FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 25

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CONTENTS

We began by recalling the distinguished affine base.

Definition. The **distinguished affine base** of a scheme X is the data of the affine open sets and the distinguished inclusions.

0.1. *Theorem. —*

(a) *A sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. (Hence if you start with a sheaf, and take the sheaf on the distinguished affine base, and then take the induced sheaf, you get the sheaf you started with.)*
(b) *A morphism of sheaves on a distinguished affine base uniquely determines a morphism of sheaves.*
(c) *An $\mathcal{O}_X$-module "on the distinguished affine base" yields an $\mathcal{O}_X$-module.*

1. QUASICOHERENT SHEAVES

We now define the notion of *quasicoherent sheaf*. In the same way that a scheme is defined by "gluing together rings", a quasicoherent sheaf over that scheme is obtained by "gluing together modules over those rings". We will give two equivalent definitions; each definition is useful in different circumstances. The first just involves the distinguished topology.

The first definition is more directly "sheafy". Given an A-module M, we defined a sheaf $\tilde{M}$ on $\operatorname{Spec} A$ long ago — the sections over $D(f)$ were M_f.

Definition A. An $\mathcal{O}_X$-module $\mathcal{F}$ is a **quasicoherent sheaf** if for every affine open $\operatorname{Spec} A$,

$$\mathcal{F}|_{\operatorname{Spec} A} \cong \widetilde{\Gamma(\operatorname{Spec} A, \mathcal{F})}.$$

Date: Wednesday, January 23, 2008.

(The "wide tilde" is supposed to cover the entire right side $\Gamma(\operatorname{Spec} A, \mathcal{F})$.) This isomorphism is as sheaves of $\mathcal{O}_X$-modules.

Hence by this definition, the sheaves on $\operatorname{Spec} A$ correspond to A-modules. Given an A-module M, we get a sheaf $\tilde{M}$. Given a sheaf $\mathcal{F}$ on $\operatorname{Spec} A$, we get an A-module $\Gamma(X, \mathcal{F})$. These operations are inverse to each other. So in the same way as schemes are obtained by gluing together rings, quasicoherent sheaves are obtained by gluing together modules over those rings.

The second definition really focuses on the distinguished affine base, and is reminiscent of the Affine Covering Lemma.

Definition B. Suppose $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A \subset X$ is a distinguished open set. Let $\phi : \Gamma(\operatorname{Spec} A, \mathcal{F}) \to \Gamma(\operatorname{Spec} A_f, \mathcal{F})$ be the restriction map. The source of ϕ is an A-module, and the target is an A_f-module, so by the universal property of localization, ϕ naturally factors as:

$$\begin{array}{ccc} \Gamma(\operatorname{Spec} A, \mathcal{F}) & \xrightarrow{\quad\phi\quad} & \Gamma(\operatorname{Spec} A_f, \mathcal{F}) \\ & \searrow \qquad\qquad \nearrow_{\alpha} & \\ & \Gamma(\operatorname{Spec} A, \mathcal{F})_f & \end{array}$$

An $\mathcal{O}_X$-module $\mathcal{F}$ is a **quasicoherent sheaf** if for each such distinguished $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, α is an isomorphism.

Thus a quasicoherent sheaf is the data of one module for each affine open subset (a module over the corresponding ring), such that the module over a distinguished open set $\operatorname{Spec} A_f$ is given by localizing the module over $\operatorname{Spec} A$. This will be an easy criterion to check.

1.1. *Proposition. — Definitions A and B are the same.*

Proof. Clearly Definition A implies Definition B. (Recall that the definition of $\tilde{M}$ was in terms of the distinguished topology on $\operatorname{Spec} A$.) We now show that Definition B implies Definition A. By Definition B, the sections over any distinguished open $\operatorname{Spec} A_f$ of $\mathcal{M}$ on $\operatorname{Spec} A$ is precisely $\Gamma(\operatorname{Spec} A, \mathcal{M})_f$, i.e. the sections of $\Gamma(\widetilde{\operatorname{Spec} A, \mathcal{M}})$ over $\operatorname{Spec} A_f$, and the restriction maps agree. Thus the two sheaves agree. □

We like Definition B because it says that to define a quasicoherent $\mathcal{O}_X$-module is that we just need to know what it is on all affine open sets, and that it behaves well under inverting a single element.

