FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 21

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CONTENTS

This class will meet 8:40-9:55. Please be sure to be on the e-mail list so I can warn you which days class will take place.

Welcome back! Where we're going this quarter: last quarter, we established the objects of study: varieties or schemes. This quarter we'll be mostly concerned with important means of studying them: ~~vector bundles~~ quasicoherent sheaves and cohomology thereof. As a punchline for this quarter, I hope to say a lot of things about curves (Riemann surfaces) at the end of the quarter. However, in keeping with the attitude of last quarter, my goal isn't to make a beeline for the punchline. Instead we'll have a scorched-earth policy and try to cover everything between here and there relatively comprehensively. We start with ~~smoothness~~ nonsingularity of schemes. Then ~~vector bundles~~ locally free sheaves, quasicoherent sheaves and coherent sheaves. Then to ~~line bundles~~ invertible sheaves, and divisors. Then we'll interpret these for projective schemes in terms of graded modules. We'll investigate pushing forward and pulling back quasicoherent sheaves. We'll construct schemes using these notions, and for example define the notion of a projective morphism. We'll study differentials (e.g. the tangent bundle of smooth schemes, but also for singular things). Then we'll discuss cohomology (both Cech cohomology and derived functor cohomology). Then curves! The punch line for today: $\operatorname{Spec} \mathbb{Z}$ is a ~~smooth~~ nonsingular curve.

1. NONSINGULARITY ("SMOOTHNESS") OF NOETHERIAN SCHEMES

One natural notion we expect to see for geometric spaces is the notion of when an object is "smooth". In algebraic geometry, this notion, called *nonsingularity* (or *regularity*, although we won't use this term) is easy to define but a bit subtle in practice. We will soon define what it means for a scheme to be *nonsingular* (or *regular*) at a point. A point that is not nonsingular is (not surprisingly) called *singular* ("not smooth"). A scheme is said *nonsingular* if all its points are nonsingular, and singular if one of its points is singular.

Date: Friday, January 11, 2008.

The notion of nonsingularity is less useful than you might think. Grothendieck taught us that the more important notions are properties of morphisms, not of objects, and there is indeed a "relative notion" that applies to a morphism of schemes $f : X \to Y$ that is much better-behaved (corresponding to the notion of submersion in differential geometry). For this reason, the word "smooth" is reserved for these morphisms. We will discuss smooth morphisms in the spring quarter. However, nonsingularity is still useful, especially in (co)dimension 1, and we shall discuss this case (of *discrete valuation rings*) next day.

2. THE ZARISKI TANGENT SPACE

We begin by defining the notion of the tangent space of a scheme at a point. It will behave like the tangent space you know and love at "smooth" points, but will also make sense at other points. In other words, geometric intuition at the smooth points guides the definition, and then the definition guides the algebra at all points, which in turn lets us refine our geometric intuition.

This definition is short but surprising. The main difficulty is convincing yourself that it deserves to be called the tangent space. I've always found this tricky to explain, and that is because we want to show that it agrees with our intuition; but unfortunately, our intuition is worse than we realize. So I'm just going to define it for you, and later try to convince you that it is reasonable.

Suppose $\mathfrak{p}$ is a prime ideal of a ring A, so $[\mathfrak{p}]$ is a point of $\operatorname{Spec} A$. Then $[\mathfrak{p}A_\mathfrak{p}]$ is a point of the scheme $\operatorname{Spec} A_\mathfrak{p}$. For convenience, we let $\mathfrak{m} := \mathfrak{p}A_\mathfrak{p} \subset A_\mathfrak{p} =: B$. Let $k = B/\mathfrak{m}$ be the residue field. Then $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over the residue field k: it is an B-module, and elements of $\mathfrak{m}$ acts like 0. This is defined to be the **Zariski cotangent space**. The dual is the **Zariski tangent space**. Elements of the Zariski cotangent space are called **cotangent vectors** or **differentials**; elements of the tangent space are called **tangent vectors**.

Note that this definition is intrinsic. It doesn't depend on any specific description of the ring itself (such as the choice of generators over a field k, which is equivalent to the choice of embedding in affine space). Notice that in some sense, the cotangent space is more algebraically natural than the tangent space. There is a moral reason for this: the cotangent space is more naturally determined in terms of functions on a space, and we are very much thinking about schemes in terms of "functions on them". This will come up later.

