INTERSECTION THEORY CLASS 13

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CONTENTS

1. Where we are: Segre classes of vector bundles, and Segre classes of cones

We first defined *Segre class of vector bundles* over an arbitrary scheme X. If E is a vector bundle, we get an operator on class on X. We define it by projectivizing E, so we have a flat and proper morphism $\mathbb{P}E \to X$, pulling back α to $\mathbb{P}E$, capping with $\mathcal{O}(1)$ a certain number of times, and pushing forward.

Hence we get $s_i(E)\cap : A_k X \to A_{k-i} X$, and for example we checked the non-immediate fact that $s_0(E)$ is the identity. (Recall s_0 involved pulling back, capping with precisely $\operatorname{rank} E - 1$ copies of $\mathcal{O}(1)$, and then pushing forward.) Note that $s_k(E) = s_k(E \oplus \mathbf{1})$, as the Whitney product formula gives $s(E \oplus \mathbf{1}) = s(E)s(\mathbf{1}) = s(E)$.

We want to generalize this to cones. Here again is the definition of a *cone* on a scheme X. Let $S^\cdot = \oplus_{i \geq 0} S^i$ be a sheaf of graded $\mathcal{O}_X$-algebras. Assume $\mathcal{O}_X \to S^0$ is surjective, S^1 is coherent, and $S^\cdot$ is generated (as an algebra) by S^1. Then you can define $\underline{\operatorname{Proj}}(S^\cdot)$, which has a line bundle $\mathcal{O}(1)$. $\underline{\operatorname{Proj}}(S^\cdot) \to X$ is a projective (hence proper) morphism, but it isn't necessarily flat! (Draw a picture, where the cone has components of different dimension.) Flat morphisms have equidimensional fibers, and cones needn't have this.

A couple of important points, brought out by Joe and Soren. I've been imprecise with terminology. Although one often sees phrases such as "the cone is $C = \operatorname{Spec}(S^\cdot)$", we lose a little information this way; the cone should be defined to be the graded sheaf $S^\cdot$. The sheaf can be recovered from $C_X Y$ along with the action of the multiplicative group $\mathcal{O}_X^*$; the nth graded piece is the part of the algebra where the multiplicative group acts with weight n.

Example 1: say let E be a vector bundle, and $S^i = \operatorname{Sym}^i(E^\vee)$. Then $\underline{\operatorname{Proj}}\, S^\cdot = \mathbb{P}E$. *Example 2:* Say $T^i = \operatorname{Sym}^i(E^\vee \oplus \mathbf{1}) = S^i \oplus S^{i-1} z$, so (better) $T^\cdot = S^\cdot[z]$. Then $\underline{\operatorname{Proj}}\, T^\cdot = \mathbb{P}E$. *Example 3:*

Date: Wednesday, November 3, 2004.

$\underline{\mathrm{Proj}}(S^\cdot[z]) = C \coprod \underline{\mathrm{Proj}}(S^\cdot) = \underline{\mathrm{Spec}}\, S^\cdot \coprod \underline{\mathrm{Proj}}(S^\cdot)$. The argument is just the same. The right term is a Cartier divisor in class $\mathcal{O}_{\underline{\mathrm{Proj}}(S^\cdot[z])}(1)$. *Example 4:* The blow-up can be described in this way, and it will be good to know this. Suppose X is a subscheme of Y, cut out by ideal sheaf $\mathcal{I}$. (In our situation where all schemes are finite type, $\mathcal{I}$ is a coherent sheaf.) Then let $S^\cdot = \oplus_i \mathcal{I}^i$, where $\mathcal{I}$ is the ith power of the ideal $\mathcal{I}$. ($\mathcal{I}^0$ is defined to be $\mathcal{O}_X$.) Then $\mathrm{Bl}_X Y \cong \underline{\mathrm{Proj}}\, S^\cdot$. A short calculation shows that the exceptional divisor class is $\mathcal{O}(-1)$. The *exceptional divisor* turns out to be $\underline{\mathrm{Proj}} \oplus \mathcal{I}^n/\mathcal{I}^{n+1}$. (Note that this is indeed a graded sheaf of algebras.) As $\oplus \mathcal{I}^n \to \oplus \mathcal{I}^n/\mathcal{I}^{n+1}$ is a surjective map of rings, this indeed describes a closed subscheme of the blow-up. (Remember this formula — it will come up again soon!)

