# INTERSECTION THEORY CLASSES 20 AND 21: BIVARIANT INTERSECTION THEORY

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CONTENTS



## 1. WHAT WE'RE DOING THIS WEEK

In this final week of class, I'll describe bivariant intersection theory, covering much of Chapter 20. Again, you should notice that given chapters 1 through 6, we can comfortably jump into chapter 20.

Suppose $f : X \to Y$ is any morphism. Throughout today and Wednesday's lectures, we'll use the following notation. Suppose we are given any $Y' \to Y$. Define $X' = X \times_Y Y$, so we have a fiber square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Recall that the final fundamental intersection construction we came up with was the following. Suppose $f$ is a local complete intersection of codimension $d$ (or more generally a local complete intersection morphism). Then we defined

$$f^! : A_k Y' \to A_{k-d} X'$$

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for all $Y' \to Y$. These Gysin pullbacks were well-behaved in all ways, and in particular compatible with proper pushforward, flat pullback, and intersection products.

An earlier example was that of a flat pullback; if $f$ if flat of relative dimension $n$, then $f'$ is too, and we got $f^* : A_k Y' \to A_{k-n} X'$, which again behaves well with respect to everything else.

We'll now generalize this notion. Define a *bivariant class* for *any* $f$ (not just lci) as follows. It is a collection of homomorphisms $A_k Y' \to A_{k-p} X'$ for all $Y' \to Y$, all $k$, again compatible with pushforward, pullback, and intersection products. We'll call the group of such things $A^p(f : X \to Y)$.

We'll see that the group $A^{-k}(X \to pt)$ will be (canonically) isomorphic to $\boxed{A_k X}$. We'll see that $A^k(id : X \to X)$ is a ring, which Fulton calls the *cohomology* group; I might call the resulting ring the Chow ring. We'll denote this by $\boxed{A^k X}$.

The ring structure is a product of the form $A^p X \otimes A^q X \to A^{p+q} X$. We'll define more generally

$$\boxed{A^p(f : X \to Y) \otimes A^q(g : Y \to Z) \to A^{p+q}(g \circ f : X \to Z)}.$$

We'll prove *Poincare duality* when $X$ is *smooth*: $\boxed{A^* X \cong A_* X}$ (as rings — recall we defined a ring structure on the latter). We'll define proper pushforward and pullback operations for bivariant groups. Basically, they'll behave the way you'd expect from homology and cohomology. This will give a cap product $A^* X \times A_* X \to A_* X$. **Alarming fact:** This ring is apparently not known to be commutative in general, because the argument requires resolution of singularities. (It is known to be commutative in characteristic 0, and for smooth things in positive characteristic, and a few more things.) I think it should be possible to show that the ring is commutative in general using technology not available when this theory was first developed, using Johan de Jong's "alteration theorem" in positive characteristic. If you would like to patch this hole, then come talk to me.

Okay, let's get started. Today I'll outline the results, and prove a few things; Wednesday I'll prove some more things.

## 2. Precise statements

Let $f : X \to Y$ be a morphism. For each $g : Y' \to Y$, form the fiber square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}.$$

A bivariant class $c$ in $A^p(f : X \to Y)$ is a collection of homomorphisms

$$c_g^{(k)} : A_k Y' \to A_{k-p} X'$$

for all $g : Y' \to Y$, and all $k$, compatible with proper pushforwards, flat pullbacks, and intersection products. I'll make that precise in a moment, by stating 3 conditions explicitly. But first I want to show you that you've seen this before in several circumstances.

Example 1. If $f$ is a local complete intersection, or more generally an lci morphism, we've defined $f^!$. This gives some inkling as to why we want to deal with maps $X \to Y$. We could have just had a class on $Y'$, but we have more refined information; we have a class on $X'$, that pushes forward to the more refined class on $Y$.

Example 2. If $f : X \to Y$ is the identity, and $V$ is a vector bundle on $Y$, then the Chern classes are of this form: $\alpha \mapsto (g^* c_k(V)) \cap \alpha$.

Example 3 (which generalizes further): pseudodivisors. Let $L$ be a line bundle on $Y$, and $X$ the zero-scheme of a section $s$ of $L$. ($s$ might cut out a Cartier divisor, i.e. $X$ will contain no associated points of $Y$; at the other extreme, $s$ might be $0$ everywhere.) Then we defined "capping with a pseudo-divisor": $f^* A_k Y \to A_{k-1} X$. Because pseudodivisors "pull back", $X'$ is a pseudodivisor on $Y'$ (with corresponding line bundle $g^* L$, and corresponding section $g^* s$), so we get a map $f^* A_k Y' \to A_{k-1} X'$, and this behaves well respect to everything else.

So we're creating a machine that in some sense incorporates most things we've done so far.

Here are the conditions.