I'll give two of plausibility arguments that this is a reasonable definition. Hopefully one will catch your fancy.

In differential geometry, the tangent space at a point is sometimes defined as the vector space of derivations at that point. A derivation is a function that takes in functions near the point that vanish at the point, and gives elements of the field k, and satisfies the Leibniz rule

$$(fg)' = f'g + g'f.$$

Translation: a derivation is a map $\mathfrak{m} \to k$. But $\mathfrak{m}^2 \to 0$, as if $f(p) = g(p) = 0$, then

$$(fg)'(p) = f'(p)g(p) + g'(p)f(p) = 0.$$

Thus we have a map $\mathfrak{m}/\mathfrak{m}^2 \to k$, i.e. an element of $(\mathfrak{m}/\mathfrak{m}^2)^\vee$.

2.A. EXERCISE. Check that this is reversible, i.e. that any map $\mathfrak{m}/\mathfrak{m}^2 \to k$ gives a derivation. In other words, verify that the Leibniz rule holds.

Here is a second vaguer motivation that this definition is plausible for the cotangent space of the origin of $\mathbb{A}^n$. Functions on $\mathbb{A}^n$ should restrict to a linear function on the tangent space. What function does $x^2 + xy + x + y$ restrict to "near the origin"? You will naturally answer: $x + y$. Thus we "pick off the linear terms". Hence $\mathfrak{m}/\mathfrak{m}^2$ are the linear functionals on the tangent space, so $\mathfrak{m}/\mathfrak{m}^2$ is the cotangent space. In particular, you should picture functions vanishing at a point (lying in $\mathfrak{m}$) as giving functions on the tangent space in this obvious a way.

2.1. *Old-fashioned example.* Here is an example to help tie this down to earth. Computing the Zariski-tangent space is actually quite hands-on, because you can compute it just as you did when you learned multivariable calculus. In $\mathbb{A}^3$, we have a curve cut out by $x + y + z^2 + xyz = 0$ and $x - 2y + z + x^2y^2z^3 = 0$. (You know enough to check that this is a curve, but it is not important to do so.) What is the tangent line near the origin? (Is it even smooth there?) Answer: the first surface looks like $x + y = 0$ and the second surface looks like $x - 2y + z = 0$. The curve has tangent line cut out by $x + y = 0$ and $x - 2y + z = 0$. It is smooth (in the analytic sense). In multivariable calculus, the students do a page of calculus to get the answer, because we aren't allowed to tell them to just pick out the linear terms.

Let's make explicit the fact that we are using. If A is a ring, $\mathfrak{m}$ is a maximal ideal, and $f \in \mathfrak{m}$ is a function vanishing at the point $[\mathfrak{m}] \in \operatorname{Spec} A$, then the Zariski tangent space of $\operatorname{Spec} A/(f)$ at $\mathfrak{m}$ is cut out in the Zariski tangent space of $\operatorname{Spec} A$ (at $\mathfrak{m}$) by the single linear equation $f \pmod{\mathfrak{m}^2}$. The next exercise will force you think this through.

2.B. IMPORTANT EXERCISE ("KRULL'S PRINCIPAL IDEAL THEOREM FOR THE ZARISKI TANGENT SPACE"). Suppose A is a ring, and $\mathfrak{m}$ a maximal ideal. If $f \in \mathfrak{m}$, show that the Zariski tangent space of A/f is cut out in the Zariski tangent space of A by $f \pmod{\mathfrak{m}^2}$. (Note: we can quotient by f and localize at $\mathfrak{m}$ in either order, as quotienting and localizing "commute".) Hence the dimension of the Zariski tangent space of $\operatorname{Spec} A$ at $[\mathfrak{m}]$ is the dimension of the Zariski tangent space of $\operatorname{Spec} A/(f)$ at $[\mathfrak{m}]$, or one less.

Here is another example to see this principle in action: $x + y + z^2 = 0$ and $x + y + x^2 + y^4 + z^5 = 0$ cuts out a curve, which obviously passes through the origin. If I asked my multivariable calculus students to calculate the tangent line to the curve at the origin, they would do a reams of calculations which would boil down to picking off the linear terms. They would end up with the equations $x + y = 0$ and $x + y = 0$, which cuts out a plane, not a line. They would be disturbed, and I would explain that this is because the curve isn't smooth at a point, and their techniques don't work. We on the other hand bravely declare that the cotangent space is cut out by $x + y = 0$, and (will soon) *define* this as a singular point. (Intuitively, the curve near the origin is very close to lying in the plane $x + y = 0$.) Notice: the cotangent space jumped up in dimension from what it was "supposed to be", not down. We'll see that this is not a coincidence soon, in Theorem 3.1.

