FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 35 AND 36

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CONTENTS

In these two lectures, we will define Cech cohomology and discuss its most important properties, although not in that order.

1. INTRODUCTION

As $\Gamma(X, \cdot)$ is a left-exact functor, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves on X, then

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X)$$

is exact. We dream that this sequence continues off to the right, giving a long exact sequence. More explicitly, there should be some covariant functors H^i ($i \geq 0$) from quasicoherent sheaves on X to groups such that $H^0 = \Gamma$, and so that there is a "long exact sequence in cohomology".

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H}) \tag{1}$$

$$\longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H}) \longrightarrow \cdots$$

(In general, whenever we see a left-exact or right-exact functor, we should hope for this, and in good cases our dreams will come true. The machinery behind this is sometimes called *derived functor cohomology*, which we will discuss shortly.)

Before defining cohomology groups of quasicoherent sheaves explicitly, we first describe their important properties. Indeed these fundamental properties are in some ways more important than the formal definition. The boxed properties will be the important ones.

Date: Friday, February 22 and Monday, February 25, 2008.

Suppose X is a separated and quasicompact A-scheme. (The separated and quasicompact hypotheses will be necessary in our construction.) For each quasicoherent sheaf $\mathcal{F}$ on X, we will define A-modules $H^i(X, \mathcal{F})$. In particular, if $A = k$, they are k-vector spaces.

(i) $\boxed{H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})}$.

(ii) Each H^i is a $\boxed{\text{covariant functor in the sheaf } \mathcal{F}}$ extending the usual covariance for $H^0(X, \cdot)$: $\mathcal{F} \to \mathcal{G}$ induces $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G})$.

(iii) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of quasicoherent sheaves on X, then we have a $\boxed{\text{long exact sequence}}$ (1). The maps $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$ come from covariance, and similarly for $H^i(X, \mathcal{G}) \to H^i(X, \mathcal{G})$. The *connecting homomorphisms* $H^i(x, \mathcal{H}) \to H^{i+1}(X, \mathcal{F})$ will have to be defined.

(iv) If $f : X \to Y$ is any morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X, then there is a natural morphism $\boxed{H^i(Y, f_*\mathcal{F}) \to H^i(X, \mathcal{F})}$ extending $\Gamma(Y, f_*\mathcal{F}) \to \Gamma(X, \mathcal{F})$. We will later see this as part of a larger story, the *Leray spectral sequence*. If $\mathcal{G}$ is a quasicoherent sheaf on Y, then setting $\mathcal{F} := f^*\mathcal{G}$ and using the adjunction map $\mathcal{G} \to f_* f^* \mathcal{G}$ and covariance of (ii) gives a natural **pullback map** $H^i(Y, \mathcal{G}) \to H^i(X, f^*\mathcal{G})$ (via $H^i(Y, \mathcal{G}) \to H^i(Y, f_* f^* \mathcal{G}) \to H^i(X, f^*\mathcal{G})$) extending $\Gamma(Y, \mathcal{G}) \to \Gamma(X, f^*\mathcal{G})$. In this way, H^i is a "contravariant functor in the space".

(v) If $f : X \hookrightarrow Y$ is an affine morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X, the natural map of (iv) is an isomorphism: $\boxed{H^i(Y, f_*\mathcal{F}) \xrightarrow{\sim} H^i(X, \mathcal{F})}$. When f is a closed immersion and $Y = \mathbb{P}^N_A$, this isomorphism will translate calculations on arbitrary projective A-schemes to calculations on $\mathbb{P}^n_A$.

(vi) If X can be covered by n affines, then $\boxed{H^i(X, \mathcal{F}) = 0}$ for $i \geq n$ for all $\mathcal{F}$. In particular, all higher ($i > 0$) quasicoherent cohomology groups on affine schemes vanish. The vanishing of H^1 in this case, along with the long exact sequence **(iii)** implies that Γ is an exact functor for quasicoherent sheaves on affine schemes, something we already knew. It is also true that if $\dim X = n$, then $H^i(X, \mathcal{F}) = 0$ for all $i > n$ and for all $\mathcal{F}$ (**dimensional vanishing**). We will prove this for quasiprojective A-schemes, but we won't use this fact in general, and hence won't prove it. (A proof is given in Hartshorne (Thm. III.2.7) for derived functors, and we show in a week or two that this agrees with Cech cohomology.)

