INTERSECTION THEORY CLASS 14

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CONTENTS

1. WHERE WE ARE: SEGRE CLASSES OF VECTOR BUNDLES, AND SEGRE CLASSES OF CONES

1.1. **Segre classes of cones.** Once again, the definition of a *cone* on a scheme X. Let $S^\cdot = \oplus_{i \geq 0} S^i$ be a sheaf of graded $\mathcal{O}_X$-algebras. Assume $\mathcal{O}_X \to S^0$ is surjective, S^1 is coherent, and $S^\cdot$ is generated (as an algebra) by S^1. I'm happy calling this the cone. $C = \underline{\mathrm{Spec}}\, S^\cdot$. $\underline{\mathrm{Proj}}(S^\cdot)$ has a line bundle $\mathcal{O}(1)$. (The "underline" under Spec and $\underline{\mathrm{Proj}}$ is meant to distinguish the "sheafy" version from the usual version of these constructions.) Define the *Segre class*

$$s(C) := q_*(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [\underline{\mathrm{Proj}}(C \oplus \mathbf{1})])$$

where q is the morphism $\underline{\mathrm{Proj}}(C \oplus \mathbf{1}) = \underline{\mathrm{Proj}}(S^\cdot[t]) \to X$.

If $X \hookrightarrow Y$ is a closed immersion of schemes, the *normal cone* is $\sum_{n=0}^{\infty} \mathcal{I}^n / \mathcal{I}^{n+1}$. The Segre class of X in Y is defined to be the Segre class of the normal cone. More on the normal cone shortly. Last day we finished proving:

Proposition ("functoriality of Segre classes of subschemes"). Let $f : Y' \to Y$ be a morphism of pure-dimensional schemes, $X \subset Y$ a closed subscheme, $X' = f^{-1}(X)$ the inverse image scheme, $g : X' \to X$ the induced morphism.

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(a) If f proper, Y irreducible, and f maps each irreducible component of Y' onto Y then

$$g_*(s(X',Y')) = \deg(Y'/Y)s(X,Y).$$

(b) If f flat, then

$$g^*(s(X',Y')) = s(X,Y).$$

2. What the "functoriality of Segre classes of subschemes" buys us

As a special case, this result shows that Segre classes have a fundamental birational invariance: if $f : Y' \to Y$ is a birational proper morphism, and $X' = f^{-1}X$, then $s(X',Y')$ pushes forward to $s(X,Y)$.

From (a), we immediately have:

Corollary. With the same assumptions as the proposition, if X' is *regular imbedded* (=lci) in Y' , with normal bundle N', then

$$g_*(c(N')^{-1} \cap [X']) = \deg(Y'/Y)s(X,Y).$$

If $X \subset Y$ is also regularly imbedded, with normal bundle N, then

$$g_*(c(N')^{-1} \cap [X']) = \deg(Y'/Y)(c(N)^{-1} \cap [X]).$$

To see why the first part might matter: Suppose $X \hookrightarrow Y$ is a very nasty closed immersion. Then blow up Y along X, to get Y' with exceptional divisor X'. Then X' *is* regularly imbedded (lci) in Y' — it is a Cartier divisor! This is the content of the next corollary.

Corollary. Let X be a closed subscheme of a variety Y. Let $\tilde{Y}$ be the blow-up of Y along X, $\tilde{X} = \mathbb{P}C$ the exceptional divisor, $\eta : \tilde{Y} \to Y$ the projection. Then

$$\begin{aligned} s(X,Y) &= \sum_{k\geq 1} (-1)^{k-1}\eta_*(\tilde{X}^k) \\ &= \sum_{i\geq 0} \eta_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}C]) \end{aligned}$$

In that first equation, the term $\tilde{X}^k$ should be interpreted as the kth self intersection of the Cartier divisor $\tilde{X}$, also known as the exceptional divisor. In other words, it should be interpreted as meaning the second line.

2.1. **The multiplicity of a variety along a subvariety.**

We'll now define them multiplicity of a scheme Y along a subvariety X. (As a special case, this will define the multiplicity of a variety at a closed (=old-fashioned) point. That special case is a fundamental commutative algebra notion due to Samuel.) If the general point of X is a smooth point of Y, we'll get 1. Definition: $s(X,Y) \in A_*X$. Then $s(X,Y) = e_XY[X]$ + lower order terms. e_XY is the multiplicity.

Useful exercise: What is the multiplicity of $(0,0)$ in the cusp $y^2 - x^3$? Here the characteristic is not 2 or 3. (Answer: 2. Hint: blow this up. The blow-up is $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2-x^3)$ given by $t \mapsto (t^2, t^3)$.)