One reason we like Definition A is that it works well in gluing arguments, as in the proof of the following fact.

1.2. *Proposition (quasicoherence is an affine-local notion). — Let* X *be a scheme, and* $\mathcal{F}$ *an* $\mathcal{O}_X$-*module. Then let* P *be the property of affine open sets that* $\mathcal{F}|_{\operatorname{Spec} A} \cong \Gamma(\widetilde{\operatorname{Spec} A, \mathcal{F}})$. *Then* P *is an affine-local property.*

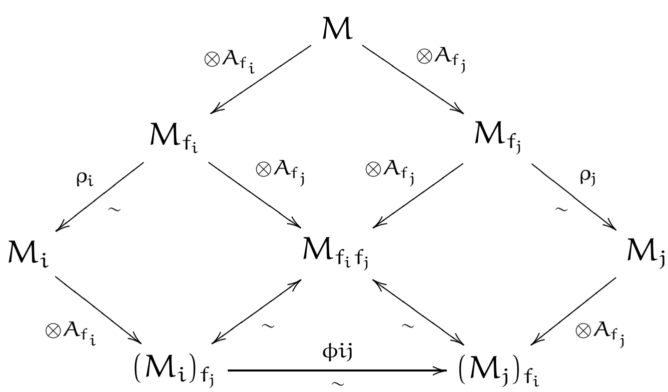
Before we prove this, we give an exercise to show its utility.

1.A. EXERCISE. Show that locally free sheaves are quasicoherent.

Proof. By the Affine Communication Lemma, we must check two things. Clearly if $\operatorname{Spec} A$ has property P, then so does the distinguished open $\operatorname{Spec} A_f$: if M is an A-module, then $\tilde{M}|_{\operatorname{Spec} A_f} \cong \tilde{M}_f$ as sheaves of $\mathcal{O}_{\operatorname{Spec} A_f}$-modules (both sides agree on the level of distinguished open sets and their restriction maps).

We next show the second hypothesis of the Affine Communication Lemma. Suppose we have modules $M_1, \ldots, M_n$, where M_i is an A_{f_i}-module, along with isomorphisms $\phi_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$ of $A_{f_i f_j}$-modules, satisfying the cocycle condition. We want to construct an M such that $\tilde{M}$ gives us $\tilde{M}_i$ on $D(f_i) = \operatorname{Spec} A_{f_i}$, or equivalently, isomorphisms $\rho_i : \Gamma(D(f_i), \tilde{M}) \to M_i$, so that the bottom triangle of

(1)



commutes.

We already know that M should be the sections of $\mathcal{F}$ over $\operatorname{Spec} A$, as $\mathcal{F}$ is a sheaf. Consider elements of $M_1 \times \cdots \times M_n$ that "agree on overlaps"; let this set be M. Then

$$0 \to M \to M_1 \times \cdots \times M_n \to M_{12} \times M_{13} \times \cdots \times M_{(n-1)n}$$

is an exact sequence (where $M_{ij} = (M_i)_{f_j} \cong (M_j)_{f_i}$, and the latter morphism is the "difference" morphism). So M is a kernel of a morphism of A-modules, hence an A-module. We are left to show that $M_i \cong M_{f_i}$ (and that this isomorphism satisfies (1)).

For convenience we assume $i = 1$. Localization is exact, so

(2)

$$0 \longrightarrow M_{f_1} \longrightarrow M_1 \times (M_2)_{f_1} \times \cdots \times (M_n)_{f_1} \longrightarrow M_{12} \times \cdots \times (M_{23})_{f_1} \times \cdots \times (M_{(n-1)n})_{f_1}$$

is an exact sequence of A_{f_1}-modules.

