FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 38

RAVI VAKIL

CONTENTS

1. A USEFUL VERY GENERAL FACT FROM HOMOLOGICAL ALGEBRA

Here is a fact that is very useful, because it applies in so many situations.

1.A. IMPORTANT EXERCISE IN ABSTRACT NONSENSE. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a covariant additive functor from one abelian category to another. Suppose $C^\bullet$ is a complex in $\mathcal{A}$.

(a) Describe a natural morphism $FH^\bullet \to H^\bullet F$.
(b) If F is exact, show that the morphism of (a) is an isomorphism.

If this makes your head spin, you may prefer to think of it in the following specific case, where both $\mathcal{A}$ and $\mathcal{B}$ are the category of A-modules, and F is $\cdot \otimes N$ for some fixed N-module. Your argument in this case will translate without change to yield a solution to Exercise 1.A. If $\otimes N$ is exact, then N is called a **flat** A-module.

For example, localization is exact, so $S^{-1}A$ is a *flat* A-algebra for all multiplicative sets S. Thus taking cohomology of a complex of A-modules commutes with localization — something you could verify directly.

2. HIGHER DIRECT IMAGE SHEAVES

Cohomology groups were defined for $X \to \operatorname{Spec} A$ where the structure morphism is quasicompact and separated; for any quasicoherent $\mathcal{F}$ on X, we defined $H^i(X, \mathcal{F})$. We'll now define a "relative" version of this notion, for quasicompact and separated morphisms $\pi : X \to Y$: for any quasicoherent $\mathcal{F}$ on X, we'll define $R^i\pi_*\mathcal{F}$, a quasicoherent sheaf on Y.

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We have many motivations for doing this. In no particular order:

(1) It "globalizes" what we did before.
(2) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of quasicoherent sheaves on X, then we know that $0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{G} \to \pi_*\mathcal{H}$ is exact, and higher pushforwards will extend this to a long exact sequence.
(3) We'll later see that this will show how cohomology groups vary in families, especially in "nice" situations. Intuitively, if we have a nice family of varieties, and a family of sheaves on them, we could hope that the cohomology varies nicely in families, and in fact in "nice" situations, this is true. (As always, "nice" usually means "flat", whatever that means.)

All of the important properties of cohomology described earlier will carry over to this more general situation. Best of all, there will be no extra work required.

In the notation $R^j f_* \mathcal{F}$ for higher pushforward sheaves, the "R" stands for "right derived functor", and corresponds to the fact that we get a long exact sequence in cohomology extending to the right (from the 0th terms). Later this year, we will see that in good circumstances, if we have a left-exact functor, there is be a long exact sequence going off to the right, in terms of right derived functors. Similarly, if we have a right-exact functor (e.g. if M is an A-module, then $\otimes_A M$ is a right-exact functor from the category of A-modules to itself), there may be a long exact sequence going off to the left, in terms of left derived functors.

Suppose $\pi : X \to Y$, and $\mathcal{F}$ is a quasicoherent sheaf on X. For each $\operatorname{Spec} A \subset Y$, we have A-modules $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})$. We will show that these patch together to form a quasicoherent sheaf. We need check only one fact: that this behaves well with respect to taking distinguished open sets. In other words, we must check that for each $f \in A$, the natural map $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F}) \to H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})_f$ (induced by the map of spaces in the opposite direction — H^i is contravariant in the space) is precisely the localization $\otimes_A A_f$. But this can be verified easily: let $\{U_i\}$ be an affine cover of $\pi^{-1}(\operatorname{Spec} A)$. We can compute $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})$ using the Cech complex. But this induces a cover $\operatorname{Spec} A_f$ in a natural way: If $U_i = \operatorname{Spec} A_i$ is an affine open for $\operatorname{Spec} A$, we define $U_i' = \operatorname{Spec}(A_i)_f$. The resulting Cech complex for $\operatorname{Spec} A_f$ is the localization of the Cech complex for $\operatorname{Spec} A$. As taking cohomology of a complex commutes with localization (as discussed in Exercise 1.A), we have defined a quasicoherent sheaf on Y by one of our definitions of quasicoherent sheaves by Definition 2′ of a quasicoherent sheaf.

Define the **ith higher direct image sheaf** or the **ith (higher) pushforward sheaf** to be this quasicoherent sheaf.

2.1. *Theorem. —*

(a) $R^0\pi_*\mathcal{F}$ *is canonically isomorphic to* $\pi_*\mathcal{F}$.
(b) $R^i\pi_*$ *is a covariant functor from the category of quasicoherent sheaves on* X *to the category of quasicoherent sheaves on* Y*, and a contravariant functor in* Y*-schemes* X.

