INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 12

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Contents

Problem sets back at end.

1. Products of projective varieties; the Segre map

Theorem. The product of two projective varieties is a projective variety.

Proof. Since a closed subvariety of a projective variety is a projective variety, it is enough to show that $\mathbb{P}^m \times \mathbb{P}^n$ is a projective variety. So we're done, modulo the following lemma.

Lemma. $\mathbb{P}^m \times \mathbb{P}^n$ is a projective variety.

Describe the image of $(x_0; \dots; x_m) \times (y_0; \dots; y_n)$ in $\mathbb{P}^{(m+1)(n+1)-1}$: $z_{ij} = x_i y_j$. This is called the *Segre embedding.*

The image is in the locus of rank 1 matrices; in fact it is *precisely* the rank 1 matrices.

Make clear that you can recover the points of $\mathbb{P}^m$ and $\mathbb{P}^n$.

You have defining equations: all the 2×2 minors. Hence the image V is a projective prevariety.

We've described the map on points one way. Describe in another way.

Then you have to check on patches. Not hard, but I'll omit it; you can read about it in any of the references. The details of the proof are important, but more

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important is the geometric insight into what's going on. So let's go through an example. $\square$

Remark (another parallel between affine and projective prevarieties). Recall that if X and Y are affine, with coordinate rings $A(X)$ and $A(Y)$, then the coordinate ring of $X \times Y$ is $A(X) \otimes_{\overline{k}} A(Y)$. Something similar happens with projective prevarieties too. Suppose X and Y are projective, and lie in $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively, so X has *graded* coordinate ring $R(X) = \overline{k}[x_0, \ldots, x_m]/I(X)$, where $I(X)$ is a *homogeneous* ideal, and $R(Y) = \overline{k}[y_0, \ldots, y_n]/I(Y)$ similarly. Then under the Segre embedding (so $X \times Y \subset \mathbb{P}^{(m+1)(n+1)-1}$), $R(X \times Y) \cong R(X) \otimes R(Y)$ where the grading behaves well under tensor product (i.e. the tensor product of the ith graded piece of $R(X)$ and the jth graded piece of $R(Y)$ lies in the $(i+j)$th graded piece of $R(X \times Y)$.

Exercise. Prove that this is the case if $X = \mathbb{P}^m$ and $Y = \mathbb{P}^n$. Caution: I've shown that the product is cut out by the equations of the 2×2 minors, but I didn't show that the ideal $I(X \times Y)$ is generated by the 2×2 minors, although that's true. (Analogy: In $\mathbb{A}^2$, the y-axis Y is cut out by the equation $x^2 = 0$, but $I(Y)$ isn't generated by x^2; it's generated by x.)

1.1. $\mathbb{P}^1 \times \mathbb{P}^1$ **and the smooth quadric surface.** When you do the numerology with $m = n = 1$, we see that $\mathbb{P}^1 \times \mathbb{P}^1$ maps into $\mathbb{P}^3$, and it is given by a single equation $wx - yz = 0$.

Draw a picture.

Remark. Almost all quadric surfaces look the same. Hence we know the (classical) topology of almost all quadric surfaces: they are products of 2-spheres.

1.2. **Rulings of the smooth quadric surface.** Show them the lines in the real picture. We'll see these algebraically. First of all, a line in $\mathbb{P}^3$ is the intersection of two distinct hyperplanes, e.g. $w = x = 0$. They are isomorphic to $\mathbb{P}^1$, e.g for $w = x = 0$, the isomorphism is given by $(0; 0; y; z) \leftrightarrow (y; z)$. (For hyperplanes with uglier coefficients, just change coordinates!)

Give the projection to the first $\mathbb{P}^1$, and to the second $\mathbb{P}^1$: $(w_0; x_0; y_0; z_0) \mapsto (w_0; y_0)$ or $(z_0; x_0)$ is the map to the first $\mathbb{P}^1$, and $(w_0; x_0; y_0; z_0) \mapsto (w_0; z_0)$ or $(y_0; x_0)$ is the map to the second $\mathbb{P}^1$.

The two one-parameter family of lines: first note that the Segre map is

$$(a; b) \times (c; d) \mapsto (ac; bd; ad; bc).$$

The first family of lines is: fix $(a; b) = (a_0; b_0)$, and consider: $(a_0c; b_0d; a_0d; b_0c)$; this is the line that is the intersection of $b_0w = a_0x$ and $b_0y = a_0z$. The second is similar; just switch the roles of $(a; b)$ and $(c; d)$.