(vii) The functor H^i behaves well under direct sums, and more generally under colimits: $\boxed{H^i(X, \varinjlim \mathcal{F}_j) = \varinjlim H^i(X, \mathcal{F}_j)}$.

(viii) We will also identify the cohomology of all $\mathcal{O}(m)$ on $\mathbb{P}^n_A$:

1.1. *Theorem.* —

- $H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ *is a free* A*-module of rank* $\binom{n+m}{n}$ *if* $i = 0$ *and* $m \geq 0$*, and* 0 *otherwise.*

- $H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ *is a free* A*-module of rank* $\binom{-m-1}{-n-m-1}$ *if* $m \leq -n-1$, *and* 0 *otherwise.*
- $H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = 0$ *if* $0 < i < n$.

We already have shown the first statement in an Essential Exercise (class 27 end of section 3).

Theorem 1.1 has a number of features that will be the first appearances of things that we'll prove later.

- The cohomology of these bundles vanish above n (**(vi)** above)
- These cohomology groups are always *finitely-generated* A-modules. This will be true for all coherent sheaves on projective A-schemes (Theorem 1.2(i)).
- The top cohomology group vanishes for $m > -n-1$. (This is a first appearance of *Kodaira vanishing*.)
- The top cohomology group is one-dimensional for $m = -n-1$ if $A = k$. This is the first appearance of the *dualizing sheaf*.
- There is a natural duality

  $$H^i(X, \mathcal{O}(m)) \times H^{n-i}(X, \mathcal{O}(-n-1-m)) \to H^n(X, \mathcal{O}(-n-1)).$$

  This is the first appearance of *Serre duality*.

Before proving these facts, let's first use them to prove interesting things, as motivation.

By an earlier Theorem from last quarter (class 30 Corollary 3.3), for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_A$ we can find a surjection $\mathcal{O}(m)^{\oplus j} \to \mathcal{F}$, which yields the exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}(m)^{\oplus j} \to \mathcal{F} \to 0 \tag{2}$$

for some coherent sheaf $\mathcal{G}$. We can use this to prove the following.

1.2. *Theorem. — (i) For any coherent sheaf* $\mathcal{F}$ *on a projective* A*-scheme where* A *is Noetherian,* $h^i(X, \mathcal{F})$ *is a coherent (finitely generated)* A*-module.*
(ii) (Serre vanishing) Furthermore, for $m \gg 0$, $H^i(X, \mathcal{F}(m)) = 0$ *for all* i, *even without Noetherian hypotheses.*

A non-Noetherian generalization of the coherence statement is given in Exercise 1.A.

Proof. Because cohomology of a closed scheme can be computed on the ambient space (see **(v)** above), we may immediately reduce to the case $X = \mathbb{P}^n_A$.

(i) Consider the long exact sequence:

$$
\begin{array}{l}
0 \longrightarrow H^0(\mathbb{P}^n_A, \mathcal{G}) \longrightarrow H^0(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^0(\mathbb{P}^n_A, \mathcal{F}) \longrightarrow \\
H^1(\mathbb{P}^n_A, \mathcal{G}) \longrightarrow H^1(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^1(\mathbb{P}^n_A, \mathcal{F}) \longrightarrow \cdots \\
\cdots \longrightarrow H^{n-1}(\mathbb{P}^n_A, \mathcal{G}) \longrightarrow H^{n-1}(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^{n-1}(\mathbb{P}^n_A, \mathcal{F}) \longrightarrow \\
H^n(\mathbb{P}^n_A, \mathcal{G}) \longrightarrow H^n(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^n(\mathbb{P}^n_A, \mathcal{F}) \longrightarrow 0
\end{array}
$$

The exact sequence ends here because $\mathbb{P}^n_A$ is covered by $n+1$ affines (**(vi)** above). Then $H^n(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j})$ is finitely generated by Theorem 1.1, hence $H^n(\mathbb{P}^n_A, \mathcal{F})$ is finitely generated for all coherent sheaves $\mathcal{F}$. Hence in particular, $H^n(\mathbb{P}^n_A, \mathcal{G})$ is finitely generated. As $H^{n-1}(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j})$ is finitely generated, and $H^n(\mathbb{P}^n_A, \mathcal{G})$ is too, we have that $H^{n-1}(\mathbb{P}^n_A, \mathcal{F})$ is finitely generated for all coherent sheaves $\mathcal{F}$. We continue inductively downwards.

