SPECTRAL SEQUENCES: FRIEND OR FOE?

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Spectral sequences are a powerful book-keeping tool for proving things involving complicated commutative diagrams. They were introduced by Leray in the 1940's at the same time as he introduced sheaves. They have a reputation for being abstruse and difficult. It has been suggested that the name 'spectral' was given because, like spectres, spectral sequences are terrifying, evil, and dangerous. I have heard no one disagree with this interpretation, which is perhaps not surprising since I just made it up.

Nonetheless, the goal of this note is to tell you enough that you can use spectral sequences without hesitation or fear, and why you shouldn't be frightened when they come up in a seminar. What is different in this presentation is that we will use spectral sequence to prove things that you may have already seen, and that you can prove easily in other ways. This will allow you to get some hands-on experience for how to use them. We will also see them only in a "special case" of double complexes (which is the version by far the most often used in algebraic geometry), and not in the general form usually presented (filtered complexes, exact couples, etc.). See chapter 5 of Weibel's marvelous book for more detailed information if you wish. If you want to become comfortable with spectral sequences, you *must* try the exercises.

For concreteness, we work in the category vector spaces over a given field. However, everything we say will apply in any abelian category, such as the category $\mathbf{Mod}_A$ of A-modules.

0.1. Double complexes.

A **first-quadrant double complex** is a collection of vector spaces $E^{p,q}$ ($p, q \in \mathbb{Z}$), which are zero unless $p, q \geq 0$, and "rightward" morphisms $d_>^{p,q} : E^{p,q} \to E^{p,q+1}$ and "upward" morphisms $d_\wedge^{p,q} : E^{p,q} \to E^{p+1,q}$. In the superscript, the first entry denotes the row number, and the second entry denotes the column number, in keeping with the convention for matrices, but opposite to how the (x, y)-plane is labeled. The subscript is meant to suggest the direction of the arrows. We will always write these as $d_>$ and $d_\wedge$ and ignore the superscripts. We require that $d_>$ and $d_\wedge$ satisfying (a) $d_>^2 = 0$, (b) $d_\wedge^2 = 0$, and one more condition: (c) either $d_> d_\wedge = d_\wedge d_>$ (all the squares commute) or $d_> d_\wedge + d_\wedge d_> = 0$ (they all anticommute). Both come up in nature, and you can switch from one to the other by replacing $d_\wedge^{p,q}$ with $d_\wedge^p (-1)^q$. So I'll assume that all the squares anticommute, but that you know how to turn the commuting case into this one. (You will see that there is no difference in the recipe, basically because the image and kernel of a homomorphism f equal the image and kernel respectively of $-f$.)

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$$
\begin{tikzcd}
E^{p+1,q} \arrow[rr, "d_>^{p+1,q}"] & & E^{p+1,q+1} \\
& \text{anticommutes} & \\
E^{p,q} \arrow[u, "d_\wedge^{p,q}"] \arrow[rr, "d_>^{p,q}"] & & E^{p,q+1} \arrow[u, "d_\wedge^{p,q+1}"']
\end{tikzcd}
$$

There are variations on this definition, where for example the vertical arrows go downwards, or some different subset of the $E^{p,q}$ are required to be zero, but I'll leave these straightforward variations to you.

From the double complex we construct a corresponding (single) complex $E^\bullet$ with $E^k = \oplus_i E^{i,k-i}$, with $d = d_> + d_\wedge$. In other words, when there is a *single* superscript k, we mean a sum of the kth antidiagonal of the double complex. The single complex is somtimes called the **total complex**. Note that $d^2 = (d_> + d_\wedge)^2 = d_>^2 + (d_> d_\wedge + d_\wedge d_>) + d_\wedge^2 = 0$, so $E^\bullet$ is indeed a complex.

The **cohomology** of the single complex is sometimes called the **hypercohomology** of the double complex. We will instead use the phrase "cohomology of the double complex".