Here is a nice consequence of the notion of Zariski tangent space.

2.2. Problem. Consider the ring $A = k[x, y, z]/(xy - z^2)$. Show that (x, z) is not a principal ideal.

As $\dim A = 2$ (by Krull's Principal Ideal Theorem), and $A/(x, z) \cong k[y]$ has dimension 1, we see that this ideal is height 1 (as codimension is the difference of dimensions for finitely generated k-domains). Our geometric picture is that $\operatorname{Spec} A$ is a cone (we can diagonalize the quadric as $xy - z^2 = ((x + y)/2)^2 - ((x - y)/2)^2 - z^2$, at least if $\operatorname{char} k \neq 2$), and that (x, z) is a ruling of the cone. (See Figure 1 for a sketch.) This suggests that we look at the cone point.

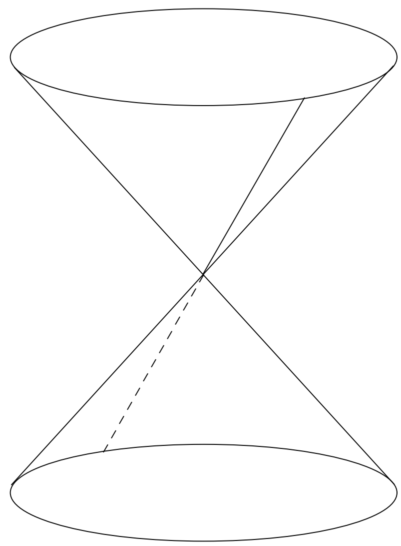

FIGURE 1. $V(x, z) \subset \operatorname{Spec} k[x, y, z]/(xy - z^2)$ is a ruling on a cone; $(x, z)^2$ is not (x, z)-primary.

Solution. Let $\mathfrak{m} = (x, y, z)$ be the maximal ideal corresponding to the origin. Then $\operatorname{Spec} A$ has Zariski tangent space of dimension 3 at the origin, and $\operatorname{Spec} A/(x, z)$ has Zariski tangent space of dimension 1 at the origin. But $\operatorname{Spec} A/(f)$ must have Zariski tangent space of dimension at least 2 at the origin by Exercise 2.B.

2.C. EXERCISE. Show that $(x, z) \subset k[w, x, y, z]/(wz - xy)$ is a codimension 1 ideal that is not principal. (See Figure 2 for a sketch.)

2.3. Morphisms and tangent spaces. Suppose $f : X \to Y$, and $f(p) = q$. Then if we were in the category of manifolds, we would expect a tangent map, from the tangent

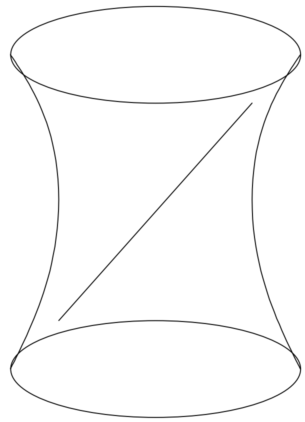

FIGURE 2. The ruling $V(x,z)$ on $V(wz-xy) \subset \mathbb{P}^3$.

space of p to the tangent space at q. Indeed that is the case; we have a map of stalks $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, which sends the maximal ideal of the former $\mathfrak{n}$ to the maximal ideal of the latter $\mathfrak{m}$ (we have checked that this is a "local morphism" when we briefly discussed local-ringed spaces). Thus $\mathfrak{n}^2 \to \mathfrak{m}^2$, from which $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$, from which we have a natural map $(\mathfrak{m}/\mathfrak{m}^2)^\vee \to (\mathfrak{n}/\mathfrak{n}^2)^\vee$. This is the map from the tangent space of p to the tangent space at q that we sought.

Here are some exercises to give you practice with the Zariski tangent space.