(ii) Twist (2) by $\mathcal{O}(N)$ for $N \gg 0$. Then

$$H^n(\mathbb{P}^n_A, \mathcal{O}(m+N)^{\oplus j}) = \oplus_j H^n(\mathbb{P}^n_A, \mathcal{O}(m+N)) = 0$$

(by **(vii)** above), so $H^n(\mathbb{P}^n_A, \mathcal{F}(N)) = 0$. Translation: for any coherent sheaf, its top cohomology vanishes once you twist by $\mathcal{O}(N)$ for N sufficiently large. Hence this is true for $\mathcal{G}$ as well. Hence from the long exact sequence, $H^{n-1}(\mathbb{P}^n_A, \mathcal{F}(N)) = 0$ for $N \gg 0$. As in (i), we induct downwards, until we get that $H^1(\mathbb{P}^n_A, \mathcal{F}(N)) = 0$. (The induction proceeds no further, as it is *not* true that $H^0(\mathbb{P}^n_A, \mathcal{O}(m+N)^{\oplus j}) = 0$ for large N — quite the opposite.) □

1.A. EXERCISE ONLY FOR THOSE WHO LIKE WORKING WITH NON-NOETHERIAN RINGS. Prove part (i) in the above result without the Noetherian hypotheses, assuming only that A is a coherent A-module (A is "coherent over itself"). (Hint: induct downwards as before. Show the following in order: $H^n(\mathbb{P}^n_A, \mathcal{F})$ finitely generated, $H^n(\mathbb{P}^n_A, \mathcal{G})$ finitely generated, $H^n(\mathbb{P}^n_A, \mathcal{F})$ coherent, $H^n(\mathbb{P}^n_A, \mathcal{G})$ coherent, $H^{n-1}(\mathbb{P}^n_A, \mathcal{F})$ finitely generated, $H^{n-1}(\mathbb{P}^n_A, \mathcal{G})$ finitely generated, etc.)

In particular, we have proved the following, that we would have cared about even before we knew about cohomology.

1.3. *Corollary. — Any projective* k*-scheme has a finite-dimensional space of global sections. More generally, if* $\mathcal{F}$ *is a coherent sheaf on a projective* A*-scheme, then* $H^0(X, \mathcal{F})$ *is a finitely generated* A*-module.*

This is true more generally for proper k-schemes, not just projective k-schemes, but this requires more work.

Here is three important consequences. They can also be shown directly, without the use of cohomology, but with much more elbow grease.

1.B. EXERCISE. Suppose X is a projective integral scheme over an algebraically closed field. Show that $h^0(X, \mathcal{O}_X) = 1$. Hint: show that $H^0(X, \mathcal{O}_X)$ is a finite-dimensional k-algebra, and a domain. Hence show it is a field. (For experts: the same argument holds with the weaker hypotheses where X is proper, geometrically connected, and reduced over an arbitrary field.)

1.C. CRUCIAL EXERCISE (PUSHFORWARDS OF COHERENTS ARE COHERENT). Suppose $f : X \to Y$ is a projective morphism, and $\mathcal{O}_Y$ is coherent over itself (true in all reasonable circumstances). Show that the pushforward of a coherent sheaf on X is a coherent sheaf on Y.

Finite morphisms are affine (from the definition) and projective (shown earlier, class 33/34 Exercise 3.A). We can now show that this is a characterization of finiteness.

1.4. *Corollary. — If $\pi : X \to Y$ is projective and affine and $\mathcal{O}_Y$ is coherent, then π is finite.*

In fact, more generally, if π is universally closed and affine, then π is finite, by Atiyah-Macdonald Exercise 5.35 (thanks Joe!). We won't use this, so I won't explain why.

Proof. By Exercise 1.C, $\pi_* \mathcal{O}_X$ is coherent and hence finitely generated. □

1.D. EXERCISE. Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on projective X with $\mathcal{F}$ coherent. Show that for $n \gg 0$,

$$0 \to H^0(X, \mathcal{F}(n)) \to H^0(X, \mathcal{G}(n)) \to H^0(X, \mathcal{H}(n)) \to 0$$

is also exact. (Hint: for $n \gg 0$, $H^1(X, \mathcal{F}(n)) = 0$.)

