this will allow us to state theorems more succinctly (e.g. "if X is quasicompact and quasiseparated" rather than "if X is quasicompact, and either this or that or the other thing hold").

1.J. EXERCISE. Show that affine schemes are quasiseparated.

"Quasicompact and quasiseparated" means something rather down to earth:

1.K. EXERCISE. Show that a scheme X is quasicompact and quasiseparated if and only if X can be covered by a finite number of affine open subsets, any two of which have intersection also covered by a finite number of affine open subsets.

2. REDUCEDNESS AND INTEGRALITY

Recall that one of the alarming things about schemes is that functions are not determined by their values at points, and that was because of the presence of *nilpotents*.

2.1. Definition. Recall that a ring is **reduced** if it has no nonzero nilpotents. A scheme X is **reduced** if $\mathcal{O}_X(U)$ has no nonzero nilpotents for any open set U of X.

An example of a nonreduced affine scheme is $\operatorname{Spec} k[x,y]/(y^2,xy)$. A useful representation of this scheme is given in Figure 1, although we will only explain in §5 why this is a good picture. The fuzz indicates that there is some nonreducedness going on at the origin. Here are two different functions: x and $x+y$. Their values agree at all points (all closed points $[(x-a,y)] = (a,0)$ and at the generic point $[(y)]$). They are actually the same function on the open set $D(x)$, which is not surprising, as $D(x)$ is reduced, as the next exercise shows. (This explains why the fuzz is only at the origin, where $y=0$.)

2.A. EXERCISE. Show that $\left(k[x,y]/(y^2,xy)\right)_x$ has no nilpotents. (Possible hint: show that it is isomorphic to another ring, by considering the geometric picture.)

2.B. EXERCISE (REDUCEDNESS IS STALK-LOCAL). Show that a scheme is reduced if and only if none of the stalks have nilpotents. Hence show that if f and g are two functions on a reduced scheme that agree at all points, then $f = g$. (Two hints: $\mathcal{O}_X(U) \hookrightarrow \prod_{x \in U} \mathcal{O}_{X,x}$ from an earlier Exercise, and the nilradical is intersection of all prime ideals.)

Warning: if a scheme X is reduced, then it is immediate from the definition that its ring of global sections is reduced. However, the converse is not true; we will meet an example later.

2.C. EXERCISE. Suppose X is quasicompact, and f is a function (a global section of $\mathcal{O}_X$) that vanishes at all points of x. Show that there is some n such that $f^n = 0$. Show that this may fail if X is not quasicompact. (This exercise is less important, but shows why we like quasicompactness, and gives a standard pathology when quasicompactness doesn't hold.) Hint: take an infinite disjoint union of $\operatorname{Spec} A_n$ with $A_n := k[\epsilon]/\epsilon^n$.

Definition. A scheme X is **integral** if $\mathcal{O}_X(U)$ is an integral domain for every open set U of X.

2.D. IMPORTANT EXERCISE. Show that a scheme X is integral if and only if it is irreducible and reduced.

2.E. EXERCISE. Show that an affine scheme $\operatorname{Spec} A$ is integral if and only if A is an integral domain.

2.F. EXERCISE. Suppose X is an integral scheme. Then X (being irreducible) has a generic point η. Suppose $\operatorname{Spec} A$ is any non-empty affine open subset of X. Show that the stalk at η, $\mathcal{O}_{X,\eta}$, is naturally $FF(A)$, the fraction field of A. This is called the **function field** $FF(X)$ of X. It can be computed on any non-empty open set of X, as any such open set contains the generic point. The symbol FFis deliberately ambiguous — it may stand for fraction field or function field.

2.G. EXERCISE. Suppose X is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are inclusions so long as $V \neq \varnothing$. Suppose $\operatorname{Spec} A$ is any non-empty affine open subset of X (so A is an integral domain). Show that the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = FF(A)$ (where U is any non-empty open set) is an inclusion. Thus irreducible varieties (an important example of integral schemes defined later) have the convenient that sections over different open sets can be considered subsets of the same thing. This makes restriction maps and gluing easy to consider; this is one reason why varieties are usually introduced before schemes.

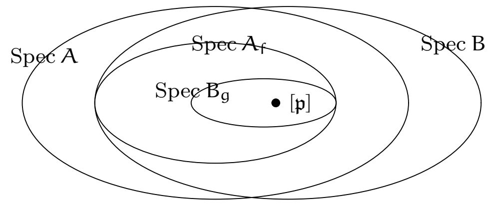

FIGURE 2. Trick to show that the intersection of two affine open sets may be covered by open sets that are simultaneously open in both affines

An almost-local criterion for integrality is given in 3.F.