Our initial goal will be to find the cohomology of the double complex. You will see later that we secretly also have other goals.

A spectral sequence is a recipe for computing some information about the cohomology of the double complex. I won't yet give the full recipe. Surprisingly, this fragmentary bit of information is sufficent to prove lots of things.

0.2. Approximate Definition. A **spectral sequence** with **rightward orientation** is a sequence of tables or **pages** ${}_>E_0^{p,q}$, ${}_>E_1^{p,q}$, ${}_>E_2^{p,q}$, $\ldots (p, q \in \mathbb{Z})$, where ${}_>E_0^{p,q} = E^{p,q}$, along with a differential

$$
{}_>d_r^{p,q} : {}_>E_r^{p,q} \to {}_>E^{p+r,q-r+1}
$$

with ${}_>d_r^{p,q} \circ {}_>d_r^{p,q} = 0$, along with an isomorphism of the cohomology of ${}_>d_r$ at ${}_>E^{p,q}$ (i.e. $\ker {}_>d_r^{p,q} / \operatorname{im} {}_>d_r^{p-r,q+r-1}$) with ${}_>E_{r+1}^{p,q}$.

The orientation indicates that our 0th differential is the rightward one: $d_0 = d_>$. The left subscript "$>$" is usually omitted.

The order of the morphisms is best understood visually:

(1)
$$\begin{tikzcd}
\bullet & & & \\
& \bullet & & \\
& & \bullet & \\
& & \bullet \arrow[r, "d_0"'] \arrow[u, "d_1"'] \arrow[uul, "d_2"] \arrow[uuull, "d_3"] & \bullet
\end{tikzcd}$$

(the morphisms each apply to different pages). Notice that the map always is "degree 1" in the grading of the single complex $E^\bullet$.

The actual definition describes what $E_r^{\bullet,\bullet}$ and $d_r^{\bullet,\bullet}$ actually are, in terms of $E^{\bullet,\bullet}$. We will describe d_0, d_1, and d_2 below, and you should for now take on faith that this sequence continues in some natural way.

Note that $E_r^{p,q}$ is always a subquotient of the corresponding term on the 0th page $E_0^{p,q} = E^{p,q}$. In particular, if $E^{p,q} = 0$, then $E_r^{p,q} = 0$ for all r, so $E_r^{p,q} = 0$ unless $p, q \in \mathbb{Z}^{\geq 0}$. Notice also that for any fixed p, q, once r is sufficiently large, $E_{r+1}^{p,q}$ is computed from $(E_r^{\bullet,\bullet}, d_r)$ using the complex

$$\begin{tikzcd}
0 & & \\
& E_r^{p,q} \arrow[ul, "d_r^{p,q}"] & \\
& & 0 \arrow[ul, "d_r^{p+r,q-r-1}"']
\end{tikzcd}$$

and thus we have canonical isomorphisms

$$E_r^{p,q} \cong E_{r+1}^{p,q} \cong E_{r+2}^{p,q} \cong \cdots$$

We denote this module $E_\infty^{p,q}$.

We now describe the first few pages of the spectral sequence explicitly. As stated above, the differential d_0 on $E_0^{\bullet,\bullet} = E^{\bullet,\bullet}$ is defined to be $d_>$. The rows are complexes:

The 0th page E_0:
$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

and so E_1 is just the table of cohomologies of the rows. You should check that there are now vertical maps $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ of the row cohomology groups, induced by $d_\wedge$, and that these make the columns into complexes. (We have "used up the horizontal morphisms", but "the vertical differentials live on".)

$$\text{The 1st page } E_1: \qquad \begin{tikzcd} \bullet & \bullet & \bullet \\ \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] \\ \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] \end{tikzcd}$$

We take cohomology of d_1 on E_1, giving us a new table, $E_2^{p,q}$. It turns out that there are natural morphisms from each entry to the entry two above and one to the left, and that the composition of these two is 0. (It is a very worthwhile exercise to work out how this natural morphism d_2 should be defined. Your argument may be reminiscent of the connecting homomorphism in the Snake Lemma 0.5 or in the long exact sequence in cohomology arising from a short exact sequence of complexes, Exercise 0.D. This is no coincidence.)

$$\text{The 2nd page } E_2: \qquad \begin{tikzcd} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet \arrow[uul] & \bullet \arrow[uul] \end{tikzcd}$$

This is the beginning of a pattern.