2. DEFINITIONS AND PROOFS OF KEY PROPERTIES

This section could be read much later; the facts we will use are all stated in the previous section. However, the arguments aren't that complicated, so you may feel like reading this right away. As you read this, you should go back and check off all the facts, to assure yourself that I've shown all that I've promised.

2.1. Cech cohomology. Cech cohomology in general settings is often defined using a limit over finer and finer covers of a space. In our algebro-geometric setting, the situation is much cleaner, and we can use a single cover.

Suppose X is quasicompact and separated, e.g. X is quasiprojective over A. In particular, X may be covered by a finite number of affine open sets, and the intersection of any two affine open sets is also an affine open set (by separatedness, Class 17 Proposition

1.2). We'll use quasicompactness and separatedness only in order to ensure these two nice properties.

Suppose $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{U} = \{U_i\}_{i=1}^n$ is a *finite* set of affine open sets of X covering U. For $I \subset \{1, \dots, n\}$ define $U_I = \cap_{i \in I} U_i$, which is affine by the separated hypothesis. Consider the **Cech complex**

$$0 \to \bigoplus_{\substack{|I|=1 \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \cdots \to \bigoplus_{\substack{|I|=i \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \bigoplus_{\substack{|I|=i+1 \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \cdots. \tag{3}$$

The maps are defined as follows, in terms of the summands. The map from $\mathcal{F}(U_I) \to \mathcal{F}(U_J)$ is 0 unless $I \subset J$, i.e. $J = I \cup \{j\}$. If j is the kth element of J, then the map is $(-1)^{k-1}$ times the restriction map $\operatorname{res}_{U_I, U_J}$.

2.A. EASY EXERCISE (FOR THOSE WHO HAVEN'T SEEN ANYTHING LIKE THE CECH COMPLEX BEFORE). Show that the Cech complex is indeed a complex, i.e. that the composition of two consecutive arrows is 0.

Define $H^i_{\mathcal{U}}(U, \mathcal{F})$ to be the ith cohomology group of the complex (3). Note that if X is an A-scheme, then $H^i_{\mathcal{U}}(X, \mathcal{F})$ is an A-module. We have almost succeeded in defining the Cech cohomology group H^i, except our definition seems to depend on a choice of a cover $\mathcal{U}$.

2.B. EASY EXERCISE. Show that $H^0_{\mathcal{U}}(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$. (Hint: use the sheaf axioms for $\mathcal{F}$.)

2.C. EXERCISE. Suppose $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves on a topological space, and $\mathcal{U}$ is an open cover such that on any intersection of open subsets in $\mathcal{U}$, the sections of $\mathcal{F}_2$ surject onto $\mathcal{F}_3$. Show that we get a long exact sequence of cohomology. (Note that this applies in our case!)

2.2. *Theorem/Definition. — Recall that X is quasicompact and separated. $H^i_{\mathcal{U}}(U, \mathcal{F})$ is independent of the choice of (finite) cover $\{U_i\}$. More precisely,*

(*) *for all k, for any two covers $\{U_i\} \subset \{V_i\}$ of size at most k, the maps $H^i_{\{V_i\}}(X, \mathcal{F}) \to H^i_{\{U_i\}}(X, \mathcal{F})$ induced by the natural maps of Cech complexes* (3) *are isomorphisms.*

Define the Cech cohomology group $H^i(X, \mathcal{F})$ to be this group.

The dependence of k in the statement is there because we will prove it by induction on k.

(For experts: maps of complexes inducing isomorphisms are called *quasiisomorphisms*. We are actually getting a finer invariant than cohomology out of this construction; we are getting an element of the *derived category of A-modules*.)

Proof. We prove this by induction on k. The base case $k = 1$ is trivial. We need only prove the result for $\{U_i\}_{i=1}^n \subset \{U_i\}_{i=0}^n$, where the case $k = n$ is assumed known. Consider the exact sequence of complexes

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & & \bigoplus_{\substack{|I|=i-1 \\ 0\in I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i \\ 0\in I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1 \\ 0\in I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & & \bigoplus_{\substack{|I|=i-1 \\ I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i \\ I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1 \\ I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & & \bigoplus_{\substack{|I|=i-1 \\ I\subset\{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i \\ I\subset\{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1 \\ I\subset\{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0 &
\end{array}$$

