# INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 14

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Problem sets back at end.

If pages 26–27 of Eisenbud-Harris are missing in your copy, let me know; Alex Ghitza has kindly made a bunch of copies.

Come by and ask me questions!

Last time I said that we would call a dominant rational map $Y \dashrightarrow X$ *finite* if the induced morphism of function fields $k(X) \dashrightarrow k(Y)$ is a finite extension. That's not really in keeping with standard usage, so let me redefine it as *generically finite*.

## 1. Dimension

I'll start by reviewing what I mentioned last time on dimension.

But first, let me make a few algebraic remarks, on things such as transcendence degrees of finitely-generated field extensions. You should think about the statements I make; I think you'll find them all believable. Of course, one needs to properly prove everything. If you've already seen these remarks, great; if you're taking Commutative Algebra, that's great too; and otherwise, you should convince yourself that the statements are reasonable, and treat them as black boxes.

*Date*: Tuesday, October 26, 1999.

1.1. **Last time.** **Definition.** If $X$ is a prevariety defined over $\overline{k}$, define $\dim X = tr.d._{\overline{k}} k(X)$. If $Z$ is a closed subset of $X$, then $Z$ has *pure dimension* $r$ if each of its components has dimension $r$. A variety of dimension 1 is a *curve*, a variety of dimension 2 is a *surface*, a variety of dimension $n$ is an $n$-fold.

*Observation.* If $U \subset X$ is a nonempty open subset, then $k(U) = k(X)$, so $\dim U = \dim X$.

I mentioned an algebraic fact, which I proved (albeit imperfectly).

**Lemma.** Let $R$ be an integral domain over $\overline{k}$, $\mathfrak{p} \subset R$ a prime ideal. Then $tr.d._{\overline{k}} R \geq tr.d._{\overline{k}} R/\mathfrak{p}$, with equality iff $P = \{0\}$ or both sides are infinite.

Geometrically, this translates into:

**Proposition.** If $Y$ is a closed subprevariety of $X$, then $\dim Y < \dim X$.

Summary of proof: can reduce to an affine open meeting $Y$ by earlier observation, then use lemma.

**Definition.** Call the difference the *codimension* of $Y$ in $X$.

So for example, if the codimension is 1, there are no other subvarieties in between $Y$ and $X$.

I then quoted a result from commutative algebra:

**Theorem (Krull's Hauptidealsatz, or Principal Ideal Theorem).** Suppose $R$ is a finitely generated integral domain over $\overline{k}$, $f \in R$, $\mathfrak{p}$ a minimal prime of $(f)$ (i.e. minimal among the prime ideals containing it). Then if $f \neq 0$, $tr.d.R/\mathfrak{p} = tr.d.R - 1$. (Proof omitted.)

Let me repeat why, in geometrical situations, this is very reasonable. For example, let $R = \overline{k}[x, y, z]$, and let $f$ be some polynomial, say $xy(x - y^3 - z^4)$. Notice that the vanishing set $V(f) = \{f = 0\}$ consists of two planes and this weird surface. The minimal primes containing $(f)$ correspond to maximum subvarieties of $\mathbb{A}^3$ contained in the vanishing set of $f$, so there are 3 of them. Intuitively, it is reasonable that all 3 have dimension 2.

The immediate geometric consequence of this is:

**Theorem.** Let $X$ be a variety, $U \subset X$ open, $g \in \mathcal{O}_X(U)$ a regular function on $U$, $Z$ an irreducible component of $V(g) \cap U$. Then if $g \neq 0$, $\dim Z = \dim X - 1$.

($U$ is a red herring, and doesn't add any complexity to the proof. Essentially: $V(g)$ has pure codimension 1 for any non-zero $g \in \mathcal{O}_X(X)$.)

*Proof.* Take $U_0 \subset U$ to be any open affine meeting $Z$. Let $R = \mathcal{O}_X(U_0)$ be its coordinate ring, and $f = \operatorname{res}_{U,U_0} g \in R$ the restriction of our function $g$.

Then $Z \cap U_0$ (being irreducible) corresponds to some prime ideal $\mathfrak{p} \subset R$. $Z$ is a maximal irreducible subset of $V(g) \subset U$, so $Z \cap U_0$ is a maximal irreducible subset of $V(f) \subset U_0$, so $\mathfrak{p}$ is a minimal prime containing $f$, and we're in the situation of the Hauptidealsatz. □

Conversely, if $Z$ is an irreducible closed subset of $X$ of codimension 1, then for any open $U$ meeting $Z$ and for all non-zero functions $f$ on $U$ vanishing on $Z$, $Z \cap U$ is a component of $f = 0$.

**Corollary.** If $X$ is a variety, with subvariety $Z$ of codimension at least 2. Then there is a subvariety of $W$ of codimension 1 containing $Z$.

