FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 46

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CONTENTS

1. CURVES OF GENUS 4 AND 5

We begin with two exercises in general genus, and then go back to genus 4.

1.A. EXERCISE. Suppose C is a genus g curve. Show that if C is not hyperelliptic, then the canonical bundle gives a closed immersion $C \hookrightarrow \mathbb{P}^{g-1}$. (In the hyperelliptic case, we have already seen that the canonical bundle gives us a double cover of a rational normal curve.) Hint: follow the genus 3 case. Such a curve is called a **canonical curve**, and this closed immersion is called the **canonical embedding** of C.

1.B. EXERCISE. Suppose C is a curve of genus $g > 1$, over a field k that is not algebraically closed. Show that C has a closed point of degree at most $2g - 2$ over the base field. (For comparison: if $g = 1$, it turns out that there is no such bound independent of k!)

We next consider nonhyperelliptic curves C of genus 4. Note that $\deg \mathcal{K} = 6$ and $h^0(C, \mathcal{K}) = 4$, so the canonical map expresses C as a sextic curve in $\mathbb{P}^3$. We shall see that all such C are complete intersections of quadric surfaces and cubic surfaces, and conversely all nonsingular complete intersections of quadrics and cubics are genus 4 nonhyperelliptic curves, canonically embedded.

By Riemann-Roch,

$$h^0(C, \mathcal{K}^{\otimes 2}) = \deg \mathcal{K}^{\otimes 2} - g + 1 = 12 - 4 + 1 = 9.$$

We have the restriction map $H^0(\mathbb{P}^3, \mathcal{O}(2)) \to H^0(C, \mathcal{K}^{\otimes 2})$, and $\dim \operatorname{Sym}^2 \Gamma(C, \mathcal{K}) = \binom{4+1}{2} = 10$. Thus there is at least one quadric in $\mathbb{P}^3$ that vanishes on our curve C. Translation: C lies on at least on quadric Q. Now quadrics are either double planes, or the union of two planes, or cones, or nonsingular quadrics. (They corresponds to quadric forms of rank 1, 2, 3, and 4 respectively.) But C can't lie in a plane, so Q must be a cone or nonsingular. In particular, Q is irreducible.

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Now C can't lie on *two* (distinct) such quadrics, say Q and Q'. Otherwise, as Q and Q' have no common components (they are irreducible and not the same!), $Q \cap Q'$ is a curve (not necessarily reduced or irreducible). By Bezout's theorem, $Q \cap Q'$ is a curve of degree 4. Thus our curve C, being of degree 6, cannot be contained in $Q \cap Q'$. (Do you see why?)

We next consider cubics surface. By Riemann-Roch again, $h^0(C, \mathcal{K}^{\otimes 3}) = \deg \mathcal{K}^{\otimes 3} - g + 1 = 18 - 4 + 1 = 15$. Now $\dim \operatorname{Sym}^3 \Gamma(C, \mathcal{K})$ has dimension $\binom{4+2}{3} = 20$. Thus C lies on at least a 5-dimensional vector space of cubics. Now a 4-dimensional subspace come from multiplying the quadric Q by a linear form $(?w + ?x + ?y + ?z)$. But hence there is still one cubic K whose underlying form is not divisible by the quadric form Q (i.e. K doesn't contain Q.) Then K and Q share no component, so $K \cap Q$ is a complete intersection. By Bezout's theorem (the degree of a complete intersection of hypersurfaces is the product of the degrees of the hypersurfaces), we obtain a curve of degree 6. Our curve C has degree 6. This suggests that $C = K \cap Q$. In fact, $K \cap Q$ and C have the same Hilbert polynomial, and $C \subset K \cap Q$. Hence $C = K \cap Q$ by the following exercise.

1.C. EXERCISE. Suppose $X \subset Y \subset \mathbb{P}^n$ are a sequence of closed subschemes, where X and Y have the same Hilbert polynomial. Show that $X = Y$. Hint: consider the exact sequence

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

Show that if the Hilbert polynomial of $\mathcal{I}_{X/Y}$ is 0, then $\mathcal{I}_{X/Y}$ must be the 0 sheaf. (Handy trick: For $m \gg 0$, $\mathcal{I}_{X/Y}(m)$ is generated by global sections and is also 0. This of course applies with $\mathcal{I}$ replaced by *any* coherent sheaf.)

We now show the converse, and that any nonsingular complete intersection C of a quadric surface with a cubic surface is a canonically embedded genus 4 curve. By an earlier exercise on computing the genus of a complete intersection (in our discussion of Hilbert functions), such a complete intersection has genus 4.

1.D. EXERCISE. Show that $\mathcal{O}_C(1)$ has at least 4 sections. (Translation: C doesn't lie in a hyperplane.)

The only degree $2g - 2$ invertible sheaf with (at least) g sections is the canonical sheaf (we've used this fact many times), so $\mathcal{O}_C(1) \cong \mathcal{K}_C$, and C is indeed canonically embedded.

1.E. EXERCISE. Give a heuristic argument suggesting that the nonhyperelliptic curves of genus 4 "form a family of dimension 9".

On to genus 5!

1.F. EXERCISE. Suppose C is a nonhyperelliptic genus 5 curve. Show that the canonical curve is degree 8 in $\mathbb{P}^4$. Show that it lies on a three-dimensional vector space of quadrics (i.e. it lies on 3 linearly independent independent quadrics). Show that a nonsingular complete intersection of 3 quadrics is a canonical genus 5 curve.

In fact a canonical genus 5 is always a complete intersection of 3 quadrics.

1.G. EXERCISE. Give a heuristic argument suggesting that the nonhyperelliptic curves of genus 5 “form a family of dimension 12”.

We have now understand curves of genus 3 through 5 by thinking of canonical curves as complete intersections. Sadly our luck has run out.

1.H. EXERCISE. Show that if $C \subset \mathbb{P}^{g-1}$ is a canonical curve of genus $g \geq 6$, then C is *not* a complete intersection. (Hint: Bezout's theorem.)

2. CURVES OF GENUS 1: THE BEGINNING

To avoid dividing up these notes too much, I've moved these into the Class 47 notes.

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