The bottom two rows are Cech complexes with respect to two covers. We get a long exact sequence of cohomology from this short exact sequence of complexes. Thus we wish to show that the top row is exact. But the ith cohomology of the top row is precisely $H^i_{\{U_i \cap U_0\}_{i>0}}(U_i, \mathcal{F})$ except at step 0, where we get 0 (because the complex starts off $0 \to \mathcal{F}(U_0) \to \oplus_{j=1}^n \mathcal{F}(U_0 \cap U_j)$). So it suffices to show that higher Cech groups of affine schemes are 0. Hence we are done by the following result. $\square$

2.3. *Theorem. — The higher Cech cohomology* $H^i_{\mathcal{U}}(X, \mathcal{F})$ *of an affine* A*-scheme* X *vanishes (for any affine cover* $\mathcal{U}$*,* $i > 0$*, and quasicoherent* $\mathcal{F}$*).*

Serre describes this as a partition of unity argument.

Proof. We want to show that the "extended" complex

$$0 \to \mathcal{F}(X) \to \oplus_{|I|=1} \mathcal{F}(U_I) \to \oplus_{|I|=2} \mathcal{F}(U_I) \to \cdots \tag{4}$$

(where the global sections are appended to the front) has no cohomology, i.e. is exact. We do this with a trick.

Suppose first that some U_i, say U_0, is X. Then the complex is the middle row of the following short exact sequence of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \oplus_{|I|=1, 0\in I}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2, 0\in I}\mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \oplus_{|I|=1}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2}\mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \oplus_{|I|=1, 0\notin I}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2, 0\notin I}\mathcal{F}(U_I) & \longrightarrow & \cdots
\end{array}$$

The top row is the same as the bottom row, slid over by 1. The corresponding long exact sequence of cohomology shows that the central row has vanishing cohomology. (Topological experts will recognize this as a *mapping cone* construction.)

We next prove the general case by sleight of hand. Say $X = \operatorname{Spec} R$. We wish to show that the complex of A-modules (4) is exact. It is also a complex of R-modules, so we wish to show that the complex of R-modules (4) is exact. To show that it is exact, it suffices to show that for a cover of $\operatorname{Spec} R$ by distinguished open sets $D(f_i)$ $(1 \leq i \leq r)$ (i.e. $(f_1, \dots, f_r) = 1$ in R) the complex is exact. (Translation: exactness of a sequence of sheaves may be checked locally.) We choose a cover so that each $D(f_i)$ is contained in some $U_j = \operatorname{Spec} A_j$. Consider the complex localized at f_i. As

$$\Gamma(\operatorname{Spec} A, \mathcal{F})_f = \Gamma(\operatorname{Spec}(A_j)_f, \mathcal{F})$$

(as this is one of the definitions of a quasicoherent sheaf), as $U_j \cap D(f_i) = D(f_i)$, we are in the situation where one of the U_i's is X, so we are done. □

We have now proved properties **(i)–(iii)** of the previous section.

2.D. EXERCISE (PROPERTY **(v)**). Suppose $f : X \to Y$ is an affine morphism, and Y is a quasicompact and separated A-scheme (and hence X is too, as affine morphisms are both quasicompact and separated). If $\mathcal{F}$ is a quasicoherent sheaf on X, describe a natural isomorphism $H^i(Y, f_*\mathcal{F}) \cong H^i(X, \mathcal{F})$. (Hint: if $\mathcal{U}$ is an affine cover of Y, "$f^{-1}(\mathcal{U})$" is an affine cover X. Use these covers to compute the cohomology of $\mathcal{F}$.)

2.E. EXERCISE (PROPERTY **(iv)**). Suppose $f : X \to Y$ is any quasicompact separated morphism, $\mathcal{F}$ is a quasicoherent sheaf on X, and Y is a quasicompact quasiseparated A-scheme. The hypotheses on f ensure that $f_*\mathcal{F}$ is a quasicoherent sheaf on Y. Describe a natural morphism $H^i(Y, f_*\mathcal{F}) \to H^i(X, \mathcal{F})$ extending $\Gamma(Y, f_*\mathcal{F}) \to \Gamma(X, \mathcal{F})$.

2.F. UNIMPORTANT EXERCISE. Prove Property **(vii)** of the previous section.