3. PROPERTIES OF SCHEMES THAT CAN BE CHECKED "AFFINE-LOCALLY"

This section is intended to address something tricky and annoying in the definition of schemes. We've defined a scheme as a topological space with a sheaf of rings, that can be covered by affine schemes. Hence we have all of the affine opens in the cover, but we don't know how to communicate between any two of them. Somewhat more explicitly, if I have an affine cover, and you have an affine cover, and we want to compare them, and I calculate something on my cover, there should be some way of us getting together, and figuring out how to translate my calculation over onto your cover. The Affine Communication Lemma 3.3 will provide a convenient machine for doing this.

Thanks to this lemma, we can define a host of important properties of schemes. All of these are "affine-local" in that they can be checked on any affine cover, i.e. a covering by open affine sets. We like such properties because we can check them using any affine cover we like. If the scheme in question is quasicompact, then we need only check a finite number of affine open sets.

3.1. *Warning.* In our limited examples so far, any time we've had an affine open subset of an affine scheme $\operatorname{Spec} B \subset \operatorname{Spec} A$, in fact $\operatorname{Spec} B = D(f)$ for some f. But this is not always true, and we will eventually have an example, using elliptic curves.

3.2. *Proposition. — Suppose* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *are affine open subschemes of a scheme* X. *Then* $\operatorname{Spec} A \cap \operatorname{Spec} B$ *is the union of open sets that are simultaneously distinguished open subschemes of* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$.

Proof. (See Figure 2 for a sketch.) Given any point $[\mathfrak{p}] \in \operatorname{Spec} A \cap \operatorname{Spec} B$, we produce an open neighborhood of $[\mathfrak{p}]$ in $\operatorname{Spec} A \cap \operatorname{Spec} B$ that is simultaneously distinguished in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$. Let $\operatorname{Spec} A_f$ be a distinguished open subset of $\operatorname{Spec} A$ contained in $\operatorname{Spec} A \cap \operatorname{Spec} B$. Let $\operatorname{Spec} B_g$ be a distinguished open subset of $\operatorname{Spec} B$ contained in $\operatorname{Spec} A_f$. Then $g \in \Gamma(\operatorname{Spec} B, \mathcal{O}_X)$ restricts to an element $g' \in \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X) = A_f$. The points of $\operatorname{Spec} A_f$ where g vanishes are precisely the points of $\operatorname{Spec} A_f$ where g' vanishes, so

$$\begin{aligned} \operatorname{Spec} B_g &= \operatorname{Spec} A_f \setminus \{[\mathfrak{p}] : g' \in \mathfrak{p}\} \\ &= \operatorname{Spec}(A_f)_{g'}. \end{aligned}$$

If $g' = g''/f^n$ ($g'' \in A$) then $\operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fg''}$, and we are done. □

The following easy result will be crucial for us.

3.3. *Affine Communication Lemma. — Let* P *be some property enjoyed by some affine open sets of a scheme* X*, such that*

(i) *if an affine open set* $\operatorname{Spec} A \hookrightarrow X$ *has* P *then for any* $f \in A$, $\operatorname{Spec} A_f \hookrightarrow X$ *does too.*
(ii) *if* $(f_1, \dots, f_n) = A$*, and* $\operatorname{Spec} A_{f_i} \hookrightarrow X$ *has* P *for all* i*, then so does* $\operatorname{Spec} A \hookrightarrow X$.

Suppose that $X = \cup_{i \in I} \operatorname{Spec} A_i$ *where* $\operatorname{Spec} A_i$ *is an affine, and* A_i *has property* P*. Then every other open affine subscheme of* X *has property* P *too.*

We say such a property is **affine-local**. Note that any property that is stalk-local (a scheme has property P if and only if all its stalks have property Q) is necessarily affine-local (a scheme has property P if and only if all of its affines have property R, where an affine scheme has property R if and only if and only if all its stalks have property Q), but it is sometimes not so obvious what the right definition of Q is; see for example the discussion of normality in the next section.

