FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 41

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CONTENTS

Welcome back!

Let's now use what we have developed to study something explicit — curves. Our motivating question is a loose one: what are the curves, by which I mean nonsingular irreducible separated curves, finite type over a field k? In other words, we'll be dealing with geometry, although possibly over a non-algebraically closed field.

Here is an explicit question: are all curves (say reduced, even non-singular, finite type over given k) isomorphic? Obviously not: some are affine, and some (such as $\mathbb{P}^1$) are not. So to simplify things — and we'll further motivate this simplification in Class 42 — are all projective curves isomorphic? Perhaps all nonsingular projective curves are isomorphic to $\mathbb{P}^1$? Once again the answer is no, but the proof is a bit subtle: we've defined an invariant, the genus, and shown that $\mathbb{P}^1$ has genus 0, and that there exist nonsingular projective curves of non-zero genus. Are all (nonsingular) genus 0 curves isomorphic to $\mathbb{P}^1$? We know there exist nonsingular genus 1 curves (plane cubics) — is there only one? If not, "how many" are there?

In order to discuss interesting questions like these, we'll develop some theory. We first show a useful result that will help us focus our attention on the projective case.

1. NORMALIZATION

I now want to tell you how to normalize a reduced Noetherian scheme, which is roughly how best to turn a scheme into a normal scheme. More precisely, a **normalization** of a scheme X is a morphism $\nu : \tilde{X} \to X$ from a normal scheme, where ν induces a bijection of irreducible components of $\tilde{X}$ and X, and ν gives a birational morphism on each of the components. It will be nicer still, as it will satisfy a universal property. (I drew a picture of a normalization of a curve.)

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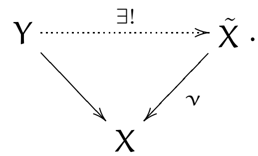
Let's begin with the case where X is irreducible, and hence integral. (We will then deal with the more general case, and also discuss normalization in a function field extension.)

In this case of X irreducible, the normalization $\nu : \tilde{X} \to X$ is an affine and surjective map, such that given any dominant morphism f from an irreducible normal scheme to X, this morphism factors uniquely through ν:



Thus if the normalization exists, then it is unique up to unique isomorphism. We now have to show that it exists, and we do this in the usual way. We deal first with the case where X is affine, say $X = \operatorname{Spec} A$, where A is an integral domain. Then let $\tilde{A}$ be the *integral closure* of A in its fraction field $\mathrm{FF}(A)$.

1.A. EXERCISE. Show that $\nu : \operatorname{Spec} \tilde{A} \to \operatorname{Spec} A$ is surjective. (Hint: use the Going-up Theorem.)

1.B. EXERCISE. Show that $\nu : \operatorname{Spec} \tilde{A} \to \operatorname{Spec} A$ satisfies the universal property.

1.C. EXERCISE. Show that normalizations exist in general.

1.D. EXERCISE. Show that $\dim \tilde{X} = \dim X$ (hint: see the discussion around the notes for the Going-Up Theorem).

1.E. EXERCISE. Explain how to generalize the notion of normalization to the case where X is a reduced Noetherian scheme (with possibly more than one component). This basically requires defining a universal property. I'm not sure what the "perfect" definition, but all reasonable universal properties should be equivalent.

1.F. EXERCISE. Show that the normalization map is an isomorphism on an open dense subset of X.

Now might be a good time to see some examples.

1.G. EXERCISE. Show that $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2 - x^2(x+1))$ given by $(x,y) \mapsto (t^2-1, t(t^2-1))$ (see Figure 1) is a normalization. (Hint: show that $k[t]$ and $k[x,y]/(y^2 - x^2(x+1))$ have the same fraction field. Show that $k[t]$ is integrally closed. Show that $k[t]/k[x,y]/(y^2 - x^2(x+1))$ is an integral extension.)