(c) *(the* **long exact sequence of higher pushforward sheaves***)* *A short exact sequence* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *of sheaves on* X *induces a long exact sequence*

$$0 \longrightarrow R^0\pi_*\mathcal{F} \longrightarrow R^0\pi_*\mathcal{G} \longrightarrow R^0\pi_*\mathcal{H} \longrightarrow$$

$$R^1\pi_*\mathcal{F} \longrightarrow R^1\pi_*\mathcal{G} \longrightarrow R^1\pi_*\mathcal{H} \longrightarrow \cdots$$

of sheaves on Y.

(d) *(projective pushforwards of coherent are coherent) If* π *is a projective morphism and* $\mathcal{O}_Y$ *is coherent on* Y *(this hypothesis is automatic for* Y *locally Noetherian), and* $\mathcal{F}$ *is a coherent sheaf on* X*, then for all* i, $R^i\pi_*\mathcal{F}$ *is a coherent sheaf on* Y.

Proof. Because it suffices to check each of these results on affine open sets, they all follow from the analogous statements in Cech cohomology. □

The following result is handy, and essentially immediate from our definition.

2.A. EXERCISE. Show that if π is affine, then for $i > 0$, $R^i\pi_*\mathcal{F} = 0$.

Remark. This is in fact a characterization of affineness. Serre's criterion for affineness states that if f is quasicompact and separated, then f is affine if and only if f_* is an exact functor from the category of quasicoherent sheaves on X to the category of quasicoherent sheaves on Y. exact on the category of quasicoherent sheaves (EGA II.5.2). We won't use this fact.

2.B. EXERCISE (HIGHER PUSHFORWARDS AND COMMUTATIVE DIAGRAMS). (a) Suppose $f : Z \to Y$ is any morphism, and $\pi : X \to Y$ as usual is quasicompact and separated. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let

$$\begin{array}{ccc} W & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{f} & Y \end{array}$$

is a fiber diagram. Describe a natural morphism $f^*(R^i\pi_*\mathcal{F}) \to R^i\pi'_*(f')^*\mathcal{F}$ of sheaves on Z. (Hint: Exercise 1.A.)

(b) If $f : Z \to Y$ is an affine morphism, and for a cover $\operatorname{Spec} A_i$ of Y, where $f^{-1}(\operatorname{Spec} A_i) = \operatorname{Spec} B_i$, B_i is a *flat* A-algebra, and the diagram in (a) is a fiber square, show that the natural morphism of (a) is an isomorphism. (You can likely generalize this immediately, but this will lead us into the concept of flat morphisms, and we'll hold off discussing this notion for a while.)

A useful special case of (a) if the following.

2.C. EXERCISE. Show that if $y \in Y$, there is a natural morphism $H^i(Y, f_*\mathcal{F})_y \to H^i(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$. (Hint: if you take a complex, and tensor it with a module, and take cohomology, there is a map from that to what you would get if you take cohomology and tensor it with a module.)

We'll later see that in good situations this is an isomorphism, and thus the higher direct image sheaf indeed "patches together" the cohomology on fibers.

2.D. EXERCISE (PROJECTION FORMULA). Suppose $\pi : X \to Y$ is quasicompact and separated, and $\mathcal{E}$, $\mathcal{F}$ are quasicoherent sheaves on X and Y respectively. (a) Describe a natural morphism

$$(R^i\pi_*\mathcal{E}) \otimes \mathcal{F} \to R^i\pi_*(\mathcal{E} \otimes \pi^*\mathcal{F}).$$

(Hint: Exercise 1.A.) (b) If $\mathcal{F}$ is locally free, show that this natural morphism is an isomorphism.

3. FUN APPLICATIONS OF THE HIGHER PUSHFORWARD

Here are a series of useful geometric facts shown using similar tricks.

3.1. *Theorem (relative dimensional vanishing). — If $f : X \to Y$ is a projective morphism and $\mathcal{O}_Y$ is coherent, then the higher pushforwards vanish in degree higher than the maximum dimension of the fibers.*

This is false without the projective hypothesis, as shown by the following exercise.

3.A. EXERCISE. Consider the open immersion $\pi : \mathbb{A}^n - 0 \to \mathbb{A}^n$. By direct calculation, show that $R^{n-1}f_*\mathcal{O}_{\mathbb{A}^n-0} \neq 0$.

Proof of Theorem 3.1. Let m be the maximum dimension of all the fibers.