Proof. Let $\operatorname{Spec} A$ be an affine subscheme of X. Cover $\operatorname{Spec} A$ with a finite number of distinguished opens $\operatorname{Spec} A_{g_j}$, each of which is distinguished in some $\operatorname{Spec} A_i$. This is possible by Proposition 3.2 and the quasicompactness of $\operatorname{Spec} A$. By (i), each $\operatorname{Spec} A_{g_j}$ has P. By (ii), $\operatorname{Spec} A$ has P. □

By choosing property P appropriately, we define some important properties of schemes.

3.4. *Proposition. — Suppose* A *is a ring, and* $(f_1, \dots, f_n) = A$.

(a) *If* A *is a Noetherian ring, then so is* A_{f_i}*. If each* A_{f_i} *is Noetherian, then so is* A.
(b) *If* A *is reduced, then* A_{f_i} *is also reduced. If each* A_{f_i} *is reduced, then so is* A.
(c) *Suppose* B *is a ring, and* A *is an* B*-algebra. (Hence* A_g *is a* B*-algebra for all* B*.) If* A *is a finitely generated* B*-algebra, then so is* A_{f_i}*. If each* A_{f_i} *is a finitely-generated* B*-algebra, then so is* A.

We'll prove these shortly. But let's first motivate you to read the proof by giving some interesting definitions *assuming* Proposition 3.4 is true.

3.5. Important Definitions. Suppose X is a scheme. If X can be covered by affine opens $\operatorname{Spec} A$ where A is Noetherian, we say that X is a **locally Noetherian scheme**. If in addition X is quasicompact, or equivalently can be covered by finitely many such affine opens, we say that X is a **Noetherian scheme**. By Exercise 1.C, the underlying topological space of a Noetherian scheme is Noetherian. (We will see a number of definitions of the form "if X has this property, we say that it is locally Q; if further X is compact, we say that it is Q.")

3.A. EXERCISE. Show that all open subsets of a Noetherian topological space (hence a Noetherian scheme) are quasicompact.

3.B. EXERCISE. Show that locally Noetherian schemes are quasiseparated.

3.C. EXERCISE. Show that a Noetherian scheme has a finite number of irreducible components. Show that a Noetherian scheme has a finite number of connected components, each a finite union of irreducible components.

3.D. EXERCISE. If X is a Noetherian scheme, show that every point p has a closed point in its closure. (In particular, every non-empty Noetherian scheme has closed points; this is not true for every scheme, as remarked in Exercise 1.H.)

3.E. EXERCISE. If X is an affine scheme or Noetherian scheme, show that it suffices to check reducedness at *closed points*. (Hint: For the Noetherian case, recall Exercise 3.D.)

Integrality is not stalk-local, but it almost is, as is shown in the following believable exercise.

3.F. UNIMPORTANT EXERCISE. Show that a locally Noetherian scheme X is integral if and only if X is connected and all stalks $\mathcal{O}_{X,p}$ are integral domains (informally: "the scheme is locally integral"). Thus in "good situations" (when the scheme is Noetherian), integrality is the union of local (stalks are domains) and global (connected) conditions.

3.6. *Remark.* Joe Rabinoff gave a great example showing that "locally Noetherian" is not a stalk-local condition. Joe's counterexample: Let k be an algebraically closed field, let $b_1, b_2, b_3, \ldots \in k$ be a sequence of distinct elements, and let

$$A = k[s, a_1, a_2, \ldots]/((s - b_i)a_{i+1} - a_i, a_i^2)_{i=1,2,\ldots}$$

I claim that A is not noetherian, but that A_p is noetherian for every prime ideal. It suffices to check for maximal ideals, as Noetherianness is preserved by localization.. The nilradical $\mathfrak{N}$ of A is $(a_1, a_2, \ldots)$ (as the a_i clearly lie in the nilradical, and $A/(a_1, \ldots)$ is a domain so we've found it all), and $A/\mathfrak{N} = k[s]$, so the maximal ideals of A are the ideals of the form $\mathfrak{m} = (s - b, a_1, a_2, \ldots)$ for $b \in k$ Let $\mathfrak{m}$ be such an ideal.

- Suppose that $b = b_n$ for some n. For $i \neq n$, we have $a_{i+1} = a_i/(s - b_i)$ in $A_\mathfrak{m}$. Hence $A_\mathfrak{m}$ is the localization of a ring generated by the two variables s and a_n, so it's Noetherian.
- If b is distinct from all the b_i, then $A_\mathfrak{m}$ is the localization of a ring generated by s and a_1, as above.

Hence all stalks are Noetherian, but clearly the nilradical of A is not finitely generated.

3.G. EXERCISE. Show that X is reduced if and only if X can be covered by affine opens $\operatorname{Spec} A$ where A is reduced (nilpotent-free).