*Proof.* We can restrict to an affine open meeting $Z$, so without loss of generality assume $X$ is affine. Then $Z$ corresponds to a prime ideal $\mathfrak{p} \neq (0)$. Let $f$ be any nonzero element of $\mathfrak{p}$. Then all the components of $V(f)$ have codimension 1 (by the Principal Ideal Theorem). Hence $Z$ isn't a component, so it must be contained in one.

**Corollary.** If $X$ is a variety, and $Z$ is a maximal closed irreducible subset, smaller than $X$. Then $\dim Z = \dim X - 1$.

**Corollary.** Suppose $\emptyset \neq Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_r = X$ (where no $\subset$ is an inclusion) is a maximal chain of closed irreducible subsets of $X$. Then $\dim X = r$.

*Proof.* By induction on $X$. □

(Discuss a bit.)

*Remark.* This is a good initial definition to make for schemes; our original variety definition involves the field $\overline{k}$. *Exercise (possibly on the next problem set).* Check that $\operatorname{Spec} \mathbb{Z}$ is a curve.

**Corollary.** Let $X$ be a variety and let $Z$ be a component of $V((f_1, \dots, f_r))$, where $f_i \in \mathcal{O}_X(X)$. Then $\operatorname{codim} Z \leq r$.

(Explain.)

**Corollary.** Let $U$ be an affine variety, $Z$ a closed irreducible subset. Let $r = \operatorname{codim} Z$. Then there are $f_1, \dots, f_r$ in $A(U) = \mathcal{O}_U(U)$ such that $Z$ is a component of $V((f_1, \dots, f_r))$.

*Proof.* We did the case $r = 1$ earlier, and the proof is the same. □

1.2. **An algebraic definition of dimension.** Given a Noetherian local ring $\mathcal{O}$, you can attach an integer called the *Krull dimension*. (This will come up towards the end of this semester's commutative algebra class.)

It's defined as the length $r$ of the longest chain of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n = \mathfrak{m} \subset \mathcal{O}$. (Here $\mathfrak{m}$ is the maximal ideal.)

**Corollary.** The dimension of a variety $X$ is the Krull dimension of any of the stalks of the structure sheaf.

*Proof.* Fix a point $p$. Translating the Krull definition into geometry, we're asking about the longest chain of subvarieties of $X$ containing $p$. But we already know that this is $\dim X$. $\square$

### 1.3. **Other facts that are not hard to prove.** (Proofs are omitted for the first 2.)

**Proposition.** If $X$ is an affine variety with coordinate ring $R$, where $R$ is a unique factorization domain. Then every closed codimension 1 subset equals $V((f))$ for some $f \in R$.

**Proposition.** $\dim X \times Y = \dim X + \dim Y$.

**Proposition.** The Zariski topology on a dimension 1 prevariety is the cofinite topology.

(Explain.) We'll use this later when we study curves.

## 2. Non-singularity: a beginning

For a reference, see Hartshorne I.5 or Shafarevich, the start of Ch. II.

Some intuition, in the classical topology. Consider the plane curve $y = x + x^2$ in $\mathbb{C}^2$. Why is it smooth? What is the tangent line? (Discuss.)

What about $y = x + z + y^2$, $x = y + z + x^3$? How about $y = x + z + y^2$, $y = x + z + y^4$?

There are no constant terms. All we care about are the linear terms. In the ring $\overline{k}[x, y, z]$ with maximal ideal $\mathfrak{m}$, we care about $\mathfrak{m}/\mathfrak{m}^2$.

Classically, something is smooth of dimension $n$ if there is a local isomorphism with $\mathbb{C}^n$. I'll let you check that this is the same as the following definition.

**Definition.** Let $Y$ be a dimension $d$ affine variety in $\mathbb{A}^n$ (with coordinates $x_1$, ..., $x_n$. Suppose $Y$ is defined by equations $f_1$, ..., $f_t$ (i.e. $I(Y)$ is generated by the $f_i$; recall that any ideal in $\overline{k}[x_1, \ldots, x_n]$ is finitely generated!). Warning: We know that $t$ is at least the codimension $n-d$, but they two aren't necessarily equal! Then $Y \subset \mathbb{A}^n$ is *nonsingular at a point* $p \in Y$ if the rank of the *Jacobian matrix* $(\partial f_i / \partial x_j(p))_{i,j}$ is $n - d$.

I still must convince you that this is a reasonable definition, and in particular, that this agrees with the old definition. I'll let you check that in the classical topology, if $Y$ is nonsingular at $a$ in this sense, then there is a (classical) neighbourhood of $a$ isomorphic to an open set in $\mathbb{C}^n$. I'll do an example first.

Consider our earlier example, $x - y + z + y^2 = 0$, $-x + y + z + x^3 = 0$. So our point is the origin, $f_1 = x - y + z + y^2$, $f_2 = -x + y + z + x^3$. The two intersect in a curve, of dimension 1. The Jacobian matrix is

$$J = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Indeed the rank of the matrix $J$ is 3-1=2.