So the same construction of Segre classes of vector bundles doesn't work: there is no flat pullback to $\underline{\mathrm{Proj}}(S^\cdot)$. So what do we do?

Idea (slightly wrong): We can't pull classes back to $\underline{\mathrm{Proj}}(S^\cdot)$. But there is a natural class up there already: the fundamental class. So we define

$$s(C) \stackrel{?}{=} q_*(\sum_{i\geq 0} c_1(\mathcal{O}(1))^i \cap [\underline{\mathrm{Proj}}\, C])$$

where q is the morphism $\underline{\mathrm{Proj}}\, C \to X$. Instead, as Segre class of vector bundles are stable with respect to adding trivial bundles, we define

$$s(C) := q_*(\sum_{i\geq 0} c_1(\mathcal{O}(1))^i \cap [\underline{\mathrm{Proj}}(C \oplus \mathbf{1})])$$

where q is the morphism $\underline{\mathrm{Proj}}(C \oplus \mathbf{1}) \to X$. Why is adding in this trivial factor the right thing to do? Partial reason: if C is the 0 cone, i.e. $S^i = 0$ for $i > 0$, then $\underline{\mathrm{Proj}}\, C$ is empty, but $\underline{\mathrm{Proj}}\, C \oplus \mathbf{1}$ is not; we get different answers. But if you add more $\mathbf{1}$'s, you will then get the same answer: $s(C \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}) = s(C)$.

(Exercise: show that $s(C \oplus \mathbf{1}) = s(C)$.)

Note: s has pieces in various dimensions.

Last time I proved:

Proposition. (a) If E is a vector bundle on X, then $s(E) = c(E)^{-1} \cap [X]$, where $c(E)$ is the total Chern class of X, $r = \mathrm{rank}(E)$. $c(E) = 1 + c_1(E) + \cdots + c_r(E)$. (I would write $s(E) = s(E) \cap [X]$, but the two uses of $s(E)$ are confusing!) This is basically our definition of Segre/Chern classes.

(b) Let $C_1, \dots, C_t$ be the irreducible components of C, m_i the geometric multiplicities of C_i in C. Then $s(C) = \sum_{i=1}^t m_i s(C_i)$. (Note that the C_i are cones as well, so $s(C_i)$ makes sense.) In other words, we can compute the Segre class piece by piece.

2. THE NORMAL CONE, AND THE SEGRE CLASS OF A SUBVARIETY

Let X be a closed subscheme of a scheme Y (not necessarily lci = local complete intersection), cut out by ideal sheaf $\mathcal{I}$.

$\mathcal{I}/\mathcal{I}^2$ is the conormal sheaf to X; it is a sheaf on X. (Why is it a sheaf on X? Locally, say $Y = \operatorname{Spec} R$, and $X = \operatorname{Spec} R/I$. Then this is the R-module I/I^2. The fact that I said that it is an R-module makes it a priori a sheaf on Y. But note that it is also an R/I module; the action of I on I/I^2 is the zero action.) If X is a local complete intersection (regular imbedding), then this turns out to be a vector bundle.