Our earlier definition required us to check that the ring of functions over *any* open set is nilpotent free. Our new definition lets us check a single affine cover. Hence for example $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are reduced.

Suppose X is a scheme, and A is a ring (e.g. A is a field k), and $\Gamma(U, \mathcal{O}_X)$ has an A-algebra for all A, and the restriction maps respect the A-algebra structure. Then we say that X is an **A-scheme**, or a **scheme over** A. Suppose X is an A-scheme. If X can be covered by affine opens $\operatorname{Spec} B_i$ where each B_i is a *finitely generated* A-algebra, we say that X is **locally of finite type over** A, or that it is a **locally of finite type** A**-scheme**. (This is admittedly cumbersome terminology; it will make more sense later, once we know about morphisms.) If furthermore X is quasicompact, X is **finite type over** A, or a **finite type** A**-scheme**. Note that a scheme locally of finite type over k or $\mathbb{Z}$ (or indeed any Noetherian ring) is locally Noetherian, and similarly a scheme of finite type over any Noetherian ring is Noetherian. As our key "geometric" example: if $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is an ideal, then $\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]/I$ is a finite-type $\mathbb{C}$-scheme.

3.7. We now make a definition to make a connection to the language of varieties. An affine scheme that is reduced and finite type k-scheme is said to be an *affine variety (over* k*)*, or an *affine k-variety*. We are not yet ready to define varieties in general; we will need the notion of separatedness first, to exclude abominations of nature like the line with the doubled origin. We will define projective k-varieties before defining varieties in general (as separated finite type k-schemes). (Warning: in the literature, it is sometimes also required that the scheme be irreducible, or that k be algebraically closed.)

3.H. EXERCISE. Show that a point of a locally finite type k-scheme is a closed point if and only if the residue field of the stalk of the structure sheaf at that point is a finite extension of k. (Recall the following form of Hilbert's Nullstellensatz, richer than the version stated before: the maximal ideals of $k[x_1, \ldots, x_n]$ are precisely those with residue of the form a finite extension of k.) Show that the closed points are dense on such a scheme. (For another exercise on closed points, see 1.H.)

3.8. *Proof of Proposition 3.4.* (a) (i) If $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of A_f, then we can verify that $J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of A, where

$$J_j = \{r \in A : r \in I_j\}$$

where $r \in I_j$ means "the image in A_f lies in I_j". (We think of this as $I_j \cap A$, except in general A needn't inject into A_{f_i}.) Clearly J_j is an ideal of A. If $x/f^n \in I_{j+1} \setminus I_j$ where $x \in A$, then $x \in J_{j+1}$, and $x \notin J_j$ (or else $x(1/f)^n \in J_j$ as well). (ii) Suppose $I_1 \subsetneq I_2 \subsetneq I_3 \subset \cdots$ is a strictly increasing chain of ideals of A. Then for each $1 \leq i \leq n$,

$$I_{i,1} \subset I_{i,2} \subset I_{i,3} \subset \cdots$$

is an increasing chain of ideals in A_{f_i}, where $I_{i,j} = I_j \otimes_A A_{f_i}$. It remains to show that for each j, $I_{i,j} \subsetneq I_{i,j+1}$ for some i; the result will then follow.

3.I. EXERCISE. Finish this argument.

3.J. EXERCISE. Prove (b).

(c) (i) is clear: if A is generated over B by $r_1, \ldots, r_n$, then A_f is generated over B by $r_1, \ldots, r_n, 1/f$.

(ii) Here is the idea. We have generators of A_i: r_{ij}/f_i^j, where $r_{ij} \in A$. I claim that $\{r_{ij}\}_{ij} \cup \{f_i\}_i$ generate A as a B-algebra. Here's why. Suppose you have any $r \in A$. Then in A_{f_i}, we can write r as some polynomial in the r_{ij}'s and f_i, divided by some huge power of f_i. So "in each A_{f_i}, we have described r in the desired way", except for this annoying denominator. Now use a partition of unity type argument to combine all of these into a single expression, killing the denominator. Show that the resulting expression you build still agrees with r in each of the A_{f_i}. Thus it is indeed r.

3.K. EXERCISE. Make this argument precise.