Then it is a theorem that there is a filtration of $H^k(E^\bullet)$ by $E_\infty^{p,q}$ where $p + q = k$. (We can't yet state it as an official **Theorem** because we haven't precisely defined the pages and differentials in the spectral sequence.) More precisely, there is a filtration

$$E_\infty^{0,k} \xhookrightarrow{E_\infty^{1,k-1}} \; ? \; \xhookrightarrow{E_\infty^{2,k-2}} \cdots \xhookrightarrow{E^{k,0}} H^k(E^\bullet) \tag{2}$$

where the quotients are displayed above each inclusion. (I always forget which way the quotients are supposed to go, i.e. whether $E^{k,0}$ or $E^{0,k}$ is the subobject. One way of remembering it is by having some idea of how the result is proved.)

We say that the spectral sequence ${}_{>}E_\bullet^{\bullet,\bullet}$ **converges** to $H^\bullet(E^\bullet)$. We often say that ${}_{>}E_2^{\bullet,\bullet}$ (or any other page) abuts to $H^\bullet(E^\bullet)$.

Although the filtration gives only partial information about $H^\bullet(E^\bullet)$, sometimes one can find $H^\bullet(E^\bullet)$ precisely. One example is if all $E_\infty^{i,k-i}$ are zero, or if all but one of them are zero (e.g. if $E_r^{i,k-i}$ has precisely one non-zero row or column, in which case one says that the spectral sequence *collapses* at the rth step, although we will not use this term). Another example is in the category of vector spaces over a field, in which case we can find the dimension of $H^k(E^\bullet)$. Also, in lucky circumstances, E_2 (or some other small page) already equals E_∞.

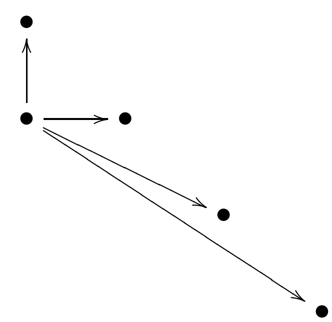
0.A. EXERCISE: INFORMATION FROM THE SECOND PAGE. Show that $H^0(E^\bullet) = E_\infty^{0,0} = E_2^{0,0}$ and

$$0 \longrightarrow E_2^{0,1} \longrightarrow H^1(E^\bullet) \longrightarrow E_2^{1,0} \xrightarrow{d_2^{1,0}} E_2^{0,2} \longrightarrow H^2(E^\bullet).$$

0.3. *The other orientation.*

You may have observed that we could as well have done everything in the opposite direction, i.e. reversing the roles of horizontal and vertical morphisms. Then the sequences of arrows giving the spectral sequence would look like this (compare to (1)).

(3)



This spectral sequence is denoted ${}_\wedge E_\bullet^{\bullet,\bullet}$ ("with the upwards orientation"). Then we would again get pieces of a filtration of $H^\bullet(E^\bullet)$ (where we have to be a bit careful with the order with which ${}_\wedge E_\infty^{p,q}$ corresponds to the subquotients — it in the opposite order to that of (2) for ${}_> E_\infty^{p,q}$). Warning: in general there is no isomorphism between ${}_> E_\infty^{p,q}$ and ${}_\wedge E_\infty^{p,q}$.

In fact, this observation that we can start with either the horizontal or vertical maps was our secret goal all along. Both algorithms compute information about the same thing ($H^\bullet(E^\bullet)$), and usually we don't care about the final answer — we often care about the answer we get in one way, and we get at it by doing the spectral sequence in the *other* way.