We now identify many of the modules appearing in (2) in terms of M_1. First of all, f_1 is invertible in A_{f_1}, so $(M_1)_{f_1}$ is canonically M_1. Also, $(M_j)_{f_1} \cong (M_1)_{f_j}$ via ϕ_{1j}. Hence if

$i, j \neq 1$, $(M_{ij})_{f_1} \cong (M_1)_{f_i f_j}$ via ϕ_{1i} and ϕ_{1j} (here the cocycle condition is implicitly used). Furthermore, $(M_{1i})_{f_1} \cong (M_1)_{f_i}$ via ϕ_{1i}. Thus we can write (2) as
(3)

$$0 \longrightarrow M_{f_1} \longrightarrow M_1 \times (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \xrightarrow{\alpha} (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \times (M_1)_{f_2 f_3} \times \cdots (M_1)_{f_{n-1} f_n}$$

By assumption, $\mathcal{F}|_{\operatorname{Spec} A_{f_1}}$ is quasicoherent, so by considering the cover of

$$\operatorname{Spec} A_{f_1} = \operatorname{Spec} A_{f_1} \cup \operatorname{Spec} A_{f_1 f_2} \cup \operatorname{Spec} A_{f_1 f_3} \cup \cdots \cup \operatorname{Spec} A_{f_1 f_n}$$

(which indeed has a "redundant" first term), and identifying sections of $\mathcal{F}$ over $\operatorname{Spec} A_{f_1}$ in terms of sections over the open sets in the cover and their pairwise overlaps, we have an exact sequence of A_{f_1}-modules

$$0 \longrightarrow M_1 \longrightarrow M_1 \times (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \xrightarrow{\beta} (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \times (M_1)_{f_2 f_3} \times \cdots (M_1)_{f_{n-1} f_n}$$

which is very similar to (3). Indeed, the final map β of the above sequence is the same as the map α of (3), so $\ker \alpha = \ker \beta$, i.e. we have an isomorphism $M_1 \cong M_{f_1}$.

Finally, the triangle of (1) is commutative, as each vertex of the triangle can be identified as the sections of $\mathcal{F}$ over $\operatorname{Spec} A_{f_1 f_2}$. $\square$

1.B. IMPORTANT EXERCISE. Suppose X is a quasicompact and quasiseparated scheme (i.e. covered by a finite number of affine open sets, the pairwise intersection of which is also covered by a finite number of affine open sets). Suppose $\mathcal{F}$ is a quasicoherent sheaf on X, and let $f \in \Gamma(X, \mathcal{O}_X)$ be a function on X. Show that the restriction map $\operatorname{res}_{X_f \subset X} : \Gamma(X, \mathcal{F}) \to \Gamma(X_f, \mathcal{F})$ (here X_f is the open subset of X where f doesn't vanish) is precisely localization. In other words show that there is an isomorphism $\Gamma(X, \mathcal{F})_f \to \Gamma(X_f, \mathcal{F})$ making the following diagram commute.

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \xrightarrow{\operatorname{res}_{X_f \subset X}} & \Gamma(X_f, \mathcal{F}) \\ & {\scriptstyle \otimes_A A_f} \searrow \qquad \nearrow {\scriptstyle \sim} & \\ & \Gamma(X, \mathcal{F})_f & \end{array}$$

All that you should need in your argument is that X admits a cover by a finite number of open sets, and that their pairwise intersections are each quasicompact. (Hint: cover by affine open sets. Use the sheaf property. A nice way to formalize this is the following. Apply the exact functor $\otimes_A A_f$ to the exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \oplus_i \Gamma(U_i, \mathcal{F}) \to \oplus \Gamma(U_{ijk}, \mathcal{F})$$

where the U_i form a finite cover of X and U_{ijk} form an affine cover of $U_i \cap U_j$.)

1.C. LESS IMPORTANT EXERCISE. Give a counterexample to show that the above statement need not hold if X is not quasicompact. (Possible hint: take an infinite disjoint union of affine schemes. The key idea is that infinite direct sums do not commute with localization.)

1.D. IMPORTANT EXERCISE (COROLLARY TO EXERCISE 1.B). Suppose $\pi : X \to Y$ is a quasicompact quasiseparated morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X. Show that $\pi_*\mathcal{F}$ is a quasicoherent sheaf on Y.