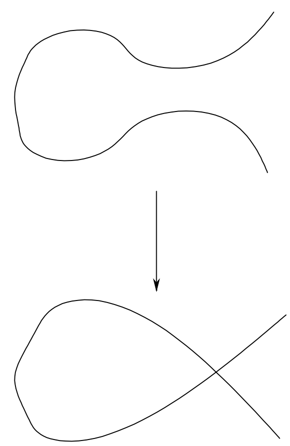


FIGURE 1. The normalization $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2 - x^2(x+1))$ given by $(x,y) \mapsto (t^2-1, t(t^2-1))$

You will see that once we guess what the normalization is, it isn't hard to verify that it is indeed the normalization. Perhaps a few words are in order as to where the polynomials t^2-1 and $t(t^2-1)$ arose in the previous exercise. The key idea is to guess $t = y/x$. (Then $t^2 = x+1$ and $y = xt$ quickly.) The key idea comes from three possible places. We begin by sketching the curve, and noticing the node at the origin. (a) The function y/x is well-defined away from the node, and at the node, the two branches have "values" $y/x = 1$ and $y/x = -1$. (b) We can also note that if $t = y/x$, then t^2 is a polynomial, so we'll need to adjoin t in order to obtain the normalization. (c) The curve is cubic, so we expect a general line to meet the cubic in three points, counted with multiplicity. (We'll make this precise when we discuss Bezout's Theorem.) There is a $\mathbb{P}^1$ parametrizing lines through the origin (with co-ordinate equal to the slope of the line, y/x), and most such lines meet the curve with multiplicity two at the origin, and hence meet the curve at precisely one other point of the curve. So this "co-ordinatizes" most of the curve, and we try adding in this co-ordinate.

1.H. EXERCISE. Find the normalization of the cusp $y^2 = x^3$.

1.I. EXERCISE. Find the normalization of the tacnode $y^2 = x^4$.

Notice that in the previous examples, normalization "resolves" the singularities of the curve. In general, it will do so in dimension one (in reasonable Noetherian circumstances, as normal Noetherian domains of dimension one are all Discrete Valuation Rings), but won't do so in higher dimension (we'll see that the cone $z^2 = x^2 + y^2$ is normal).

1.J. EXERCISE. Suppose $X = \operatorname{Spec} \mathbb{Z}[15i]$. Describe the normalization $\tilde{X} \to X$. (Hint: it isn't hard to find an integral extension of $\mathbb{Z}[15i]$ that is integrally closed.) Over what points of X is the normalization not an isomorphism?

The following fact is useful.

1.1. *Theorem (finiteness of integral closure). — Suppose* A *is a domain,* $K = FF(A)$, L/K *is a finite separable field extension, and* B *is the integral closure of* A *in* L *("the integral closure of* A *in the field extension* L/K*", i.e. those elements of* L *integral over* A*).*
(a) if A *is integrally closed, then* B *is a finitely generated* A*-module.*
(b) if A *is a finitely generated* k*-algebra, then* B *(the integral closure of* A *in its fraction field) is a finitely generated* A*-module.*

I hope to type up a proof of these facts at some point to show you that they are not that bad. Much of part (a) was proved by Greg Brumfiel in 210B.

Warning: (b) does *not* hold for Noetherian A in general. I find this very alarming. I don't know an example offhand, but one is given in Eisenbud's book. This is a sign that this Theorem is not easy.

1.K. EXERCISE. Show that if X is an integral finite-type k-scheme, then its normalization $\nu : \tilde{X} \to X$ is a finite morphism.

1.L. EXERCISE. (This is an important generalization!) Suppose X is an integral scheme. Define the *normalization of* X, $\nu : \tilde{X} \to X$, *in a given finite field extension of the function field of* X. Show that $\tilde{X}$ is normal. (This will be hard-wired into your definition.) Show that if either X is itself normal, or X is finite type over a field k, then the normalization in a finite field extension is a finite morphism. Again, this is a finite morphism. (Again, for this we need finiteness of integral closure 1.1.)