$(C_1)$: If $h : Y'' \to Y$ is proper, $g : Y' \to Y$ is arbitrary, and one forms the fiber diagram

(1)
$$\begin{tikzcd} X'' \arrow[r, "f''"] \arrow[d, "h'"'] & Y'' \arrow[d, "h"] \\ X' \arrow[r, "f'"'] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd},$$

then for all $\alpha \in A_k Y''$,

$$c_g^{(k)}(h_* \alpha) = h'_* c_{gh}^{(k)} \alpha$$

in $A_{k-p} X'$.

$(C_2)$: If $h : Y'' \to Y$ is flat of relative dimension $n$, and $g : Y' \to Y$ is arbitrary, and one forms the fiber diagram (1), then, for all $\alpha \in A_k Y'$,

$$c_{gh}^{(k+n)}(h^* \alpha) = h'^* c_g^{(k)} \alpha$$

in $A_{k+n-p} X''$.

$(C_3)$: If $g : Y' \to Y$, $h : Y' \to Z'$ are morphisms, and $i : Z'' \to Z'$ is a local complete intersection of codimension $e$, and one forms the diagram

(2)
$$\begin{tikzcd} X'' \arrow[r, "f''"'] \arrow[d, "i''"'] & Y'' \arrow[r, "h'"'] \arrow[d, "i'"] & Z'' \arrow[d, "i"] \\ X' \arrow[r, "f'"'] \arrow[d, "g'"'] & Y' \arrow[r, "h"'] \arrow[d, "g"] & Z' \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

then, for all $\alpha \in A_k Y'$,

$$i^! c_g^{(k)}(\alpha) = c_{gi'}^{(k-e)}(i^! \alpha)$$

in $A_{k-p-e} X''$.

2.1. **Basic operations and properties.** Here are some basic operations on bivariant Chow groups $A^*(X \to Y)$.

$(P_1)$ *Product*: For all $f : X \to Y$, $g : Y \to Z$, we have

$$\cdot : A^p(f : X \to Y) \otimes A^q(g : Y \to Z) \to A^{p+q}(gf : X \to Z).$$

It is pretty immediate to show this: given any $Z' \to Z$, form the fiber diagram

(3)
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[r, "g'"] \arrow[d] & Z' \arrow[d] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z. \end{tikzcd}$$

If $\alpha \in A_k Z$, then $d(\alpha) \in A_{k-1} Y'$ and $c(d(\alpha)) \in A_{k-q-p} X'$, so we define $c \cdot d$ by

$$c \cdot d(\alpha) := c(d(\alpha)).$$

$(P_2)$ *Pushforward*: Let $f : X \to Y$ be proper, $g : Y \to Z$ any morphism. Then there is a homomorphism ("proper pushforward"):

$$f_* : A^p(gf : X \to Z) \to A^p(g : Y \to Z).$$

Again, it's straightforward: given $Z' \to Z$, form the fiber diagram (3). If $c \in A^p gf)$, and $\alpha \in A_k(Z')$, then $c(\alpha) \in A_{k-p}(X')$. Since $f'$ is proper, $f'_*(c(\alpha)) \in A_{k-p}(Y')$. Define $f_*(c)$ by

$$f_*(c)(\alpha) = f'_*(c(\alpha)).$$

$(P_3)$: *Pullback* (not necessarily flat!!): Given $f : X \to Y$, $g : Y_1 \to Y$, form the fiber square

(4)
$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

For each $p$ there is a homomorphism

$$g^* : A^p(f : X \to Y) \to A^p(f_1 : X_1 \to Y_1).$$

Again, we just follow our nose. Given $c \in A^p(f)$, $Y' \to Y_1$, then composing with $g$ gives a morphism $Y' \to$. Therefor $c(\alpha) \in A_{k-p}(X')$, $X' = X \times_Y Y' = X_1 \times_{Y_1} Y'$. Set

$$g * (c)(\alpha) = c(\alpha).$$

Here are seven more axioms, which can also be easily verified.

$(A_{pr})$ *Associativity of products*. If $c \in A(X \to Y)$, $d \in A(Y \to Z)$, $e \in A(Z \to W)$, then

$$(c \cdot d) \cdot e = c \cdot (d \cdot e) \in A(X \to W).$$

$(A_{pf})$ *Functoriality of proper pushforward.* If $f : X \to Y$ and $g : Y \to Z$ are proper, $Z \to W$ arbitrary, and $c \in A(X \to W)$, then

$$(gf)_*(c) = g_*(f_*c) \in A(Z \to W).$$

$(A_{pb})$ *Functoriality of pullbacks.* If $c \in A(X \to Y)$, $g : Y_1 \to Y$, $h : Y_2 \to Y_1$, then