Consider $\sum_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}$. (Recall that $\underline{\mathrm{Proj}}$ of this sheaf gives us the exceptional divisor of the blow-up.) *Define* the normal cone $C = C_X Y$ by

$$C = \underline{\mathrm{Spec}} \sum_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

Define the *Segre class* of X in Y as the Segre class of the normal cone:

$$s(X, Y) = s(C_X Y) \in A_* X.$$

If X is regularly imbedded (=lci) in Y, then the definition of $s(X, Y)$ is

$$s(X, Y) = s(N) \cap [X] = c(N)^{-1} \cap [X].$$

The following geometric picture will come up in the central construction in intersection (the deformation to the normal cone). $X \times \mathbb{A}^1 \hookrightarrow Y \times \mathbb{A}^1$. Then blow up $X \times 0$ in $Y \times \mathbb{A}^1$. The ideal sheaf of $X \times 0$ is $\mathcal{I}[t]$, where t is the coordinate on $\mathbb{A}^1$. Thus the normal cone to $X \times 0$ in $Y \times \mathbb{A}^1$ is $C_X Y[t]$. Hence the exceptional divisor is $\underline{\mathrm{Proj}}(C_X Y[t])$ (draw a picture). Inside it is the Cartier divisor $t = 0$, which is $\underline{\mathrm{Proj}}(C_X Y)$.

3. SEGRE CLASSES BEHAVE WELL WITH RESPECT TO PROPER AND FLAT MORPHISMS

This is the key result of the chapter.

Proposition. Let $f : Y' \to Y$ be a morphism of pure-dimensional schemes, $X \subset Y$ a closed subscheme, $X' = f^{-1}(X)$ the inverse image scheme, $g : X' \to X$ the induced morphism.

(a) If f proper, Y irreducible, and f maps each irreducible component of Y' onto Y then

$$g_*(s(X', Y')) = \deg(Y'/Y) s(X, Y).$$

(b) If f flat, then

$$g^*(s(X', Y')) = s(X, Y).$$

Let me repeat why I find this a remarkable result. X' is a priori some nasty scheme; even if it is nice, its codimension in Y' isn't necessarily the same as the codimension of X in Y. The argument is quite short, and shows that what we've proved already is quite sophisticated.

As a special case, this result shows that Segre classes have a fundamental birational invariance: if $f : Y' \to Y$ is a birational proper morphism, and $X' = f^{-1}X$, then $s(X', Y')$ pushes forward to $s(X, Y)$.

Proof. Let me assume that Y' is irreducible. (It's true in general, and I may deal with the general case later.)

Let me first write the diagram on the board, and then explain it.

$$\begin{tikzcd}
 & \mathcal{O}_{\underline{\mathrm{Proj}}(C'\oplus\mathbf{1})}(1) = G^*\mathcal{O}_{\underline{\mathrm{Proj}}(C\oplus\mathbf{1})}(1) \arrow[dr] & & \\
 & \mathcal{O}_{\underline{\mathrm{Proj}}(C\oplus\mathbf{1})}(1) \arrow[dr] & \underline{\mathrm{Proj}}(C' \oplus \mathbf{1}) \arrow[r, "\text{Cartier div}"] \arrow[d, "G"] \arrow[ddll, "q'"'] & \mathrm{Bl}_{X'\times 0}(Y' \times \mathbb{A}^1) \arrow[d, "F"] \\
 & & \underline{\mathrm{Proj}}(C \oplus \mathbf{1}) \arrow[r, "\text{Cartier div}"] \arrow[ddll, "q"] & \mathrm{Bl}_{X\times 0}(Y \times \mathbb{A}^1) \\
X' \arrow[d, "g"] & & & \\
X & & &
\end{tikzcd}$$

We blow up $Y \times \mathbb{A}^1$ along $X \times 0$, and similarly for Y' and X'. The exceptional divisor of $\mathrm{Bl}_{X\times 0}(Y \times \mathbb{A}^1)$ is $\underline{\mathrm{Proj}}(C \oplus \mathbf{1})$, and similarly for Y' and X'. The universal property of blowing up $Y \times \mathbb{A}^1$ shows that there exists a unique morphism G from the top exceptional divisor to the bottom. Moreover, by construction, the exceptional divisor upstairs is the pullback of the exceptional divisor downstairs (that's the statement about the two $\mathcal{O}(1)$'s in the diagram). Let q be the morphism from the exceptional divisor $\underline{\mathrm{Proj}}(C \oplus \mathbf{1})$ to X, and similarly for q'. That square commutes: $q \circ G = g \circ q'$ (basically because that morphism G was defined by the universal property of blowing up).