2.D. USEFUL EXERCISE (THE JACOBIAN CRITERION FOR COMPUTING THE ZARISKI TANGENT SPACE). Suppose k is an algebraically closed field, and X is a finite type k-scheme. Then locally it is of the form $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Show that the Zariski tangent space at the closed point p (with residue field k, by the Nullstellensatz) is given by the cokernel of the Jacobian map $k^r \to k^n$ given by the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}. \tag{1}$$

(This is just making precise our example of a curve in $\mathbb{A}^3$ cut out by a couple of equations, where we picked off the linear terms, see Example 2.1 .) You might be alarmed: what does $\frac{\partial f}{\partial x_1}$ mean?! Do you need deltas and epsilons? No! Just define derivatives formally, e.g.

$$\frac{\partial}{\partial x_1}(x_1^2 + x_1 x_2 + x_2^2) = 2x_1 + x_2.$$

(Hint: Do this first when p is the origin, and consider linear terms, just as in Example 2.1. Note for future reference that you have not yet used the algebraic closure of k. Then in the general case (with k algebraically closed), "translate p to the origin."

2.E. LESS IMPORTANT EXERCISE ("HIGHER-ORDER DATA"). In an earlier exercise, you computed the equations cutting out the three coordinate axes of $\mathbb{A}^3_k$. (Call this scheme X.) Your ideal should have had three generators. Show that the ideal can't be generated by fewer than three elements. (Hint: working modulo $\mathfrak{m} = (x, y, z)$ won't give any useful information, so work modulo $\mathfrak{m}^2$.)

2.F. EXERCISE. Suppose X is a k-scheme. Describe a natural bijection $\operatorname{Mor}_k(\operatorname{Spec} k[\epsilon]/(\epsilon^2), X)$ to the data of a point with residue field is k, necessarily a closed point.

2.G. EXERCISE. Find the dimension of the Zariski tangent space at the point $[(2, x)]$ of $\mathbb{Z}[2i] \cong \mathbb{Z}[x]/(x^2 + 4)$. Find the dimension of the Zariski tangent space at the point $[(2, x)]$ of $\mathbb{Z}[\sqrt{-2}] \cong \mathbb{Z}[x]/(x^2 + 2)$.

3. THE LOCAL DIMENSION IS AT MOST THE DIMENSION OF THE TANGENT SPACE

We are ready to define nonsingularity. The key idea is contained in the title of this section.

3.1. *Theorem. — Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring. Then* $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$.

If equality holds, we say that A is a **regular local ring**. If a Noetherian ring A is regular at all of its primes, we say that A is a **regular ring**.

A locally Noetherian scheme X is **regular** or **nonsingular** at a point p if the local ring $\mathcal{O}_{X,p}$ is regular. It is **singular** at the point otherwise. A scheme is **regular** or **nonsingular** if it is regular at all points. It is **singular** otherwise (i.e. if it is singular at *at least one* point).

Proof of Theorem 3.1: Note that $\mathfrak{m}$ is finitely generated (as A is Noetherian), so $\mathfrak{m}/\mathfrak{m}^2$ is a finitely generated $(A/\mathfrak{m} = k)$-module, hence finite-dimensional. Say $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$. Choose a basis of $\mathfrak{m}/\mathfrak{m}^2$, and lift them to elements $f_1, \ldots, f_n$ of $\mathfrak{m}$. Then by Nakayama's lemma (version 4), $(f_1, \ldots, f_n) = \mathfrak{m}$.

We need here one fancy fact that I forgot to say last quarter. Krull's Principal Ideal Theorem states that the codimension of any irreducible component of the locus cut out by one equation is at most one. There is a generalization to an arbitrary number of equations: if A is a Noetherian ring, then any irreducible component of $V(f_1, \ldots, f_n)$ has codimension at most n. The proof isn't much harder than Krull, but I haven't given it to you. Sorry! You can read a proof in Eisenbud (Theorem 10.2, p. 235).

3.A. EXERCISE. Prove this if A is an irreducible variety over a field. (Hint: this isn't that hard. Use the fact that codimension is the difference of dimensions in this happy case.)

In our case, $V((f_1, \dots, f_n)) = V(\mathfrak{m})$ is just the point $[\mathfrak{m}]$, so the codimension of $\mathfrak{m}$ is at most n. Thus the longest chain of prime ideals contained in $\mathfrak{m}$ is at most $n+1$. But this is also the longest chain of prime ideals in A (as $\mathfrak{m}$ is the unique maximal ideal), so $n \geq \dim A$. □

3.B. EXERCISE. Show that if A is a Noetherian local ring, then A has finite dimension. (Noetherian rings in general could have infinite dimension, as we saw in an earlier exercise.)

