INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 11

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Problem sets can be handed in and picked up at the end of class.

Discuss late policy.

Aside. There were several points last day that I wish I'd described differently. Here is one. (Again, please tune out.)

Given a ring R, and a prime ideal $\mathfrak{p}$, I described the residue field as: (i) localize the ring R at $\mathfrak{p}$, and then mod out by the maximal ideal: $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. You can quickly check that this is equivalent to: (ii) Mod out by $\mathfrak{p}$ to get a domain, and then take the fraction field.

The second point of view is often much more convenient. For example, an element f of R, which you should think of as being a function, vanishes at a point $\mathfrak{p}$ if $f \in \mathfrak{p}$. (Describe this in the case of $\mathfrak{p} = (y - x^2)$ in $\overline{k}[x, y]$.) In particular, we get the fact that $V(f)$ is the set of prime ideals containing f, and more generally $V(I)$ is the set of prime ideals containing I.

Back to our regularly-scheduled program.

1. Products

We know what we mean when we discuss products of sets. We'll now define products more generally.

Categorical products. Suppose you have two sets X and Y. Then the product set Z has two natural "projection" maps p_X and p_Y to X and Y respectively. Moreover, if you have any other set W with maps to both X and Y, then there

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is a unique map $W \to Z$ such that the maps to X and Y can be obtained by composition with the projections.

Definition. Suppose X and Y are objects in a category. Then a *product* of X and Y is the data of another object Z, along with projection maps $p_X : Z \to X$ and $p_Y : Z \to Y$, such that for any other object W, maps from W to Z correspond to the data of maps from W to X, and maps from W to Y. Equivalently,

$$Mor(Z, W) \to Mor(Z, X) \times Mor(Z, Y)$$

is a bijection.

You should check that this works with sets, i.e. in the category of sets.

Remark. Then the product is defined up to unique isomorphism.

Products in the category of topological spaces. Define product topology. Explain why this is called the product topology: this is a product in the category of topological spaces.

Products in the cateogry of prevarieties?

If there is a product of X and Y in the category of prevarieties, then the points of the product are the products of the points. Reason: take Z to be a point.

To save you the suspense: products indeed exist, and we'll construct them.

Topologies will be weird. It will turn out that $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, but the Zariski topology on $\mathbb{A}^2$ isn't the same as the product of the Zariski topologies on the factor (*Exercise*).

1.1. **Products in the category of affine varieties, and in the category of varieties.** Recall that the category of affine varieties (over $\overline{k}$) is just the category of "nice" rings, with the arrows reversed, where nice means "finitely generated algebra over $\overline{k}$ that is an integral domain".

Let X and Y be affine varieties (over $\overline{k}$). Let's diagram-chase.

If we have some "nice" ring $A(W)$, the question is: given maps (ring morphisms over $\overline{k}$) $A(X) \to A(W)$ and $A(Y) \to A(W)$, is there some ring $A(X \times Y)$ along with morphisms $A(X), A(Y) \to A(X \times Y)$ through which this has to factor?

Answer: Yes, $A(X) \otimes_{\overline{k}} A(Y)$, with the morphism $A(X) \to A(X) \otimes_{\overline{k}} A(Y)$ given by $x \mapsto x \otimes 1$, and similarly for $A(Y)$. (Remind them what tensor product is; show that tensor product has this property.)

All that's left to show is that the tensor product is also "nice".

Finitely-generated is easiest: if $f_1, \ldots, f_m$ are generators for $A(X)$ and $g_1, \ldots,$ g_n are generators for $A(Y)$, then $f_i \otimes 1$ and $1 \otimes g_j$ generate the tensor product.

Then we invoke a fact from commutative algebra: Let R and S be integral domains over $\overline{k}$. Then $R \otimes_{\overline{k}} S$ is also an integral domain.

Remarks. Hence $\otimes$ is the *coproduct* in the category of rings. And we've also shown that the product in the category of affine schemes is given by tensor product.

Theorem. Let X and Y be affine varieties. Then there is product prevariety $Z := X \times Y$; it is affine with coordinate ring $A(X) \otimes_{\overline{k}} A(Y)$.

Proof. We've shown that this is the product in the category of affine varieties; what's different is that W may now be *any* prevariety. Suppose we have morphism $W \to X$ and $W \to Y$. How do we get the morphism $W \to Z$?

(This is where patching arguments make life really easy.) Cover W with affines U_i. We have maps $U_i \to X, Y$ so (as Z is a product in the category of affine varieties) we get map $U_i \to Z$.

We now need to show that these maps "glue together" to give us a map from W to Z, i.e. that if you consider the overlap $U_{ij} := U_i \cap U_j$, then the induced morphism $U_{ij} \to U_i \to Z$ is the same as the induced morphism $U_{ij} \to U_j \to Z$.

Cover U_{ij} with affines V_k; then again because Z is a product in the category of affine varieties, and we have a morphism $V_k \to X, Y$, there is only one morphism $V_k \to Z$ compatible with them. □

Examples. It isn't hard working out what products of affine varieties actually are. For example, $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$. If X is the affine variety in $\mathbb{A}^2$ cut out by $v^3 + w^3 = 1$, and Y is the affine variety in $\mathbb{A}^3$ cut out by $xyz = 3$, then $X \times Y$ is the affine variety in $\mathbb{A}^5$ cut out by $v^3 + w^3 = 1$ and $xyz = 3$. Keep these examples in mind.