0.4. Examples.

We're now ready to see how this is useful. The moral of these examples is the following. In the past, you may have proved various facts involving various sorts of diagrams, which involved chasing elements around. Now, you'll just plug them into a spectral sequence, and let the spectral sequence machinery do your chasing for you.

0.5. *Example: Proving the Snake Lemma.* Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

where the rows are exact and the squares commute. (Normally the Snake Lemma is described with the vertical arrows pointing downwards, but I want to fit this into my spectral sequence conventions.) We wish to show that there is an exact sequence

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{im}\alpha \to \operatorname{im}\beta \to \operatorname{im}\gamma \to 0. \tag{4}$$

We plug this into our spectral sequence machinery. We first compute the cohomology using the rightwards orientation, i.e. using the order (1). Then because the rows are exact, $E_1^{p,q} = 0$, so the spectral sequence has already converged: $E_\infty^{p,q} = 0$.

We next compute this "0" in another way, by computing the spectral sequence using the upwards orientation. Then ${}_\wedge E_1^{\bullet,\bullet}$ (with its differentials) is:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}\alpha \arrow[r] & \operatorname{im}\beta \arrow[r] & \operatorname{im}\gamma \arrow[r] & 0 \\
0 \arrow[r] & \ker\alpha \arrow[r] & \ker\beta \arrow[r] & \ker\gamma \arrow[r] & 0.
\end{tikzcd}$$

Then ${}_\wedge E_2^{\bullet,\bullet}$ is of the form:

$$\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & & & & & \\
& 0 \arrow[drr] & ?? \arrow[drr] & ? \arrow[drr] & ? & 0 & \\
& 0 & ? & ? \arrow[drr] & ?? \arrow[drr] & 0 & \\
& & & & & 0 & 0
\end{tikzcd}$$

We see that after ${}_\wedge E_2$, all the terms will stabilize except for the double-question-marks — all maps to and from the single question marks are to and from 0-entries. And after ${}_\wedge E_3$, even these two double-quesion-mark terms will stabilize. But in the end our complex must be the 0 complex. This means that in ${}_\wedge E_2$, all the entries must be zero, except for the two double-question-marks, and these two must be the isormorphic. This means that $0 \to \ker\alpha \to \ker\beta \to \ker\gamma$ and $\operatorname{im}\alpha \to \operatorname{im}\beta \to \operatorname{im}\gamma \to 0$ are both exact (that comes from the vanishing of the single-question-marks), and

$$\operatorname{coker}(\ker\beta \to \ker\gamma) \cong \ker(\operatorname{im}\alpha \to \operatorname{im}\beta)$$

is an isomorphism (that comes from the equality of the double-question-marks). Taken together, we have proved the exactness of (4), and hence the Snake Lemma!

Spectral sequences make it easy to see how to generalize results further. For example, if $A \to B$ is no longer assumed to be injective, how would the conclusion change?

0.6. *Example: the Five Lemma.* Suppose

(5)
$$\begin{tikzcd} F \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & J \\ A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & D \arrow[r] \arrow[u, "\delta"] & E \arrow[u, "\epsilon"] \end{tikzcd}$$

where the rows are exact and the squares commute.

Suppose α, β, δ, ϵ are isomorphisms. We'll show that γ is an isomorphism.

We first compute the cohomology of the total complex using the rightwards orientation (1). We choose this because we see that we will get lots of zeros. Then ${}_{>}E_1^{\bullet,\bullet}$ looks like this:

$$\begin{tikzcd} ? & 0 & 0 & 0 & ? \\ ? \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & ? \arrow[u] \end{tikzcd}$$

Then ${}_{>}E_2$ looks similar, and the sequence will converge by E_2, as we will never get any arrows between two non-zero entries in a table thereafter. We can't conclude that the cohomology of the total complex vanishes, but we can note that it vanishes in all but four degrees — and most important, it vanishes in the two degrees corresponding to the entries C and H (the source and target of γ).