Now $f_*[Y' \times \mathbb{A}^1] = d[Y \times \mathbb{A}^1]$ (where I am sloppily using the name f for the morphism $Y' \times \mathbb{A}^1 \to Y \times \mathbb{A}^1$). This is computed on a dense open set, so blow-up doesn't change this fact:

$$F_*[\mathrm{Bl}_{X'\times 0} Y' \times \mathbb{A}^1] = d[\mathrm{Bl}_{X\times 0} Y \times \mathbb{A}^1].$$

Now we've shown that proper pushforward commutes with intersecting with a (pseudo-)Cartier divisor. Hence

$$G_*[\underline{\mathrm{Proj}}(C' \oplus \mathbf{1})] = d[\underline{\mathrm{Proj}}(C \oplus \mathbf{1})].$$

Now I'm going to prove (a), and I'm going to ask you to prove (b) with me, so pay attention!

$$\begin{aligned}
g_* s(X', Y') &= g_* q'_* \left(\sum_i c_1(G^*(\mathcal{O}(1))^i \cap [\mathbb{P}(C' \oplus \mathbf{1})]) \right) \quad \text{(by def'n)} \\
&= q_* G_* \left(\sum_i c_1(G^*(\mathcal{O}(1))^i \cap [\mathbb{P}(C' \oplus \mathbf{1})]) \right) \quad \text{(prop. push. commute)} \\
&= q_* \left(\sum_i c_1((\mathcal{O}(1))^i \cap d[\mathbb{P}(C \oplus \mathbf{1})]) \right) \quad \text{(proj. form.)} \\
&\qquad\qquad \text{(i.e. } c_1 \text{ commutes with prop. pushforward)} \\
&= ds(X, Y) \quad \text{(by def'n)}
\end{aligned}$$

Now (b) is similar:

$$\begin{aligned}
g^* s(X, Y) &= g^* q_* \left(\sum_i c_1((\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus \mathbf{1})]) \right) \quad \text{(by def'n)} \\
&= q'_* G^* \left(\sum_i c_1((\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus \mathbf{1})]) \right) \quad \text{(push/pull commute)} \\
&= q'_* \left(\sum_i c_1((G^*\mathcal{O}(1))^i \cap G^*[\mathbb{P}(C \oplus \mathbf{1})]) \right) \\
&= s(X, Y) \quad \text{(by def'n)}
\end{aligned}$$

□

We immediately have:

Corollary. With the same assumptions as the proposition, if X' is *regular imbedded* (=lci) in Y' , with normal bundle N', then

$$g_*(c(N')^{-1} \cap [X']) = \deg(Y'/Y) s(X, Y).$$

If $X \subset Y$ is also regularly imbedded, with normal bundle N, then

$$g_*(c(N')^{-1} \cap [X']) = \deg(Y'/Y)(c(N)^{-1} \cap [X]).$$

To see why the first part might matter: Suppose $X \hookrightarrow Y$ is a very nasty closed immersion. Then blow up Y along X, to get Y' with exceptional divisor X'. Then X' *is* regularly imbedded (lci) in Y' — it is a Cartier divisor! This is the content of the next corollary.

Corollary. Let X be a open closed subscheme of a variety Y. Let $\tilde{Y}$ be the blow-up of Y along X, $\tilde{X} = \mathbb{P}C$ the exceptional divisor, $\eta : \tilde{X} \to X$ the projection. Then

$$\begin{aligned} s(X, Y) &= \sum_{k \geq 1} (-1)^{k-1} \eta_*(\tilde{X}^k) \\ &= \sum_{i \geq 0} \eta_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}C]) \end{aligned}$$

In that first equation, the term $\tilde{X}^k$ should be interpreted as the kth self intersection of the Cartier divisor $\tilde{X}$, also known as the exceptional divisor.

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