This concludes the proof of Proposition 3.4 □

4. NORMALITY AND FACTORIALITY

4.1. Normality.

We can now define a property of schemes that says that they are "not too far from smooth", called *normality*, which will come in very handy. We will see later that "locally Noetherian normal schemes satisfy Hartogs' theorem": functions defined away form a set of codimension 2 extend over that set, (2) Rational functions that have no poles are defined everywhere. We need definitions of dimension and/or poles to make this precise.

A scheme X is **normal** if all of its stalks $\mathcal{O}_{X,x}$ are normal (i.e. are domains, and integrally closed in their fraction fields). As reducedness is a stalk-local property (Exercise 2.B), normal schemes are reduced.

4.A. EXERCISE. Show that integrally closed domains behave well under localization: if A is an integrally closed domain, and S is a multiplicative subset, show that $S^{-1}A$ is an integrally closed domain. (The domain portion is easy. Hint for integral closure: assume that $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in S^{-1}A$ has a root in the fraction field. Turn this into another equation in A[x] that also has a root in the fraction field.)

It is no fun checking normality at every single point of a scheme. Thanks to this exercise, we know that if A is an integrally closed domain, then $\operatorname{Spec} A$ is normal. Also, for Noetherian schemes, normality can be checked at closed points, thanks to this exercise, and the fact that for such schemes, any point is a generization of a closed point (see Exercise 3.D)

It is not true that normal schemes are integral. For example, the disjoint union of two normal schemes is normal. Thus $\operatorname{Spec} k \coprod \operatorname{Spec} k \cong \operatorname{Spec}(k \times k) \cong \operatorname{Spec} k[x]/(x(x-1))$ is normal, but its ring of global sections is not a domain.

4.B. UNIMPORTANT EXERCISE. Show that a Noetherian scheme is normal if and only if it is the finite disjoint union of integral Noetherian normal schemes.

We are close to proving a useful result in commutative algebra, so we may as well go all the way.

4.2. *Proposition. — If A is an integral domain, then the following are equivalent.*

(1) A *integrally closed.*
(2) $A_{\mathfrak{p}}$ *is integrally closed for all prime ideals* $\mathfrak{p} \subset A$.
(3) $A_{\mathfrak{m}}$ *is integrally closed for all maximal ideals* $\mathfrak{m} \subset A$.

Proof. Clearly (2) implies (3). Exercise 4.A shows that integral closure is preserved by localization, so (1) implies (2).

It remains to show that (3) implies (1). This argument involves a very nice construction that we will use again. Suppose A is not integrally closed. We show that there is some $\mathfrak{m}$ such that $A_{\mathfrak{m}}$ is also not integrally closed. Suppose

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \tag{1}$$

(with $a_i \in A$) has a solution s in $\mathrm{FF}(A)$. Let I be the **ideal of denominators of** s:

$$I := \{r \in A : rs \in A\}.$$

(Note that I is clearly an ideal of A.) Now $I \neq A$, as $1 \notin I$. Thus there is some maximal ideal $\mathfrak{m}$ containing I. Then $s \notin A_{\mathfrak{m}}$, so equation (1) in $A_{\mathfrak{m}}[x]$ shows that $A_{\mathfrak{m}}$ is not integrally closed as well, as desired. □

4.C. UNIMPORTANT EXERCISE. If A is an integral domain, show that $A = \cap A_{\mathfrak{m}}$, where the intersection runs over all maximal ideals of A. (We won't use this exercise, but it gives good practice with the ideal of denominators.)

4.D. UNIMPORTANT EXERCISE RELATING TO THE IDEAL OF DENOMINATORS. One might naively hope from experience with unique factorization domains that the ideal of denominators is principal. This is not true. As a counterexample, consider our new friend

$A = k[a,b,c,d]/(ad - bc)$ (which we will later recognize as the cone over the quadric surface), and $a/c = b/d \in FF(A)$. Show that $I = (c,d)$.

4.3. Factoriality.

We define a notion which implies normality.

4.4. Definition. If all the stalks of a scheme X are unique factorization domains, we say that X is **factorial**.

4.E. EXERCISE. Show that any localization of a Unique Factorization Domain is a Unique Factorization Domain.

Thus if A is a unique factorization domain, then $\operatorname{Spec} A$ is factorial. (The converse need not hold. Hence this property is *not* affine-local, as we will verify later. Here is a counterexample without proof: $\mathbb{Z}[\sqrt{17}]$.) Hence it suffices to check factoriality by finding an appropriate affine cover.

One of the reasons we like factoriality is that it implies normality.