$$(gh)^*(c) = h^*g^*(c) \in A(X \times_Y Y_2 \to Y_2).$$

$(A_{prpf})$ *Product and pushforward commute.* If $f : X \to Y$ is proper, $Y \to Z$ and $Z \to W$ are arbitrary and $c \in A(X \to Z)$, $d \in A(Z \to W)$, then

$$f_*(c) \cdot d = f_*(c \cdot d) \in A(Y \to W).$$

$(A_{prpb})$ *Product and pullback commute.* If $c \in A(f : X \to Y)$, $d \in A(Y \to Z)$, and $g : Z_1 \to Z$ is a morphism, form the fiber diagram

$$\begin{tikzcd} X_1 \arrow[r, "f'"] \arrow[d] & Y_1 \arrow[r] \arrow[d, "g'"] & Z_1 \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \arrow[r] & Z. \end{tikzcd}$$

Then

$$g^*(c \cdot d) = g'^*(c) \cdot^* (d) \in A(X_1 \to Z_1).$$

$(A_{pfpb})$ *Proper pushforward and pullback commute.* If $f : X \to Y$ is proper, $Y \to Z$, $g : Z_1 \to Z$, and $c \in A(X \to Z)$ are given, then, with notation as in the preceding diagram

$$g^*f_*c = f'_*(g^*c) \in A(Y_1 \to Z_1).$$

$(A_?)$: *Projection formula.* Given a diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] & \\ X \arrow[r, "f"] & Y \arrow[r, "h"] & Z. \end{tikzcd}$$

with $g$ proper, the square a fiber square, and $c \in A(X \to Y)$, $d \in A(Y' \to Z)$, then

$$c \cdot g_*(d) = g'_*(g^*(c) \cdot d) \in A(X \to Z).$$

## 3. Proving things

Let $S = \operatorname{Spec} K$, where $K$ is some base field. There is a canonical homomorphism

$$\phi : A^{-p}(X \to S) \to A_p(X)$$

given by $c \mapsto c([S])$.

**Proposition.** This is an isomorphism.

*Proof.* We will define the inverse morphism. Given $a \in A_p(X)$, define a bivariant class $\psi(a) \in A^{-p}(X \to S)$ as follows: for any $Y \to S$, and any $\alpha \in A_k Y$, define

$$\psi(a)(\alpha) = a \times \alpha \in A_{p+k}(X \times_S Y).$$

(Here $a \times \alpha$ is the exterior product.) Since exterior products are compatible with proper pushforward, flat pullback, and intersections, $\psi(a)$ is a bivariant class.

Let's check that this really is an inverse to $\phi$. $\psi(a)([S]) = a$ immediately, so $\phi \circ \psi$ is the identity. To show that $\psi \circ \phi$ is the identity, we have to show that $c(\alpha) = \phi(c) \times \alpha \in A_{k+p}(X \times_S Y)$ for all $\alpha \in A_k Y$. By compatibility with pushforward, we can assume $\alpha = [V]$, and $V = Y$ a variety of dimension $k$:

$$\begin{tikzcd}
X \times_S V \arrow[r] \arrow[d] & V \arrow[d, "\text{cl. imm.}"] \\
X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

Then $\alpha = p^*[S]$, where $p : V \to S$ is the morphism from $V$ to $S$. Since $c$ commutes with flat pullback,

$$c(\alpha) = c(p^*[S]) = p^*c([S]) = \phi(c) \times [V]$$

as desired. □

### 3.1. **The Chow ("cohomology") ring.**

Define $A^p X := A^p(\mathrm{id} : X \to X)$. We have a cup product. We also have an element $1 \in A^0 X$, which acts as the identity. We have a cap product

$$\cap : A^p X \times A_q X \to A_{q-p} X$$

determined by

$$A^p(X \to X) \times A^{-q}(X \to S) \to A^{-(q-p)}(X \to S)$$

which makes $A_* X$ into a left $A^* X$-module. All of this follows formally from our axioms.

## 4. Things you might want to be true

4.1. **Poincare duality. Theorem.** Let $Y$ be a smooth, purely $n$-dimensional scheme (variety) — not necessarily proper (compact).
(a) The canonical homomorphism $\cap[Y] : A^p Y \to A_{n-p} Y$ is an isomorphism.
(b) The ring structure on $A^* Y$ is compatible with that defined on $A_* Y$ earlier. More generally, if $f : X \to Y$ is a morphism, $\beta \in A^* Y$, $\alpha \in A_* X$, then the class $f^*(\beta) \cap \alpha \in A_* X$ coincides with that constructed earlier.

We'll show something more general.

**Theorem.** Let $g : Y \to Z$ be a smooth morphism of relative dimension $n$, and let $[g] \in A^{-n}(g : Y \to Z)$ be the bivariant class corresponding to "flat pullback". Then for any morphism $f : X \to Y$ and any $p$,

$$\cdot[g] : A^p(f : X \to Y) \to A^{p-n}(gf : X \to Z)$$

is an isomorphism.