Example. If $\operatorname{codim}(X,Y) = n > 0$, define the multiplicity $e_X Y$ as follows. Let q be the projection $\underline{\operatorname{Proj}}(C \oplus \mathbf{1}) \to X$ and p be the projection $\underline{\operatorname{Proj}}\, C \to X$.

$$\begin{aligned} e_X Y[X] &= q_*(c_1(1))^n \cap [\underline{\operatorname{Proj}}(C \oplus \mathbf{1})] \\ &= p_*(c_1(\mathcal{O}(1))^{n-1} \cap [\underline{\operatorname{Proj}}\, C] \\ &= (-1)^{n-1} p_*(\tilde{X}^n) \end{aligned}$$

Here $\tilde{X}$ is the exceptional divisor of the blow-up.

Back to the multiplicity of a variety at a closed (=old-fashioned) point: Let A be the local ring of Y at our point, $\mathfrak{m}$ the maximal ideal of A, $A/\mathfrak{m} = k$. *Fact:*

$$\dim_k \left(\sum_{i=1}^{t} \mathfrak{m}^{i-1}/\mathfrak{m}^i \right) = l_A(A/\mathfrak{m}^t)$$

is a polynomial of degree $n = \dim Y$ in t for $t \gg 0$, whose leading term is $(e_X Y) t^n / n!$. This even works at a (non-closed) point; just take A to be the local ring of Y along X, and $n = \operatorname{codim}(X,Y)$.

Useful exercise: See that this works in for the cusp point (the previous useful exercise). Note that as a vector space $k[x,y]/(y^2-x^3) = \oplus_{n \geq 0, n \neq 1} kt^n$; note that the $n=2$ term is kx, the $n=3$ term is ky, the $n=4$ term is kx^2, the $n=5$ term is kxy, and the $n=6$ term is $kx^3 = ky^2$.

3. Deformation to the normal cone

We next come to the central construction. There's not much for us to do here, as we've built up all the necessary machinery, and even seen the construction.

Here is the main goal. Suppose $X \to Y$ is a closed immersion of schemes. The idea is that $C = C_X Y$ "looks like Y near X"; it "is like a tubular neighborhood". But it is nicer than Y near X; in particular it is a cone.

Goal: We will define a *specialization homomorphism* $\sigma : A_k Y \to A_k C$.

I'll try to give you an intuitive idea for what this means. (Try it.)

3.1. **The construction.** Here's how we do it. Let me set some notation. If $W \hookrightarrow Z$ is a closed immersion, recall that $\operatorname{Bl}_W Z$ is the blow-up of Z along W. For the purposes of the next few lectures, let $E_W Z$ be the exceptional divisor, and let $\mathcal{I}_W Z$ be the ideal sheaf. Then recall:

- $\operatorname{Bl}_W Z = \underline{\operatorname{Proj}} \oplus (\mathcal{I}_W Z)^n$
- $E_W Z = \underline{\operatorname{Proj}} \oplus (\mathcal{I}_W Z)^n / (\mathcal{I}_W Z)^{n+1}$

- $E_W Z \hookrightarrow Bl_W Z$ is a closed immersion, and describes $E_W Z$ as an effective Cartier divisor, in fact in class $\mathcal{O}_{\underline{\mathrm{Proj}}\oplus(\mathcal{I}_W Z)^n}(1)$. The closed immersion is visible at the level of graded algebras

Suppose now that $X \hookrightarrow Y$ is a closed immersion. (Other notation: when Fulton says "imbedding", we will say "closed immersion".) Let's blow up $Y \times \mathbb{P}^1$ along $X \times 0$ and see what we get. (Here let t be a coordinate on $\mathbb{P}^1$. Notational caution: Fulton prefers to blow up $X \times \infty$.) We certainly have a morphism to $\mathbb{P}^1$:

$$\mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1) \to Y \times \mathbb{P}^1 \to \mathbb{P}^1.$$

Away from $t = 0$, the blow-up doesn't do anything: $\mathrm{Bl}_{X\times 0} Y|_{t\neq 0} = Y \times (\mathbb{P}^1 - 0)$.

So what is the fiber over $t = 0$? I claim it is the union of two things, that we can identify. One "piece" is $\mathrm{Bl}_X Y$, with exceptional (Cartier) divisor $E_X Y$. The other piece is $\underline{\mathrm{Proj}}(C_X Y \oplus \mathbf{1})$; this has a Cartier divisor "$\mathbb{P}(C_X Y \oplus \mathbf{1})$" $= C_X Y \oplus \mathbb{P} C_X Y \cong E_X Y$. We glue these two pieces together along $E_X Y$.

I want to convince you that we really get these two pieces. If you've never seen this before, I want to convince you that we get those two pieces, and you can be happy with that. At the end I'll explain how to verify that we get nothing else.