Let's try this in a few cases.

1.M. EXERCISE. Suppose $X = \operatorname{Spec} \mathbb{Z}$ (with function field $\mathbb{Q}$). Find its integral closure in the field extension $\mathbb{Q}(i)$. (There is no "geometric" way to do this; it is purely an algebraic problem, although the answer should be understood geometrically.)

A finite extension K of $\mathbb{Q}$ is called a *number field*, and the integral closure of $\mathbb{Z}$ in K the *ring of integers of* K, denoted $\mathcal{O}_K$. (This notation is a little awkward given our other use of the symbol $\mathcal{O}$.) By the previous exercises, $\operatorname{Spec} \mathcal{O}_K$ is a Noetherian normal domain of dimension 1 (hence regular). This is called a *Dedekind domain*. We think of it as a smooth curve.

1.N. EXERCISE. (a) Suppose $X = \operatorname{Spec} k[x]$ (with function field $k(x)$). Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Again we get a Dedekind domain.)
(b) Suppose $X = \mathbb{P}^1$, with distinguished open $\operatorname{Spec} k[x]$. Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Part (a) involves computing the normalization over one affine open set; now figure out what happens over the "other" affine open set.)

2. Extending maps to projective schemes over smooth codimension one points: the "clear denominators" theorem

2.1. *The "curve to projective" extension Theorem. — Suppose* C *is a pure dimension* 1 *Noetherian scheme over a base* S*, and* $p \in C$ *is a nonsingular closed point of it. Suppose* Y *is a projective* S*-scheme. Then any morphism* $C - p \to Y$ *extends to* $C \to Y$*.*

I often called this the "clear denominators" theorem because it reminds me of the central simple idea in the proof. Suppose you have a map from $\mathbb{A}^1 - \{0\}$ to projective space, and you wanted to extend it to $\mathbb{P}^n$. Say for example the map was given by $t \mapsto [t^4 + t^{-3}; t^{-2} + 4t]$. Then of course you would "clear the denominators", and replace the map by $t \mapsto [t^7 + 1; t + t^4]$.

In practice, we'll use this theorem when $S = k$, and C is a k-variety.

Note that if such an extension exists, then it is unique: The non-reduced locus of C is a closed subset (we checked this earlier for any Noetherian scheme), not including p, so by replacing C by an open neighborhood of p that is reduced, we can use the theorem that maps from reduced schemes to separated schemes are determined by their behavior on a dense open set.

I will give three proofs, which I find enlightening in different ways.

Proof 1. By restricting to an affine neighborhood of C, we can reduce to the case where C is affine. We can similarly assume S is affine.

We next reduce to the case where $Y = \mathbb{P}^n_S$. Choose a closed immersion $Y \to \mathbb{P}^n_S$. If the result holds for $\mathbb{P}^n$, and we have a morphism $C \to \mathbb{P}^n$ with $C - p$ mapping to Y, then C must map to Y as well. Reason: we can reduce to the case where the source is an affine open subset, and the target is $\mathbb{A}^n_k \subset \mathbb{P}^n_k$ (and hence affine). Then the functions vanishing on $Y \cap \mathbb{A}^n_k$ pull back to functions that vanish at the generic point of C and hence vanish everywhere on C, i.e. C maps to Y.

Choose a uniformizer $t \in \mathfrak{m} - \mathfrak{m}^2$ in the local ring of C at p. By discarding the points of the vanishing set $V(t)$ aside from p, and taking an affine open subset of p in the remainder we reduce to the case where t cuts out precisely $\mathfrak{m}$ (i.e. $\mathfrak{m} = (y)$). Choose a dense open subset U of $C - p$ where the pullback of $\mathcal{O}(1)$ is trivial. Take an affine open neighborhood $\operatorname{Spec} A$ of p in $U \cup \{p\}$. Then the map $\operatorname{Spec} A - p \to \mathbb{P}^n$ corresponds to $n+1$ functions, say $f_0, \dots, f_n \in A_m$, not all zero. Let m be the smallest valuation of all the f_i. Then $[t^{-m} f_0; \dots; t^{-m} f_n]$ has all entries in A, and not all in the maximal ideal, and thus is defined at p as well. $\square$