We next compute this using the upwards orientation (3). Then ${}_{\wedge}E_1$ looks like this:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

and the spectral sequence converges at this step. We wish to show that those two question marks are zero. But they are precisely the cohomology groups of the total complex that we just showed *were* zero — so we're done!

The best way to become comfortable with this sort of argument is to try it out yourself several times, and realize that it really is easy. So you should do the following exercises!

0.B. EXERCISE: THE SUBTLE FIVE LEMMA. By looking at the spectral sequence proof of the Five Lemma above, prove a subtler version of the Five Lemma, where one of the isomorphisms can instead just be required to be an injection, and another can instead just be required to be a surjection. (I am deliberately not telling you which ones, so you can see how the spectral sequence is telling you how to improve the result.)

0.C. EXERCISE. If β and δ (in (5)) are injective, and α is surjective, show that γ is injective. State the dual statement (whose proof is of course essentially the same).

0.D. EXERCISE. Use spectral sequences to show that a short exact sequence of complexes gives a long exact sequence in cohomology.

0.E. EXERCISE (THE MAPPING CONE). Suppose $\mu : A^\bullet \to B^\bullet$ is a morphism of complexes. Suppose $C^\bullet$ is the single complex associated to the double complex $A^\bullet \to B^\bullet$. ($C^\bullet$ is called the *mapping cone* of μ.) Show that there is a long exact sequence of complexes:

$$\cdots \to H^{i-1}(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots.$$

(There is a slight notational ambiguity here; depending on how you index your double complex, your long exact sequence might look slightly different.) In particular, we will use the fact that μ induces an isomorphism on cohomology if and only if the mapping cone is exact.

You are now ready to go out into the world and use spectral sequences to your heart's content!

0.7. $\star\star$ Complete definition of the spectral sequence, and proof.

You should most definitely not read this section any time soon after reading the introduction to spectral sequences above. Instead, flip quickly through it to convince yourself that nothing fancy is involved.

We consider the rightwards orientation. The upwards orientation is of course a trivial variation of this.

0.8. *Goals.* We wish to describe the pages and differentials of the spectral sequence explicitly, and prove that they behave the way we said they did. More precisely, we wish to:

(a) describe $E_r^{p,q}$,
(b) verify that $H^k(E^\bullet)$ is filtered by $E_\infty^{p,k-p}$ as in (2),
(c) describe d_r and verify that $d_r^2 = 0$, and
(d) verify that $E_{r+1}^{p,q}$ is given by cohomology using d_r.

Before tacking these goals, you can impress your friends by giving this short description of the pages and differentials of the spectral sequence. We say that an element of $E^{\bullet,\bullet}$ is a (p,q)-*strip* if it is an element of $\oplus_{l\geq 0} E^{p+l,q-l}$ (see Fig. 1). Its non-zero entries lie on a semi-infinite antidiagonal starting with position (p,q). We say that the (p,q)-entry (the projection to $E^{p,q}$) is the *leading term* of the (p,q)-strip. Let $\boxed{S^{p,q}} \subset E^{\bullet,\bullet}$ be the submodule of all the (p,q)-strips. Clearly $S^{p,q} \subset E^{p+q}$, and $S^{0,k} = E^k$.

Note that the differential $d = d_\wedge + d_>$ sends a (p,q)-strip x to a $(p,q+1)$-strip dx. If dx is furthermore a $(p+r, q+r+1)$-strip ($r \in \mathbb{Z}^{\geq 0}$), we say that x is an r-*closed* (p,q)-*strip*. We denote the set of such $\boxed{S_r^{p,q}}$ (so for example $S_0^{p,q} = S^{p,q}$, and $S_0^{0,k} = E^k$). An element of

$$
\begin{array}{ccccc}
\ddots & 0 & 0 & 0 & 0 \\
0 & *^{p+2,q-2} & 0 & 0 & 0 \\
0 & 0 & *^{p+1,q-1} & 0 & 0 \\
0 & 0 & 0 & *^{p,q} & 0 \\
0 & 0 & 0 & 0 & 0^{p-1,q+1}
\end{array}
$$

FIGURE 1. A (p,q)-strip (in $S^{p,q} \subset E^{p+q}$). Clearly $S^{0,k} = E^k$.