The question is local on Y, so we'll show that the result holds near a point p of Y. We may assume that Y is affine, and hence that $X \hookrightarrow \mathbb{P}^n_Y$.

Let k be the residue field at p. Then $f^{-1}(p)$ is a projective k-scheme of dimension at most m. Thus we can find affine open sets $D(f_1), \ldots, D(f_{m+1})$ that cover $f^{-1}(p)$. In other words, the intersection of $V(f_i)$ does not intersect $f^{-1}(p)$.

If $Y = \operatorname{Spec} A$ and $p = [\mathfrak{p}]$ (so $k = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$), then arbitrarily lift each f_i from an element of $k[x_0, \ldots, x_n]$ to an element f'_i of $A_\mathfrak{p}[x_0, \ldots, x_n]$. Let F be the product of the denominators of the f'_i; note that $F \notin \mathfrak{p}$, i.e. $p = [\mathfrak{p}] \in D(F)$. Then $f'_i \in A_F[x_0, \ldots, x_n]$. The intersection of their zero loci $\cap V(f'_i) \subset \mathbb{P}^n_{A_F}$ is a closed subscheme of $\mathbb{P}^n_{A_F}$. Intersect it with X to get another closed subscheme of $\mathbb{P}^n_{A_F}$. Take its image under f; as projective morphisms are closed, we get a closed subset of $D(F) = \operatorname{Spec} A_F$. But this closed subset does not include p; hence we can find an affine neighborhood $\operatorname{Spec} B$ of p in Y missing the image. But if f''_i are the restrictions of f'_i to $B[x_0, \ldots, x_n]$, then $D(f''_i)$ cover $f^{-1}(\operatorname{Spec} B)$; in other words, over $f^{-1}(\operatorname{Spec} B)$ is covered by $m + 1$ affine open sets, so by the affine-cover vanishing theorem, its cohomology vanishes in degree at least $m + 1$. But the higher-direct image sheet is computed using these cohomology groups, hence the higher direct image sheaf $R^i f_* \mathcal{F}$ vanishes on $\operatorname{Spec} B$ too. ☐

3.B. IMPORTANT EXERCISE. Use a similar argument to prove *semicontinuity of fiber dimension of projective morphisms*: suppose $\pi : X \to Y$ is a projective morphism where $\mathcal{O}_Y$ is coherent. Show that $\{y \in Y : \dim f^{-1}(y) > k\}$ is a Zariski-closed subset. In other words, the dimension of the fiber "jumps over Zariski-closed subsets". (You can interpret the case $k = -1$ as the fact that projective morphisms are closed.) This exercise is rather important for having a sense of how projective morphisms behave!

Here is another handy theorem, that is proved by a similar argument. We know that finite morphisms are projective, and have finite fibers. Here is the converse.

3.2. *Theorem (projective + finite fibers = finite). — Suppose $\pi : X \to Y$ is such that $\mathcal{O}_Y$ is coherent. Then π is projective and finite fibers if and only if it is finite. Equivalently, π is projective and quasifinite if and only it is finite.*

(Recall that quasifinite = finite fibers + finite type. But projective includes finite type.)

It is true more generally that proper + quasifinite = finite.

Proof. We show it is finite near a point $y \in Y$. Fix an affine open neighborhood $\operatorname{Spec} A$ of y in Y. Pick a hypersurface H in $\mathbb{P}^n_A$ missing the preimage of y, so $H \cap X$ is closed. (You can take this as a hint for Exercise 3.B!) Let $H' = \pi_*(H \cap X)$, which is closed, and doesn't contain y. Let $U = \operatorname{Spec} R - H'$, which is an open set containing y. Then above U, π is projective and affine, so we are done by the Corollary from last day (that projective + affine = finite). ☐

Here is one last potentially useful fact.

3.C. EXERCISE. Suppose $f : X \to Y$ is a projective morphism, with $\mathcal{O}(1)$ the invertible sheaf on X. Suppose Y is quasicompact and $\mathcal{O}_Y$ is coherent. Let $\mathcal{F}$ be coherent on X. Show that

(a) $f^* f_* \mathcal{F}(n) \to \mathcal{F}(n)$ is surjective for $n \gg 0$. (First show that there is a natural map for any n! Hint: by adjointness of f_* with f_*.) [Should I relate this to fact 1.A?] Translation: for $n \gg 0$, $\mathcal{F}(n)$ is relatively generated by global sections.

(b) For $i > 0$ and $n \gg 0$, $R^i f_* \mathcal{F}(n) = 0$.

E-mail address: `vakil@math.stanford.edu`