2.4. Useful facts about cohomology for k-schemes.

2.G. EXERCISE (COHOMOLOGY AND CHANGE OF BASE FIELD). Suppose X is a projective k-scheme, and $\mathcal{F}$ is a coherent sheaf on X. Show that

$$h^0(X, \mathcal{F}) = h^0(X \times_{\operatorname{Spec} k} \operatorname{Spec} K, \mathcal{F} \otimes_k K)$$

where K/k is any field extension. Here $\mathcal{F} \otimes_k k$ means the pullback of $\mathcal{F}$ to $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. Note: the two sides of this equality are dimensions of vector spaces over different fields! (This is useful for relating facts about k-schemes to facts about schemes over algebraically closed fields.)

2.5. *Theorem. — Suppose* X *is a projective* k*-scheme, and* $\mathcal{F}$ *is a quasicoherent sheaf on* X*. Then* $H^i(X, \mathcal{F}) = 0$ *for* $i > \dim X$*.*

In other words, cohomology vanishes above the dimension of X. We will later show that this is true when X is a *quasiprojective* k-scheme.

Proof. Suppose $X \hookrightarrow \mathbb{P}^N$, and let $n = \dim X$. We show that X may be covered by n affine open sets. A key Exercise from a couple of months ago shows that there are n effective Cartier divisors on $\mathbb{P}^N$ such that their complements $U_0, \dots, U_n$ cover X. Then U_i is affine, so $U_i \cap X$ is affine, and thus we have covered X with n affine open sets. $\square$

Remark. We actually *need* n affine open sets to cover X, but I don't see an easy way to prove it. One way of proving it is by showing that the complement of an affine set is always pure codimension 1.

3. COHOMOLOGY OF LINE BUNDLES ON PROJECTIVE SPACE

We will finally prove the last promised basic fact about cohomology, property **(viii)** of §1, Theorem 1.1.

We saw earlier (Essential Exercise in class 27, end of section 3, and the ensuing discussion) that $H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ should be interpreted as the homogeneous degree m polynomials in $x_0, \dots, x_n$ (with A-coefficients). Similarly, $H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ should be interpreted as the homogeneous degree m Laurent polynomials in $x_0, \dots, x_n$, where in each monomial, each x_i appears with degree at most -1.

Proof of Theorem 1.1. As stated above, we showed the H^0 case earlier.

Rather than consider $\mathcal{O}(m)$ for various m, we consider them all at once, by considering $\mathcal{F} = \oplus_m \mathcal{O}(m)$.

We take the standard cover $U_0 = D(x_0), \dots, U_n = D(x_n)$ of $\mathbb{P}^n_A$. Notice that if $I \subset \{1, \dots, n\}$, then $\mathcal{F}(U_I)$ corresponds to the Laurent monomials where each x_i for $i \notin I$ appears with non-negative degree.

We first consider the H^n statement. $H^n(\mathbb{P}^n_A, \mathcal{F})$ is the cokernel of the surjection

$$\oplus_{i=0}^n \mathcal{F}(U_{\{1,\dots,n\}-\{i\}}) \to \mathcal{F}_{U_{\{1,\dots,n\}}}$$

i.e.

$$\oplus_{i=0}^{n} A[x_0, \ldots, x_n, x_0^{-1}, \ldots, \widehat{x_i^{-1}}, \ldots x_n^{-1}] \to A[x_0, \ldots, x_n, x_0^{-1}, \ldots, x_n^{-1}].$$

This cokernel is precisely as described.

We last consider the H^i statement ($0 < i < n$). (Strangely, the vanishing of these H^i is the hardest part of the Theorem.) We prove this by induction on n. The cases $n = 0$ and 1 are trivial. Consider the exact sequence of quasicoherent sheaves:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\times x_n} \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0$$

where $\mathcal{F}'$ is analogous sheaf on the hyperplane $x_n = 0$ (isomorphic to $\mathbb{P}_A^{n-1}$). (This exact sequence is just the direct sum over all m of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_A^n}(m-1) \xrightarrow{\times x_n} \mathcal{O}_{\mathbb{P}_A^n}(m) \longrightarrow \mathcal{O}_{\mathbb{P}_A^{n-1}}(m) \longrightarrow 0\ ,$$

which in turn is obtained by twisting the closed subscheme exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_A^n}(-1) \xrightarrow{\times x_n} \mathcal{O}_{\mathbb{P}_A^n} \longrightarrow \mathcal{O}_{\mathbb{P}_A^{n-1}} \longrightarrow 0$$

by $\mathcal{O}_{\mathbb{P}_A^n}(m)$.)