$S_r^{p,q}$ may be depicted as:

$$
\begin{tikzcd}
\ddots \arrow[r] & ? & & & \\
& *^{p+2,q-2} \arrow[u] \arrow[r] & 0 & & \\
& & *^{p+1,q-1} \arrow[u] \arrow[r] & 0 & \\
& & & *^{p,q} \arrow[u] \arrow[r] & 0
\end{tikzcd}
$$

0.9. *Preliminary definition of* $E_r^{p,q}$. We are now ready to give a first definition of $E_r^{p,q}$, which by construction should be a subquotient of $E^{p,q} = E_0^{p,q}$. We describe it as such by describing two submodules $Y_r^{p,q} \subset X_r^{p,q} \subset E^{p,q}$, and defining $E_r^{p,q} = X_r^{p,q}/Y_r^{p,q}$. Let $X_r^{p,q}$ be those elements of $E^{p,q}$ that are the leading terms of r-closed (p,q)-strips. Note that by definition, d sends $(r-1)$-closed $S^{p-(r-1),q+(r-1)-1}$-strips to (p,q)-strips. Let $Y_r^{p,q}$ be the leading $((p,q))$-terms of the differential d of $(r-1)$-closed $(p-(r-1), q+(r-1)-1)$-strips (where the differential is considered as a (p,q)-strip).

We next give the definition of the differential d_r of such an element $x \in X_r^{p,q}$. We take *any* r-closed (p,q)-strip with leading term x. Its differential d is a $(p+r, q-r+1)$-strip, and we take its leading term. The choice of the r-closed (p,q)-strip means that this is not a well-defined element of $E^{p,q}$. But it is well-defined modulo the $(r-1)$-closed $(p+1, r+1)$-strips, and hence gives a map $E_r^{p,q} \to E_r^{p+r,q-r+1}$.

This definition is fairly short, but not much fun to work with, so we will forget it, and instead dive into a snakes' nest of subscripts and superscripts.

We begin with making some quick but important observations about (p, q)-strips.

0.F. EXERCISE. Verify the following.

(a) $S^{p,q} = S^{p+1,q-1} \oplus E^{p,q}$.
(b) (Any closed (p, q)-strip is r-closed for all r.) Any element x of $S^{p,q} = S_0^{p,q}$ that is a cycle (i.e. $dx = 0$) is automatically in $S_r^{p,q}$ for all r. For example, this holds when x is a boundary (i.e. of the form dy).
(c) Show that for fixed p, q,

$$S_0^{p,q} \supset S_1^{p,q} \supset \cdots \supset S_r^{p,q} \supset \cdots$$

stabilizes for $r \gg 0$ (i.e. $S_r^{p,q} = S_{r+1}^{p,q} = \cdots$). Denote the stabilized module $S_\infty^{p,q}$. Show $S_\infty^{p,q}$ is the set of closed (p, q)-strips (those (p, q)-strips annihilated by d, i.e. the cycles). In particular, $S_r^{0,k}$ is the set of cycles in E^k.

0.10. *Defining* $E_r^{p,q}$.

Define $X_r^{p,q} := S_r^{p,q}/S_{r-1}^{p+1,q-1}$ and $Y := dS_{r-1}^{p-(r-1),q+(r-1)-1}/S_{r-1}^{p+1,q-1}$.