4.F. IMPORTANT EXERCISE. Show that unique factorization domains are integrally closed. Hence factorial schemes are are normal, and if A is a unique factorization domain, then $\operatorname{Spec} A$ is normal. (However, rings can be integrally closed without being unique factorization domains, as we'll see in Exercise 4.I. An example without proof: $\mathbb{Z}[\sqrt{17}]$ again.)

4.G. EASY EXERCISE. Show that the following schemes are normal: $\mathbb{A}^n_k$, $\mathbb{P}^n_k$, $\operatorname{Spec} \mathbb{Z}$.

4.H. EXERCISE (WHICH WILL GIVE US A NUMBER OF ENLIGHTENING EXAMPLES LATER). Suppose A is a Unique Factorization Domain with 2 invertible, $f \in A$ has no repeated prime factors, and $z^2 - f$ is irreducible in $A[z]$. Show that $\operatorname{Spec} A[z]/(z^2 - f)$ is normal. Show that if f is *not* square-free, then $\operatorname{Spec} A[z]/(z^2 - f)$ is *not* normal. (Hint: $B := A[z]/(z^2 - f)$ is a domain, as $(z^2 - f)$ is prime in $A[z]$. Suppose we have monic $F(T) = 0$ with $F(T) \in B[T]$ which has a solution α in $FF(B)$. Then by replacing $F(T)$ by $\overline{F}(T)F(T)$, we can assume $F(T) \in A[T]$. Also, $\alpha = g + hz$ where $g, h \in FF(A)$. Now α is the solution of monic $Q(T) = T^2 - 2gT + (g^2 - h^2 f)T \in FF(A)[T]$, so we can factor $F(T) = P(T)Q(T)$ in $K[T]$. By Gauss' lemma, $2g$, $g^2 - h^2 f \in A$. Say $g = r/2$, $h = s/t$ (s and t have no common factors, $r, s, t \in A$). Then $g^2 - h^2 f = (r^2 t^2 - rs^2 f)/4t^2$. Then $t = 1$, and r is even.)

4.I. EXERCISE. Show that the following schemes are normal:

(a) $\operatorname{Spec} \mathbb{Z}[x]/(x^2 - n)$ where n is a square-free integer congruent to $3 \pmod 4$;
(b) $\operatorname{Spec} k[x_1, \dots, x_n]/x_1^2 + x_2^2 + \cdots + x_m^2$ where $\operatorname{char} k \neq 2$, $m \geq 3$;

(c) $\operatorname{Spec} k[w,x,y,z]/(wz-xy)$ where $\operatorname{char} k \neq 2$ and k is algebraically closed. (This is our cone over a quadric surface example from Exercise 4.D.)

4.J. EXERCISE. Suppose A is a k-algebra where $\operatorname{char} k = 0$, and l/k is a finite field extension. Show that A is normal if and only if $A \otimes_k l$ is normal. Show that $\operatorname{Spec} k[w,x,y,z]/(wz-xy)$ is normal if k is characteristic 0. (In fact the hypothesis on the characteristic is unnecessary.) Possible hint: reduce to the case where l/k is Galois.

5. ASSOCIATED POINTS OF (LOCALLY NOETHERIAN) SCHEMES, AND DRAWING FUZZY PICTURES

Recall from just after Definition 2.1 (of *reduced*) our "fuzzy" pictures of the non-reduced scheme $\operatorname{Spec} k[x,y]/(y^2,xy)$ (see Figure 1). When this picture was introduced, we mentioned that the "fuzz" at the origin indicated that the non-reduced behavior was concentrated there; this was verified in Exercise 2.A, and indeed the origin is the only point where the stalk of the structure sheaf is non-reduced.

You might imagine that in a bigger scheme, we might have different closed subsets with different amount of "non-reducedness". This intuition will be made precise in this section. We will define *associated points* of a scheme, which will be the most important points of a scheme, encapsulating much of the interesting behavior of the structure sheaf. These will be defined for any locally Noetherian scheme. The primes corresponding to the associated points of an affine scheme $\operatorname{Spec} A$ will be called *associated primes of* A. (In fact this is backwards; we will define associated primes first, and then define associated points.)

The four properties about associated points that it will be most important to remember are as follows. Frankly, it is much more important to remember these four facts than it is to remember their proofs.

(1) *The generic points of the irreducible components are associated points.* The other associated points are called **embedded points**.

(2) *If X is reduced, then X has no embedded points.* (This jibes with the intuition of the picture of associated points described earlier.)