The long exact sequence in cohomology yields

$$0 \longrightarrow H^0(\mathbb{P}_A^n, \mathcal{F}) \xrightarrow{\times x_n} H^0(\mathbb{P}_A^n, \mathcal{F}) \longrightarrow H^0(\mathbb{P}_A^{n-1}, \mathcal{F}') \quad .$$

$$\longrightarrow H^1(\mathbb{P}_A^n, \mathcal{F}) \xrightarrow{\times x_n} H^1(\mathbb{P}_A^n, \mathcal{F}) \longrightarrow H^1(\mathbb{P}_A^{n-1}, \mathcal{F}')$$

$$\cdots \longrightarrow H^{n-1}(\mathbb{P}_A^n, \mathcal{F}) \xrightarrow{\times x_n} H^{n-1}(\mathbb{P}_A^n, \mathcal{F}) \longrightarrow H^{n-1}(\mathbb{P}_A^{n-1}, \mathcal{F}')$$

$$\longrightarrow H^n(\mathbb{P}_A^n, \mathcal{F}) \xrightarrow{\times x_n} H^n(\mathbb{P}_A^n, \mathcal{F}) \longrightarrow 0$$

We will now show that this gives an isomorphism

$$\times x_n : H^i(\mathbb{P}_A^n, \mathcal{F}) \xrightarrow{\sim} H^i(\mathbb{P}_A^n, \mathcal{F}) \tag{5}$$

for $0 < i < n$. The inductive hypothesis gives us this except for $i = 1$ and $i = n-1$, where we have to be more careful. For the first, note that $H^0(\mathbb{P}_A^n, \mathcal{F}) \longrightarrow H^0(\mathbb{P}_A^{n-1}, \mathcal{F}')$ is surjective: this map corresponds to taking the set of all polynomials in $x_0, \ldots, x_n$, and setting $x_n = 0$. The last is slightly more subtle: $H^{n-1}(\mathbb{P}_A^{n-1}, \mathcal{F}') \to H^n(\mathbb{P}_A^n, \mathcal{F})$ is injective, and corresponds to taking a Laurent polynomial in $x_0, \ldots, x_{n-1}$ (where in each monomial, each x_i appears with degree at most -1) and multiplying by x_n^{-1}, which indeed describes the kernel of $H^n(\mathbb{P}_A^n, \mathcal{F}) \xrightarrow{\times x_n} H^n(\mathbb{P}_A^n, \mathcal{F})$. (This is a worthwhile calculation! See Exercise 3.A below.) We have thus established (5) above.

We will now show that the localization $H^i(\mathbb{P}_A^n, \mathcal{F})_{x_n} = 0$. (Here's what we mean by localization. Notice $H^i(\mathbb{P}_A^n, \mathcal{F})$ is naturally a module over $A[x_0, \ldots, x_n]$ — we know how to multiply by elements of A, and by (5) we know how to multiply by x_i. Then we localize this at x_n to get an $A[x_0, \dots, x_n]_{x_n}$-module.) This means that each element $\alpha \in H^i(\mathbb{P}^n_A, \mathcal{F})$ is killed by some power of x_i. But by (5), this means that $\alpha = 0$, concluding the proof of the theorem.

Consider the Cech complex computing $H^i(\mathbb{P}^n_A, \mathcal{F})$. Localize it at x_n. Localization and cohomology commute (basically because localization commutes with operations of taking quotients, images, etc.), so the cohomology of the new complex is $H^i(\mathbb{P}^n_A, \mathcal{F})_{x_n}$. But this complex computes the cohomology of $\mathcal{F}_{x_n}$ on the affine scheme U_n, and the higher cohomology of *any* quasicoherent sheaf on an affine scheme vanishes (by Theorem 2.3 which we've just proved — in fact we used the same trick there), so $H^i(\mathbb{P}^n_A, \mathcal{F})_{x_n} = 0$ as desired. □

3.A. EXERCISE. Verify that $H^{n-1}(\mathbb{P}^{n-1}_A, \mathcal{F}') \to H^n(\mathbb{P}^n_A, \mathcal{F})$ is injective (likely by verifying that it is the map on Laurent monomials we claimed above).

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