Looking at the product of affines more closely.

i) Topology. Note that the distinguished opens on $X \times Y$ are of the form $D(\sum f_i \otimes g_i)$ where $f_i \in A(X)$, $g_i \in A(Y)$; this gives the beginning of an insight as to why the topology on the product is not the product of the topologies.

ii) Function field. Note also that the function field of the product $k(X \times Y)$ is the *quotient field* of the tensor product of the function fields: $k(X) \otimes_{\overline{k}} k(Y)$.

iii) Stalks of the structure sheaf. Let's interpret the local ring $\mathcal{O}_{X \times Y,(x,y)}$ in terms of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$. Let the maximal ideals of these local rings be $\mathfrak{m}_x$ and $\mathfrak{m}_y$ respectively. (This is concrete! The local rings are quotients of polynomials where the denominator *doesn't* vanish at the point, and the maximal ideal is where the numerator *does* vanish at the point.)

Lemma. Then $\mathcal{O}_{X\times Y,(x,y)}$ is the localization of $\mathcal{O}_{X,x}\otimes_{\overline{k}}\mathcal{O}_{Y,y}$ at the maximal ideal $\mathfrak{m}_x\mathcal{O}_{Y,y}+\mathfrak{m}_y\mathcal{O}_{X,x}$.

We'll need this technical fact (which isn't too hard) only once, in a few minutes, and then you can forget about it.

If you try to parse what this means, you'll realize that it's reasonable.

Proof. Here's a real explanation. $\mathcal{O}_{X\times Y,(x,y)}$ is the localization of $A(X)\otimes_{\overline{k}}A(Y)$ at the maximal ideal of all functions vanishing at (x,y). Now $A(X)\times A(Y)\subset\mathcal{O}_{X,x}\otimes_{\overline{k}}\mathcal{O}_{Y,y}\subset\mathcal{O}_{X\times Y,x\times y}$, so we can describe the last term by localizing at the maximal ideal of all functions vanishing at (x,y), so we need to check that this really is $\mathfrak{m}_x\mathcal{O}_{Y,y}+\mathfrak{m}_y\mathcal{O}_{X,x}$. One inclusion is clear. Conversely, if

$$h=\sum f_i\otimes_{\overline{k}}g_i\in\mathcal{O}_{X,x}\otimes_{\overline{k}}\mathcal{O}_{Y,y}$$

vanishes at (x,y), with $f_i(x)=a_i$ and $g_i(y)=b_i$, then

$$h=h-\sum a_ib_i=\sum(f_i-a_i)\otimes g_i+\sum a_i\otimes(g_i-b_i)$$

which lies in $\mathfrak{m}_x\otimes\mathcal{O}_{Y,y}+\mathcal{O}_{X,x}\otimes\mathfrak{m}_y$ as desired. □

Theorem. Let X and Y be prevarieties over $\overline{k}$. Then they have a product.

Proof. Let's first build a reasonable candidate from affines, and then later check that it really is a product. We'll start with the set, add the topology, and finally the structure sheaf.

For the underlying set, we just take the product set.

For the topology, we give a base: For all open affines $U\subset X$, $V\subset Y$, and all finite sets of elements $f_i\in A(X)$, $g_i\in A(Y)$, consider $D(\sum f_i\otimes g_i)\subset U\times V$. (Small check required to make sure that this really gives the topology you want on $U\times V$.)

Now for the structure sheaf. Let K be the quotient field of $k(X)\otimes_{\overline{k}}k(Y)$, which is our candidate for the function field of the product. For $x\in X$, $y\in Y$, let $\mathcal{O}_{X\times Y(x,y)}\subset K$ be the localization of $\mathcal{O}_{X,x}\otimes_{\overline{k}}\mathcal{O}_{Y,y}$ at the ideal $\mathfrak{m}_x\otimes\mathcal{O}_{Y,y}+\mathcal{O}_{X,x}\otimes\mathfrak{m}_y$, and define

$$\mathcal{O}_{X\times Y}(U)=\bigcap_{(x,y)\in U}\mathcal{O}_{X\times Y,(x,y)}.$$

This is a sheaf of functions, which coincides on each $U\times V$ (U, V affine) with the structure sheaf of $\mathcal{O}_{U\times V}$ (by our analysis of the affine case).

Then this is a prevariety! (Check: covered by finitely many affines, and connected.)

Next to check: that this prevariety is a product. We have our projection maps $X \times Y$ to X and Y. Suppose we're given some morphisms $f_X : W \to X$, $f_Y : W \to Y$. Then we automatically get a map of sets $W \to X \times Y$, as the underlying set o f $X \times Y$ is just the product of the underlying sets of X and Y. We just need to check that this is a morphism. To do that, we can cover X and Y by affines U_i and V_j respectively, and cover W with $f_X^{-1}U_i \cap f_Y^{-1}V_j$; we need only show that $f_X^{-1}U_i \cap f_Y^{-1}V_j \to U_i \times V_j \subset U \times V$ is a morphism; but it is because we've already shown that $U_i \times V_j$ is a product in the category of prevarieties when U_i and V_j are affines. □

It isn't hard to check:

Corollary. If U is an open subprevariety of X, then $U \times Y$ is an open subprevariety of $X \times Y$. If Z is a closed subprevariety of X, then $Z \times Y$ is a closed subprevariety of $X \times Y$.

Remark. If $\overline{k} = \mathbb{C}$, then you might reasonably have the classical topology in mind. It is true that if X and Y are complex varieties, then the classical topology on $X \times Y$ is the same as the product of the classical topologies.

2. Coming soon

1. products of projective prevarieties are projective prevarieties; the Segre map
2. rational maps; open sets of definition; birational maps