In the case of finite type schemes over an algebraically closed field k, the Jacobian criterion (Exercise 2.D) gives a hands-on method for checking for singularity at closed points.

3.C. EXERCISE. Suppose k is algebraically closed. Show that the singular *closed* points of the hypersurface $f(x_1, \dots, x_n) = 0$ in $\mathbb{A}^n_k$ are given by the equations $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$.

3.D. EXERCISE. Suppose k is algebraically closed. Show that $\mathbb{A}^1_k$ and $\mathbb{A}^2_k$ are nonsingular. (Make sure to check nonsingularity at the non-closed points! Fortunately you know what all the points of $\mathbb{A}^2_k$ are; this is trickier for $\mathbb{A}^3_k$.) Show that $\mathbb{P}^1_k$ and $\mathbb{P}^2_k$ are nonsingular. (This holds even if k isn't algebraically closed, and in higher dimension.)

Let's apply this technology to an arithmetic situation.

3.E. EASY EXERCISE. Show that $\operatorname{Spec} \mathbb{Z}$ is a nonsingular curve.

Here are some fun comments: What is the derivative of 35 at the prime 5? Answer: $35 \pmod{25}$, so 35 has the same "slope" as 10. What is the derivative of 9, which doesn't vanish at 5? Answer: the notion of derivative doesn't apply there. You'd think that you'd want to subtract its value at 5, but you can't subtract "$4 \pmod 5$" from the integer 9. Also, $35 \pmod{25}$ you might *think* you want to restate as $7 \pmod 5$, by dividing by 5, but that's morally wrong — you're dividing by a particular choice of generator 5 of the maximal ideal of $\mathbb{Z}_5$ (the 5-adics); in this case, one appears to be staring you in the face, but in general that won't be true. Follow-up fun: you can talk about the derivative of a function only for functions vanishing at a point. And you can talk about the second derivative of a function only for functions that vanish, and whose first derivative vanishes. For example, 75 has second derivative $75 \pmod{125}$ at 5. It's pretty flat.

3.F. EXERCISE. (This exercise is for those who know about the primes of the Gaussian integers $\mathbb{Z}[i]$.) Note that $\mathbb{Z}[i]$ is dimension 1, as $\mathbb{Z}[x]$ has dimension 2 (problem set exercise), and is a domain, and x^2+1 is not 0, so $\mathbb{Z}[x]/(x^2+1)$ has dimension 1 by Krull's Principal Ideal Theorem. Show that $\operatorname{Spec} \mathbb{Z}[i]$ is a nonsingular curve.

3.G. EXERCISE. Show that there is one singular point of $\operatorname{Spec} \mathbb{Z}[5i]$, and describe it.

Let's return to more geometric examples.

3.H. EXERCISE (THE EULER TEST FOR PROJECTIVE HYPERSURFACES). There is an analogous Jacobian criterion for hypersurfaces $f = 0$ in $\mathbb{P}^n_k$. Suppose k is algebraically closed. Show that the singular *closed* points correspond to the locus $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. If the degree of the hypersurface is not divisible by the characteristic of any of the residue fields (e.g. if we are working over a field of characteristic 0), show that it suffices to check $\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. (Hint: show that f lies in the ideal $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$). (Fact: this will give the singular points in general, not just the closed points. I don't want to prove this, and I won't use it.)

3.I. EXERCISE. Suppose that k is algebraically closed. Show that $y^2 z = x^3 - xz^2$ in $\mathbb{P}^2_k$ is an irreducible nonsingular curve. (This is for practice.) Warning: I didn't say $\operatorname{char} k = 0$, so be careful when using the Euler test.

3.J. EXERCISE. Find all the singular closed points of the following plane curves. Here we work over an algebraically closed field.

(a) $y^2 = x^2 + x^3$. This is called a *node*.
(b) $y^2 = x^3$. This is called a *cusp*.
(c) $y^2 = x^4$. This is called a *tacnode*.

(I haven't given a precise definition of a node, etc.)

3.K. EXERCISE. Show that the twisted cubic $\operatorname{Proj} k[w, x, y, z]/(wz - xy, wy - x^2, xz - y^2)$ is nonsingular. (You can do this by using the fact that it is isomorphic to $\mathbb{P}^1$. I'd prefer you to do this with the explicit equations, for the sake of practice.)

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