In general, if $f : X \to Y$ is a flat morphism, or a local complete intersection, or a local complete intersection morphism, the (flat or Gysin) pullback we've defined earlier defines a bivariant class, which we'll denote $[f]$. ($[f^*]$ might be better.) Fulton calls this bivariant class a *canonical orientation*. I'm not sure of the motivation for this terminology, so I'll avoid it.

*Proof.* We'll define the inverse homomorphism

$$A^{p-n}(gf : X \to Z) \to A^p(f : X \to Y).$$

Consider the fiber diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\gamma"'] & Y \arrow[d, "\delta"] & \\
X \times_Z Y \arrow[r, "f'"] \arrow[d, "p'"'] & Y \times_Z Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

where $\delta$ is the diagonal map, and $p$ and $q$ are the first and second projections. Here $\gamma$ is the "graph" of the morphism $X \to Y$ over $Z$. Define

$$L : A^{p-n}(gf : X \to Z) \to A^P(X \to Y)$$

by $L(c) = [\gamma] \cdot g^*(c)$. Notice that $\gamma$ and $\delta$ is a local complete intersection of codimension $n$, with $f'^*[\delta] = [\gamma]$. (This requires a check in the case of $\gamma$.)

Notice that $p' \circ \gamma : X \to X$ and $q \circ \delta : Y \to Y$ are both the identity morphisms (on $X$ and $Y$ respectively).

Let's verify that $L$ and "multiplication by $[g]$" are inverse homomorphisms. (This is easier to understand if you see someone pointing at diagrams!) First,

$$\begin{aligned} L(c)\cdot[g] &= [\gamma]\cdot(g^*[c]\cdot[g]) \quad (\text{axiom } (A_{pr})) \\ &= [\gamma]\cdot[p']\cdot c \quad (\text{axiom } (C_2)) \\ &= [p'\circ\gamma]\cdot c = 1\cdot c = c \quad (\text{axiom } (A_{pr})) \end{aligned}$$

Second,

$$\begin{aligned} L(c\cdot[g]) &= f'^*[\delta]\cdot p^*(c)\cdot g^*[g] \quad (\text{axioms } (A_{prpb}), (A_{pr})) \\ &= (p\circ\delta)^*(c)\cdot[\delta][q] \quad (\text{axiom } (C_2)) \\ &= c\cdot[\delta\circ q] = c\cdot 1 = c \quad (\text{axiom } (A_{pr})). \end{aligned}$$

□

4.2. **Chern classes commute with all bivariant classes.** Put another way, any operation which commutes with proper pushforward, pullback, and intersections, automatically commutes with Chern classes. Precisely:

**Proposition.** Let $c \in A^q(f : X \to Y)$, $Y' \to Y$, $\alpha \in A_k(Y')$, $E$ a vector bundle on $Y'$. Then

$$c(c_p(E)\cap\alpha) = c_p(f'^*E)\cap c(\alpha) \in A_{k-q-p}X'$$

where $f' : X' = X \times_Y Y' \to Y$.

*Proof.* Recall our definition of Chern classes. They are certain polynomials in Segre classes. Segre classes are defined using operations of the form $\alpha \mapsto p_*(c_1(\mathcal{O}(1))^i \cap p^*\alpha)$, and since $c$ commutes with $p_*$ and $p^*$, we just have to show that $c$ commutes with $c_1(L)\cap$, where $L$ is a line bundle on $Y'$. We may assume $\alpha = [V]$. Because $c$ commutes with proper pushforward, we may assume $V = Y'$. Let $L = \mathcal{O}(D)$, $D$ a Cartier divisor on $V$.

We can replace $V$ by $V'$, where $V' \to V$ is proper and birational, so we may assume $D = D_1 - D_2$, where $D_1$ and $D_2$ are effective. (Recall our trick in chapter 2: it isn't true that a Cartier divisor is a difference of two effective Cartier divisors, but we can do a clever blow-up and turn it into a difference of two effective Cartier divisors.) Let $i : D \hookrightarrow V$ be the inclusion. Then we've shown that $c_1(L)\cap\alpha = i_* i^! \alpha$, and since $c$ commutes with $i_*$ and $i^!$, $c$ commutes with $c(L)$. □

4.3. **Bivariant classes vanish in dimensions that you'd expect them to. Proposition.** Let $f : X \to Y$ be a morphism. Let $m = \dim Y$, and let $n$ be the largest dimension of any fiber $f^{-1}y$, $y \in Y$.

$$A^p(f : X \to Y) = 0 \quad \text{if } p < -n \text{ or } p > m.$$

(Think about why this is what you'd expect!) In particular, for any $X$, $A^pX = 0$ unless $0 \le p \le \dim X$.

(Proof omitted.)

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