Then $Y_r^{p,q} \subset X_r^{p,q}$ by Exercise 0.F(b). We define

$$\boxed{E_r^{p,q} = \frac{X_r^{p,q}}{Y_r^{p,q}} = \frac{S_r^{p,q}}{dS_{r-1}^{p-(r-1),q+(r-1)-1} + S_{r-1}^{p+1,q-1}}} \tag{6}$$

We have completed Goal 0.8(a).

You are welcome to verify that these definitions of $X_r^{p,q}$ and $Y_r^{p,q}$ and hence $E_r^{p,q}$ agree with the earlier ones of §0.9 (and in particular $X_r^{p,q}$ and $Y_r^{p,q}$ are both submodules of $E^{p,q}$), but we won't need this fact.

0.G. EXERCISE: $E_\infty^{p,k-p}$ GIVES SUBQUOTIENTS OF $H^k(E^\bullet)$. By Exercise 0.F(c), $E_r^{p,q}$ stabilizes as $r \to \infty$. For $r \gg 0$, interpret $S_r^{p,q}/dS_{r-1}^{p-(r-1),q+(r-1)-1}$ as the cycles in $S_\infty^{p,q} \subset E^{p+q}$ modulo those boundary elements of dE^{p+q-1} contained in $S_\infty^{p,q}$. Finally, show that $H^k(E^\bullet)$ is indeed filtered as described in (2).

We have completed Goal 0.8(b).

0.11. *Definition of* d_r.

We shall see that the map $d_r : E_r^{p,q} \to E^{p+r,q-r+1}$ is just induced by our differential d. Notice that d sends r-closed (p, q)-strips $S_r^{p,q}$ to $(p + r, q - r + 1)$-strips $S^{p+r,q-r+1}$, by the definition "r-closed". By Exercise 0.F(b), the image lies in $S_r^{p+r,q-r+1}$.

0.H. EXERCISE. Verify that d sends

$$dS_{r-1}^{p-(r-1),q+(r-1)-1} + S_{r-1}^{p+1,q-1} \to dS_{r-1}^{(p+r)-(r-1),(q-r+1)+(r-1)-1} + S_{r-1}^{(p+r)+1,(q-r+1)-1}.$$

(The first term on the left goes to 0 from $d^2 = 0$, and the second term on the left goes to the first term on the right.)

Thus we may define

$$\boxed{d_r : E_r^{p,q} = \frac{S_r^{p,q}}{dS_{r-1}^{p-(r-1),q+(r-1)-1} + S_{r-1}^{p+1,q-1}} \to}$$

$$\boxed{\frac{S_r^{p+r,q-r+1}}{dS_{r-1}^{p+1,q-1} + S_{r-1}^{p+r+1,q-r}} = E_r^{p+r,q-r+1}}$$

and clearly $d_r^2 = 0$ (as we may interpret it as taking an element of $S_r^{p,q}$ and applying d twice).

We have accomplished Goal 0.8(c).

0.12. *Verifying that the cohomology of* d_r *at* $E_r^{p,q}$ *is* $E_{r+1}^{p,q}$. We are left with the unpleasant job of verifying that the cohomology of

$$\frac{S_r^{p-r,q+r-1}}{dS_{r-1}^{p-2r+1,q-3} + S_{r-1}^{p-r+1,q+r-2}} \xrightarrow{d_r} \frac{S_r^{p,q}}{dS_{r-1}^{p-r+1,q+r-2} + S_{r-1}^{p+1,q-1}} \xrightarrow{d_r} \frac{S_r^{p+r,q-r+1}}{dS_{r-1}^{p+1,q-1} + S_{r-1}^{p+r+1,q-r}} \tag{7}$$

is naturally identified with

$$\frac{S_{r+1}^{p,q}}{dS_r^{p-r,q+r-1} + S_r^{p+1,q-1}}$$

and this will conclude our final Goal 0.8(d).