Consider the morphism $\mathrm{Bl}_{X\times 0} Y \times \mathbb{P}^1 \to Y \times \mathbb{P}^1$. Away from the $X \times 0$ on the target, this is an isomorphism. The exceptional divisor is

$$\begin{aligned} E_{X\times 0}\left(Y \times \mathbb{P}^1\right) &= \underline{\mathrm{Proj}} \oplus \left((I_{X\times 0} Y \times \mathbb{P}^1)^n / (I_{X\times 0} Y \times \mathbb{P}^1)^{n+1}\right) \\ &\cong \underline{\mathrm{Proj}}\left(\oplus (I_X Y)^n / (I_X Y)^{n+1}\right)[t] \\ &\cong \mathbb{P}(C_X Y \oplus \mathbf{1}). \end{aligned}$$

So we see the projective completion of the normal cone in this blow-up.

Let's next see the piece $\mathrm{Bl}_X Y$. Translation: we want a morphism $\mathrm{Bl}_X Y$ to $\mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1)$ that lies in the scheme-theoretic fiber $t = 0$, and we want this to be a closed immersion. I will just show you that the morphism exists; as usual we use the universal property. Consider the map $\mathrm{Bl}_X Y \to Y \times \mathbb{P}^1$ obtained via $\mathrm{Bl}_X Y \to Y \times 0 \hookrightarrow Y \times \mathbb{P}^1$. The pullback of $X \times 0$ is an effective Cartier divisor $E_X Y$. Thus by the universal property of blowing-up, we get a morphism $\mathrm{Bl}_X Y \to \mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1)$.

So I've given you an indication that we see both the projective completion of the normal cone, and $\mathrm{Bl}_X Y$, in the central fiber ($t = 0$) of $\mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1)$. How would you show that this is all we get, and that they are glued together along $E_X Y$? This is a local question, so we can take $Y = \mathrm{Spec}\, A$, and $X = \mathrm{Spec}\, A/I$. Then the question becomes completely explicit: $Y \times \mathbb{A}^1 = \mathrm{Spec}\, A[t]$. (We can work locally in $\mathbb{P}^1$ as well.) Then $\mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1)$ locally is $\mathrm{Proj} \oplus (I, t)^n$. We are interested in the fiber over $t = 0$, so we mod out by t:

$$\rho^{-1}(0) = \mathrm{Proj}\left(\left(\oplus (I, t)^n\right) / \left(t\left(\oplus (I, t)^n\right)\right)\right).$$

We want to show that this is the union of

$$E_{X\times 0}(Y \times \mathbb{P}^1) = \mathrm{Proj}\left(\oplus (I, t)^n / (I, t)^{n+1}\right)$$

and

$$\mathrm{Bl}_X Y = \operatorname{Proj} \oplus I^n$$

glued along

$$E_X Y = \operatorname{Proj} \oplus \left(I^n/I^{n+1}\right).$$

Consider ordered pairs of elements of the second and third graded rings, that are required to give the same element in the fourth graded ring. Show that this ring is the same as the first graded ring. Finally, realize that this algebraic statement is precisely the geometric statement you want to prove. (I'm not going to give the details.)

4. Specialization to the normal cone

Let $X \hookrightarrow Y$ be a closed subscheme of a scheme, and $C = C_X Y$ the normal cone to X in Y. Recall our goal: to define *specialization homomorphism* $\sigma : A_k Y \to A_k C$.

Let me now do it. Let $M^\circ = \mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1) - \mathrm{Bl}_X Y$. A picture is helpful here. Away from 0, M° is still $Y \times \mathbb{A}^1$. Over 0, the big blow-up was the projectivized completion of the normal cone $C_X Y \coprod \mathbb{P} C_X Y$ glued to $\mathrm{Bl}_X Y$ along $E_X Y = \mathbb{P} C_X Y$. We're throwing out $\mathrm{Bl}_X Y$, so the central fiber is now just the normal cone $C_X Y$. So we have really deformed Y to the normal cone. Hence this scheme M° is often called the "deformation to the normal cone". Let $i : C \hookrightarrow M^\circ$ be the closed immersion of the normal cone, and let $j : Y \times (\mathbb{P}^1 - 0) \hookrightarrow M^\circ$ be the open immersion of the complement.

Consider the following diagram:

$$\begin{tikzcd}
A_{k+1} C \arrow[r, "i_*"] & A_{k+1} M^\circ \arrow[r, "j^*"] \arrow[d, "i^*", "\text{Gysin map for divisors}"'] & A_{k+1}(Y \times \mathbb{A}^1) \arrow[r] & 0 \\
& A_k C & A_k Y. \arrow[u, "\sim"] &
\end{tikzcd}$$

The top row is the excision exact sequence. The right column is flat pullback and is an isomorphism, as flat pullback to the total space of a line bundle is always an isomorphism. The left column is the Gysin pullback map to divisors.