1.E. UNIMPORTANT EXERCISE (NOT EVERY $\mathcal{O}_X$-MODULE IS A QUASICOHERENT SHEAF).
(a) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the origin $[(t)]$, with group $k(t)$ and the usual $k[t]$-module structure. Show that this is an $\mathcal{O}_X$-module that s not a quasicoherent sheaf. (More generally, if X is an integral scheme, and $p \in X$ that is not the generic point, we could take the skyscraper sheaf at p with group the function field of X. Except in a silly circumstances, this sheaf won't be quasicoherent.)
(b) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the generic point $[(0)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this *is* a quasicoherent sheaf. Describe the restriction maps in the distinguished topology of X.

2. QUASICOHERENT SHEAVES FORM AN ABELIAN CATEGORY

The category of A-modules is an abelian category. Indeed, this is our motivating example of our notion of abelian category. Similarly, quasicoherent sheaves form an abelian category. I'll explain how.

When you show that something is an abelian category, you have to check many things, because the definition has many parts. However, if the objects you are considering lie in some ambient abelian category, then it is much easier. As a metaphor, there are several things you have to do to check that something is a group. But if you have a subset of group elements, it is much easier to check that it forms a subgroup.

You can look at back at the definition of an abelian category, and you'll see that in order to check that a subcategory is an abelian subcategory, you need to check only the following things:

(i) 0 is in your subcategory
(ii) your subcategory is closed under finite sums
(iii) your subcategory is closed under kernels and cokernels

In our case of

$$\{\text{quasicoherent sheaves}\} \subset \{\mathcal{O}_X\text{-modules}\},$$

the first two are cheap: 0 is certainly quasicoherent, and the subcategory is closed under finite sums: if $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, and over $\operatorname{Spec} A$, $\mathcal{F} \cong \tilde{M}$ and $\mathcal{G} \cong \tilde{N}$, then $\mathcal{F} \oplus \mathcal{G} = \widetilde{M \oplus N}$, so $\mathcal{F} \oplus \mathcal{G}$ is a quasicoherent sheaf.

We now check (iii). Suppose $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves. Then on any affine open set U, where the morphism is given by $\beta : M \to N$, define $(\ker \alpha)(U) = \ker \beta$ and $(\operatorname{coker} \alpha)(U) = \operatorname{coker} \beta$. Then these behave well under inversion of a single element: if

$$0 \to K \to M \to N \to P \to 0$$

is exact, then so is

$$0 \to K_f \to M_f \to N_f \to P_f \to 0,$$

from which $(\ker \beta)_f \cong \ker(\beta_f)$ and $(\operatorname{coker} \beta)_f \cong \operatorname{coker}(\beta_f)$. Thus both of these define quasicoherent sheaves. Moreover, by checking stalks, they are indeed the kernel and cokernel of α (exactness can be checked stalk-locally). Thus the quasicoherent sheaves indeed form an abelian category.

2.A. EXERCISE. Show that a sequence of quasicoherent sheaves $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ on X is exact if and only if it is exact on each open set in an affine cover of X. (In particular, taking sections over an affine open $\operatorname{Spec} A$ is an exact functor from the category of quasicoherent sheaves on X to the category of A-modules. Recall that taking sections is only left-exact in general.) In particular, we may check injectivity or surjectivity of a morphism of quasicoherent sheaves by checking on an affine cover.

Warning: If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of quasicoherent sheaves, then for any open set

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact, and we have exactness on the right is guaranteed to hold only if U is affine. (To set you up for cohomology: whenever you see left-exactness, you expect to eventually interpret this as a start of a long exact sequence. So we are expecting H^1's on the right, and now we expect that $H^1(\operatorname{Spec} A, \mathcal{F}) = 0$. This will indeed be the case.)

2.B. EXERCISE (CONNECTION TO ANOTHER DEFINITION). Show that an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is quasicoherent if and only if there exists an open cover by U_i such that on each U_i, $\mathcal{F}|_{U_i}$ is isomorphic to the cokernel of a map of two free sheaves:

$$\mathcal{O}_{U_i}^{\oplus I} \to \mathcal{O}_{U_i}^{\oplus J} \to \mathcal{F}|_{U_i} \to 0$$

is exact. We have thus connected our definitions to the definition given at the very start of the chapter.

We then began to discuss module-like constructions for quasicoherent sheaves, and I've left these for the next day's notes, so all of our discussion on that topic is in one place.

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