(3) Recall that one nice property of integral schemes X (such as irreducible affine varieties) not shared by all schemes is that for any open $U \subset X$, the natural map $\Gamma(U, \mathcal{O}_X) \to FF(X)$ is an inclusion (Exercise 2.G). Thus all sections over any open set (except $\varnothing$) and stalks can be thought of as lying in a single field $FF(X)$, which is the talk at the generic point.

More generally, if X is a locally Noetherian scheme, then for any $U \subset X$, the natural map

$$\Gamma(U, \mathcal{O}_X) \to \prod_{\text{associated } p \text{ in } U} \mathcal{O}_{X,p} \tag{2}$$

is an injection.

We define a **rational function** on a locally Noetherian scheme to be an element of the image of $\Gamma(U, \mathcal{O}_U)$ in (2) for some U containing all the associated points. The rational functions form a ring, called the **total fraction ring** of X, denoted $FF(X)$. If $X = \operatorname{Spec} A$ is affine, then this ring is called the **total fraction ring** of A, $FF(A)$. Note that if X is integral, this is the function field $FF(X)$, so this extends our earlier definition 2.F of $FF(\cdot)$. It can be more conveniently interpreted as follows, using the injectivity of (2). A rational function is a function defined on an open set containing all associated points, i.e. and ordered pair (U, f), where U is an open set containing all associated points, and $f \in \Gamma(U, \mathcal{O}_X)$. Two such data (U, f) and (U', f') define the same open rational function if and only if the restrictions of f and f' to $U \cap U'$ are the same. If X is reduced, this is the same as requiring that they are defined on an open set of each of the irreducible components. A rational function has a maximal domain of definition, because any two actual functions on an open set (i.e. sections of the structure sheaf over that open set) that agree as "rational functions" (i.e. on small enough open sets containing associated points) must be the same function, by the injectivity of (2). We say that a rational function f is **regular** at a point p if p is contained in this maximal domain of definition (or equivalently, if there is some open set containing p where f is defined).

The previous facts are intimately related to the following one.

(4) *A function on X is a zero divisor if and only if it vanishes at an associated point of X.*

Motivated by the above four properties, when sketching (locally Noetherian) schemes, we will draw the irreducible components (the closed subsets corresponding to maximal associated points), and then draw "additional fuzz" precisely at the closed subsets corresponding to embedded points. All of our earlier sketches were of this form.

Let's now get down to business of defining associated points, and showing that they the desired properties **(1)** through **(4)**.

We say an ideal $I \subset A$ in a ring is **primary** if $I \neq A$ and if $xy \in I$ implies either $x \in I$ or $y^n \in I$ for some $n > 0$.

It is useful to interpret maximal ideals as "the quotient is a field", and prime ideals as "the quotient is an integral domain". We can interpret primary ideals similarly as "the quotient is not 0, and every zero-divisor is nilpotent".

5.A. EXERCISE. Show that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime. If $\mathfrak{p} = \sqrt{\mathfrak{q}}$, we say that $\mathfrak{q}$ is $\mathfrak{p}$-*primary*. (Caution: $\sqrt{\mathfrak{q}}$ can be prime without $\mathfrak{q}$ being primary — consider our example (y^2, xy) in $k[x, y]$.)

5.B. EXERCISE. Show that if $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.

5.C. EXERCISE (REALITY CHECK). Find all the primary ideals in $\mathbb{Z}$. (Answer: (0) and (p^n).)

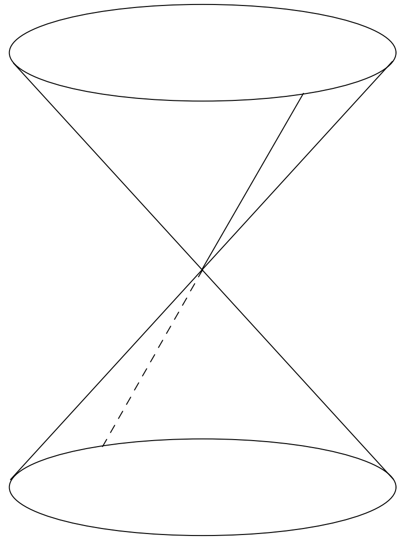

FIGURE 3. $V(x,z) \subset \operatorname{Spec} k[x,y,z]/(xy-z^2)$ is a ruling on a cone; $(x,z)^2$ is not (x,z)-primary.