2. Defining varieties

Definition. A prevariety X is a *variety* if for all prevarieties Y and for all morphisms f and g from Y to X, the locus where they agree $\{y \in Y | f(y) = g(y)\}$ is a closed subset of Y.

This is often called the *separatedness* condition. Mumford calls this the *Hausdorff* axiom, because it is the analogue of the Hausdorff condition in the definition of a manifold.

Note that the line with the doubled origin is not a variety.

Remark. An open (resp. closed) subprevariety of a variety is a variety.

Remark. An affine variety is a variety. (Sorry for nasty notation!) Reason: Suppose X is affine, and Y is any prevariety, and f, g are two morphisms $Y \to X$. Then the subset of Y where $f(y) = g(y)$, $\{y \in Y | f(y) = g(y)\}$ is as follows.

$x_1 = x_2$ in affine X iff for all regular functions $s \in A(X)$, $s(x_1) = s(x_2)$. Hence $\{y \in Y | f(y) = g(y)\}$ is the locus where all regular functions $s(f(y)) - s(g(y))$ vanish (where s runs through all of $A(X)$). This is a closed set.

Remark. We will soon see that projective prevarieties are varieties.

Special case: if $Y = X \times X$, and f and g are the projections. $\Delta(X)$ is the locus where f and g agree. Hence if X is separated, then $\Delta(X)$ is closed in $X \times X$.

This special case is all you need to check:

Proposition (Criterion for separatedness). A prevariety X is a variety iff $\Delta(X)$ is closed in $X \times X$.

Proof. Neat trick, which is a recurring theme. Suppose you have two f and g, which induce a morphism $(f, g) : Y \to X \times X$. Then

$$\{y \in Y | f(y) = g(y)\} = (f, g)^{-1}(\Delta(X)).$$

□

Exercise. Show that the line with the doubled origin is not separated, using this condition.

Proposition (Another criterion for separatedness). Let X be a prevariety. Assume that for all $x, y \in X$ there is an open affine U containing both x and y. Then X is a variety.

Proof. We use the definition. Consider two functions $f, g : Y \to X$, and let Z be the locus where they agree $Z = \{y \in Y | f(y) = g(y)\}$. Let z be in the closure of Z, and let $x_1 = f(z)$, $x_2 = g(z)$. We want to show that Z is closed, so we want to show that $x_1 = x_2$. By assumption, there is an open affine $V \subset X$ containing x_1 and x_2. Let $U = f^{-1}(V) \cap g^{-1}(V)$; it is an open neighbourhood of Z. Then consider the "restricted morphisms" $f, g : U \to V$; now V is affine (hence a variety), so

$$Z \cap U = \{y \in U | f(y) = g(y)\}$$

is closed in U. Thus $z \in Z \cap U$, and Z is indeed closed. □

Corollary. Every quasiprojective prevariety X (i.e. open subset of a projective prevariety) is a variety.

Proof. As every projective prevariety X is a closed subprevariety of $\mathbb{P}^n$ (for some n, by definition), we just need to show that $\mathbb{P}^n$ is a variety. So given any two points $y, z \in \mathbb{P}^n$, we just need to find an affine open containing both. Consider any hyperplane H not meeting y or z; there are lots! (Instead of proving this, let me just convince you that it is obvious by example. If $y = (1;0;0)$ and $z = (0;1;1)$, take the hyperplane $x_1 - x_2 = 0$.) The complement of a hyperplane is affine (proved earlier), so we're done. □

Another nice property of varieties: the intersection of any two affine opens is another affine open. I don't foresee using this, so I won't prove it, but you can find a proof in Mumford (p. 55) or Hartshorne (Exercise II.4.4).

This isn't a criterion (as it also holds for the line with the doubled origin), but it can be strengthed a little into a criterion.

Here's a prevariety that doesn't have this property: the *plane* with the doubled origin. The intersection of the two elements of the "obvious" affine cover is the plane minus the origin, which you've shown earlier is not affine.

3. Rational maps

I said a few words about rational maps, the topic we'll address on Thursday.

We can reinterpret the definition of separatedness as follows. Suppose I'm thinking of a morphism $f : Y \to X$, where X is a *variety.* And suppose I tell you what the morphism is on a non-empty open set $U \subset Y$, i.e. I tell you $f|_U : U \to X$. Then there is only one way for you to recover the "full" morphism f. Because if you have two different morphisms f_1 and f_2 extending f, then you have two morphisms $f_1, f_2 : Y \to X$ which agree on a dense open set (the set U; recall that dense means that the closure of U is Y), and agree on a closed set (as X is separated). Hence they have to agree everywhere.

Coming soon: completeness (roughly, compactness), dimension, smoothness. Then we can talk about curves.