Proof 2. We first extend the map $\operatorname{Spec} FF(C) \to Y$ to $\operatorname{Spec} \mathcal{O}_{C,p} \to Y$. We do this as follows. Note that $\mathcal{O}_{C,p}$ is a discrete valuation ring. We show first that there is a morphism $\operatorname{Spec} \mathcal{O}_{C,p} \to \mathbb{P}^n$. The rational map can be described as $[a_0; a_1; \cdots; a_n]$ where $a_i \in \mathcal{O}_{C,p}$. Let m be the minimum valuation of the a_i, and let t be a uniformizer of $\mathcal{O}_{C,p}$ (an element of valuation 1). Then $[t^{-m}a_0; t^{-m}a_1; \dots t^{-m}a_n]$ is another description of the morphism $\operatorname{Spec} FF(\mathcal{O}_{C,p}) \to \mathbb{P}^n$, and each of the entries lie in $\mathcal{O}_{C,p}$, and not all entries lie in $\mathfrak{m}$ (as one of the entries has valuation 0). This same expression gives a morphism $\operatorname{Spec} \mathcal{O}_{C,p} \to \mathbb{P}^n$.

Our intuition now is that we want to glue the maps $\operatorname{Spec} \mathcal{O}_{C,p} \to Y$ (which we picture as a map from a germ of a curve) and $C - p \to Y$ (which we picture as the rest of the curve). Let $\operatorname{Spec} R \subset Y$ be an affine open subset of Y containing the image of $\operatorname{Spec} \mathcal{O}_{C,p}$. Let $\operatorname{Spec} A \subset C$ be an affine open of C containing p, and such that the image of $\operatorname{Spec} A - p$ in Y lies in $\operatorname{Spec} R$, and such that p is cut out scheme-theoretically by a single equation (i.e. there is an element $t \in A$ such that (t) is the maximal ideal corresponding to p. Then R and A are domains, and we have two maps $R \to A_{(t)}$ (corresponding to $\operatorname{Spec} \mathcal{O}_{C,p} \to \operatorname{Spec} R$) and $R \to A_t$ (corresponding to $\operatorname{Spec} A - p \to \operatorname{Spec} R$) that agree "at the generic point", i.e. that give the same map $R \to FF(A)$. But $A_t \cap A_{(t)} = A$ in $FF(A)$ (e.g. by algebraic Hartogs' theorem — elements of the fraction field of A that don't have any poles away from t, nor at t, must lie in A), so we indeed have a map $R \to A$ agreeing with both morphisms. □

Proof 3. As $Y \to S$, by the valuative criterion of properness, the map $\operatorname{Spec} FF(C) \to Y$ extends to $\operatorname{Spec} \mathcal{O}_{C,p} \to Y$. Then proceed as in Proof 2. □

The third proof is quite short, and indeed extends the statement of Theorem 2.1 to the proper case. The only downside is that the previous proofs are straightforward, while the proof of the valuative criterion is highly nontrivial.

2.A. EXERCISE (USEFUL PRACTICE!). Suppose X is a Noetherian k-scheme, and Z is an irreducible codimension 1 subvariety whose generic point is a nonsingular point of X (so the local ring $\mathcal{O}_{X,Z}$ is a discrete valuation ring). Suppose $X \dashrightarrow Y$ is a rational map to a projective k-scheme. Show that the domain of definition of the rational map includes a dense open subset of Z. In other words, rational maps from Noetherian k-schemes to projective k-schemes can be extended over nonsingular codimension 1 sets. (By the easy direction of the valuative criterion of separatedness, or the theorem of uniqueness of extensions of maps from reduced schemes to separated schemes.) this map is unique.)

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