Let's begin by understanding the kernel of the right map of (7). Suppose $a \in S_r^{p,q}$ is mapped to 0. This means that $da = db + c$, where $b \in S_{r-1}^{p+1,q-1}$. If $u = a - b$, then $u \in S^{p,q}$, while $du = c \in S_{r-1}^{p+r+1,q-r} \subset S^{p+r+1,q-r}$, from which u is r-closed, i.e. $u \in S_{r+1}^{p,q}$. Hence $a = b + u + x$ where $dx = 0$, from which $a - x = b + c \in S_{r-1}^{p+1,q-1} + S_{r+1}^{p,q}$. However, $x \in S^{p,q}$, so $x \in S_{r+1}^{p,q}$ by Exercise 0.F(b). Thus $a \in S_{r-1}^{p+1,q-1} + S_{r+1}^{p,q}$. Conversely, any $a \in S_{r-1}^{p+1,q-1} + S_{r+1}^{p,q}$ satisfies

$$da \in dS_{r-1}^{p+r,q-r+1} + dS_{r+1}^{p,q} \subset dS_{r-1}^{p+r,q-r+1} + S_{r-1}^{p+r+1,q-r}$$

(using $dS_{r+1}^{p,q} \subset S_0^{p+r+1,q-r}$ and Exercise 0.F(b)) so any such a is indeed in the kernel of

$$S_r^{p,q} \to \frac{S_r^{p+r,q-r+1}}{dS_{r-1}^{p+1,q-1} + S_{r-1}^{p+r+1,q-r}}.$$

Hence the kernel of the right map of (7) is

$$\ker = \frac{S_{r-1}^{p+1,q-1} + S_{r+1}^{p,q}}{dS_{r-1}^{p-r+1,q+r-2} + S_{r-1}^{p+1,q-1}}.$$

Next, the image of the left map of (7) is immediately

$$\mathrm{im} = \frac{dS_r^{p-r,q+r-1} + dS_{r-1}^{p-r+1,q+r-2} + S_{r-1}^{p+1,q-1}}{dS_{r-1}^{p-r+1,q+r-2} + S_{r-1}^{p+1,q-1}} = \frac{dS_r^{p-r,q+r-1} + S_{r-1}^{p+1,q-1}}{dS_{r-1}^{p-r+1,q+r-2} + S_{r-1}^{p+1,q-1}}$$

(as $S_r^{p-r,q-r+1}$ contains $S_{r-1}^{p-r+1,q+r-1}$).

Thus the cohomology of (7) is

$$\ker / \operatorname{im} = \frac{S_{r-1}^{p+1,q-1} + S_{r+1}^{p,q}}{dS_r^{p-r,q+r-1} + S_{r-1}^{p+1,q-1}} = \frac{S_{r+1}^{p,q}}{S_{r+1}^{p,q} \cap (dS_r^{p-r,q+r-1} + S_{r-1}^{p+1,q-1})}$$

where the equality on the right uses the fact that $dS_r^{p-r,q+r+1} \subset S_{r+1}^{p,q}$ and an isomorphism theorem. We thus must show

$$S_{r+1}^{p,q} \cap (dS_r^{p-r,q+r-1} + S_{r-1}^{p+1,q-1}) = dS_r^{p-r,q+r-1} + S_r^{p+1,q-1}.$$

However,

$$S_{r+1}^{p,q} \cap (dS_r^{p-r,q+r-1} + S_{r-1}^{p+1,q-1}) = dS_r^{p-r,q+r-1} + S_{r+1}^{p,q} \cap S_{r-1}^{p+1,q-1}$$

and $S_{r+1}^{p,q} \cap S_{r-1}^{p+1,q-1}$ consists of (p,q)-strips whose differential vanishes up to row $p+r$, from which $S_{r+1}^{p,q} \cap S_{r-1}^{p+1,q-1} = S_r^{p,q}$ as desired.

This completes the explanation of how spectral sequences work for a first-quadrant double complex. The argument applies without significant change to more general situations, including filtered complexes.

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