5.1. ⋆ *Unimportant warning for experts (all others should skip this).* A prime power need not be primary. For example, let $A = k[x,y,z]/(xy-z^2)$, and $\mathfrak{p} = (x,z)$. Then $\mathfrak{p}$ is prime but $\mathfrak{p}^2$ is not primary. (Verify this — the algebra is easy! Why is $(x^2, xz, z^2, xy-z^2)$ not primary in $k[x,y,z]$?) We will soon be able to interpret $\operatorname{Spec} A$ as a "cone", and $V(x,z)$ as the "ruling" of the cone, see Figure 3, and the corresponding picture gives a geometric hint that there is something going on. We'll come back to this at a later date.

5.2. Primary decompositions.

A **primary decomposition** of an ideal $I \subset A$ is an expression of the ideal as a finite intersection of primary ideals.

$$I = \cap_{i=1}^n \mathfrak{q}_i$$

If there are "no redundant elements" (the $\sqrt{\mathfrak{q}_i}$ are all distinct, and for no i is $\mathfrak{q}_i \supset \cap_{j\neq i}\mathfrak{q}_j$), we say that the decomposition is **minimal**. Clearly any ideal with a primary decomposition has a minimal primary decomposition (using Exercise 5.B).

5.D. IMPORTANT EXERCISE (EXISTENCE OF PRIMARY DECOMPOSITION FOR NOETHERIAN RINGS). Suppose A is a Noetherian ring. Show that every proper ideal $I \subset A$ has a primary decomposition. (Hint: mimic the Noetherian induction argument we saw last week.)

5.E. IMPORTANT EXERCISE. (a) Find a minimal primary decomposition of (y^2, xy). (b) Find another one. (Possible hint: see Figure 1. You might be able to draw sketches of your different primary decompositions.)

In order to study these objects, we'll need a useful fact and a definition.

5.F. ESSENTIAL EXERCISE. (a) If $\mathfrak{p}$, $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are prime ideals, and $\mathfrak{p} = \cap \mathfrak{p}_i$, show that $\mathfrak{p} = \mathfrak{p}_i$ for some i. (Hint: assume otherwise, choose $f_i \in \mathfrak{p}_i - \mathfrak{p}$, and consider $\prod f_i$.)
(b) If $\mathfrak{p} \supset \cap \mathfrak{p}_i$, then $\mathfrak{p} \supset \mathfrak{p}_i$ for some i.
(c) Suppose $I \subseteq \cup_{i=1}^n \mathfrak{p}_i$. (The right side is not an ideal!) Show that $I \subset \mathfrak{p}_i$ for some i. (Hint: by induction on n. Don't look in the literature — you might find a much longer argument!)

Parts (a) and (b) are "geometric facts"; try to draw pictures of what they mean.

If $I \subset A$ is an ideal, and $x \in A$, then define the **colon ideal** $(I : x) := \{a \in A : ax \in I\}$. (We will use this terminology only for this section.) For example, x is a *zero-divisor* if $(0 : x) \neq 0$.

5.3. *Theorem ("uniqueness" of primary decomposition). — Suppose $I \subset A$ has a minimal primary decomposition*

$$I = \cap_{i=1}^n \mathfrak{q}_i.$$

(For example, this is always true if A is Noetherian.) Then the $\sqrt{\mathfrak{q}_i}$ are precisely the prime ideals that are of the form

$$\sqrt{(I : x)}$$

for some $x \in A$. Hence this list of primes is independent of the decomposition.

These primes are called the **associated primes** of the ideal I. The **associated primes of** A are the associated primes of 0.

Proof. We make a very useful observation: for any $x \in A$,

$$(I : x) = (\cap \mathfrak{q}_i : x) = \cap(\mathfrak{q}_i : x),$$

from which

$$\sqrt{(I : x)} = \cap \sqrt{(\mathfrak{q}_i : x)} = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j. \qquad (3)$$

Now we prove the result.

Suppose first that $\sqrt{(I : x)}$ is prime, say $\mathfrak{p}$. Then $\mathfrak{p} = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j$ by (3), and by Exercise 5.F(a), $\mathfrak{p} = \mathfrak{p}_j$ for some j.

Conversely, given $\mathfrak{q}_i$, we find an x such that $\sqrt{(I : x)} = \sqrt{\mathfrak{q}_i}$ $(= \mathfrak{p}_i)$. Take $x \in \cap_{j \neq i} \mathfrak{q}_j - \mathfrak{q}_i$ (which is possible by minimality of the primary decomposition). Then by (3), we're done. □

E-mail address: vakil@math.stanford.edu