Now we have shown $i^* i_* : A_{k+1} C \to A_k C$ is the same as capping with c_1 of the normal (line) bundle to the divisor C in M°. But in this case the normal line bundle is trivial: it is the pullback of the normal bundle to $t = 0$ in $\mathbb{P}^1$. Thus $i^* i_* = 0$. Hence $A_{k+1} M^\circ \to A_k C$ descends to a map $A_{k+1}(Y \times \mathbb{A}^1) \to A_k C$, and hence we get a map $\sigma : A_k Y \to A_k C$, which is what we wanted! Here's the final diagram:

$$\begin{tikzcd}
A_{k+1} C \arrow[r, "i_*"] \arrow[dr, "i^* i_* = 0"'] & A_{k+1} M^\circ \arrow[r, "j^*"] \arrow[d, "i^*"] & A_{k+1}(Y \times \mathbb{A}^1) \arrow[r] \arrow[dl, "\therefore"'] & 0 \\
& A_k C & A_k Y. \arrow[u, "\sim"] \arrow[l, "\therefore \sigma"] &
\end{tikzcd}$$

Remark We could define this morphism more explicitly as follows. Define $\sigma : Z_k Y \to Z_k C$ by $\sigma([V]) = [C_{V \cap X} V]$ where V is a subvariety of Y. (Extend this to $Z_k Y$ by linearity.) Note that $C_{V\cap X}V \hookrightarrow C_XY$, so this makes sense. **Proposition.** This descends to the morphism $A_kY \to A_kC$ I just defined. *Sketch of proof.* In the bottom row of that last big diagram, it suffices to verify that $[V] \mapsto [C_{V\cap X}V]$. Hence it suffices to show that in the "southwest" morphism in the big diagram (marked "$\therefore$"), $[V \times \mathbb{A}^1]$ maps to $[C_{V\cap X}V]$. We take the subvariety $V \times \mathbb{A}^1 \hookrightarrow Y \times (\mathbb{P}^1 - 0)$, take its closure in M°, and intersect with the Cartier divisor $(t = 0) = C$. We can do this explicitly locally on Y, using $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} A/I$, etc.; I'll omit this since I don't think we'll need this fact.

Corollary. Suppose $i : X \hookrightarrow Y$ is a locally complete intersection (regular imbedding) of codimension d, with normal bundle N. Define the *Gysin homomorphism* or *Gysin pullback*

$$i^* : A_kY \to A_{k-d}X$$

as the composition

$$A_kY \xrightarrow{\sigma} A_kN \xrightarrow{s_N^*} A_{k-d}X.$$

4.1. **Gysin pullback for local complete intersections.** We already had defined the Gysin pullback or Gysin homomorphism in the case where Y is a vector bundle over X. This extends it to when "Y looks like a vector bundle over X". Notice that the two definitions agree; one needs to check that the normal cone to a the zero section of a vector bundle is the vector bundle itself (which is true). Also, we showed that the Gysin pullback for vector bundles satisfied all sorts of nice properties; if we show that σ satisfies these nice properties too, then we'll know it for Gysin pullbacks to local complete intersections.

Note: $i^*i_*(\alpha) = c_d(N) \cap \alpha$. Reason: we know this for vector bundles.

Note also: If Y is purely n-dimensional, notice that $i^*[Y] = [X]$. Because $\sigma[Y] = [C]$, and $s_N^*[C] = [X]$.

4.2. **Intersection products on smooth varieties!** If X is an n-dimensional variety which is smooth over the ground field, then the diagonal morphism $\Delta : X \to X \times X$ is a local complete intersection of codimension n. Then we get an intersection product on A_*X!

$$A_pX \otimes A_qX \xrightarrow{\times} A_{p+q}(X \times X) \xrightarrow{\Delta^*} A_{p+q-n}X.$$

(Notice that we don't need X to be proper!)

I should probably be a bit clearer about that first map, which might reasonably be called $\boxtimes$. (You can see a discussion in Chapter 1 if you want.) Here's what we need: consider the map

$$Z_pX \otimes Z_qY \xrightarrow{\times} Z_{p+q}(X \times Y)$$

defined on varieties by $[V] \times [W] = [V \times W]$, and defined generally by linearity. (We'll take $X = Y$, but we might as well do this in some generality.)

Lemma. If $\alpha \sim 0$ (or, symmetrically, $\beta \sim 0$) then $\alpha \times \beta \sim 0$.

(This is Prop. 1.10 (a) in the book.)

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