I argued (with too much hand-waving) that the implicit function theorem shows (in the case $\overline{k} = \mathbb{C}$) that the Jacobian condition implies that $X$ is a manifold at $p$; in the case above, consider the inverse of the projection $(x, y, z) \mapsto t = x + y + z$.

2.1. **A more algebraic definition of nonsingularity; hence nonsingularity is intrinsic.** You'd think that the nonsingularity of a point of $Y$ wouldn't depend on how you stuck it in an affine space, and you'd be right; but the above definition *does* depend on that, so it isn't clear that nonsingularity really is intrinsic. We'll show this now.

**Algebraic Definition.** Let $A$ be a noetherian local ring with maximal ideal $\mathfrak{m}$ and algebraically residue field $\overline{k}$. Then $A$ is a *regular local ring* if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

The reason this will be relevant is:

**Theorem.** Let $Y \subset \mathbb{A}^n$ be an affine variety. Let $p \in Y$ be a point. Then $Y$ is nonsingular at $p$ if and only if the local ring $\mathcal{O}_{Y,p}$ is a regular local ring.

We'll prove this soon.

Thus the concept of nonsingularity is intrinsic, so we can make the following definitions:

**Definition.** Let $Y$ be any prevariety. Then $Y$ is *nonsingular at a point* $p \in Y$ if the local ring $\mathcal{O}_{Y,p}$ is a regular local ring; otherwise it is *singular at* $p$. $Y$ is *nonsingular* if it is nonsingular at any point. Otherwise it is *singular*.

*Remark.* To check that something is singular, it is still easier to use the Jacobian definition. We make this more general definition because it is, well, more general.

**Theorem.** Let $A$ be the localization of $\overline{k}[x_1, \ldots, x_n]$ at the origin, so $A$ has dimension $n$. Then $\mathfrak{m}/\mathfrak{m}^2$ are naturally isomorphic to the vector space $(\alpha_1, \ldots, \alpha_n) \in \overline{k}^n$ (call it $V$), where points of the vector space can be associated with linear forms $\alpha_1 x_1 + \ldots \alpha_n x_n$. Hence $A$ is a regular local ring.

The proof will be enlightening (hopefully) for several reasons.

*Proof.* The morphism $V \to \mathfrak{m}/\mathfrak{m}^2$ is just given by

$$(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1 x_1 + \ldots \alpha_n x_n.$$

The morphism $\mathfrak{m}/\mathfrak{m}^2$ is given by

$$f \in A \mapsto (\partial f/\partial x_1(0,...,0), \ldots, \partial f/\partial x_n(0,...,0))$$

(where $f$ vanishes at the origin). To show that this is well-defined, we need to check that if $f \in \mathfrak{m}^2$, then $f \mapsto (0,...,0)$. But if $f \in \mathfrak{m}^2$, then $f = \sum_i g_i h_i$, where $g_i$, $h_i$ are in $\mathfrak{m}$. Then by the chain rule, $\partial(g_i h_i)/\partial x_1 = g_i \partial h_i/\partial x_1 + h_i \partial g_i/\partial x_1$, so indeed if $f \in \mathfrak{m}^2$ then $f \mapsto (0,...,0)$

Finally, we need to show that they compositions are the identity. The map $V \mapsto \mathfrak{m}/\mathfrak{m}^2 \mapsto V$ is the identity (show it), so what's left is to show that $\mathfrak{m}/\mathfrak{m}^2 \mapsto V \mapsto \mathfrak{m}/\mathfrak{m}^2$ is also the identity; this comes down to the fact that if $f \mapsto 0$, then $f \in \mathfrak{m}^2$. □

*Important observation.* Notice that the elements of $\mathfrak{m}/\mathfrak{m}^2$ are naturally identified with linear functions on $\mathbb{A}^n$. Now $\mathbb{A}^n$ can canonically be identified with the tangent space of $\mathbb{A}^n$ at the origin. So we've made an identification of $\mathfrak{m}/\mathfrak{m}^2$ with the *cotangent space* of $\mathbb{A}^n$ at the origin.

Based on this observation:

**Definition.** Let $(A, \mathfrak{m})$ be the local ring of a point $p \in Y$. Call $\mathfrak{m}/\mathfrak{m}^2$ the *Zariski co-tangent space* to $Y$ at $p$, and $(\mathfrak{m}/\mathfrak{m}^2)^*$ the *Zariski tangent space.*

*Exercise (that I will give on Thursday).* Suppose $f : X \to Y$ is a morphism of varieties, with $f(p) = q$. Show that there are natural morphisms $f^* : \mathfrak{m}_q/\mathfrak{m}_q^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2$ (the induced map on cotangent spaces) and $f^* : (\mathfrak{m}_p/\mathfrak{m}_p^2)^* \to (\mathfrak{m}_q/\mathfrak{m}_q^2)^*$ (the induced map on tangent spaces). (If you imagine what is happening on the level of tangent spaces and cotangent spaces of smooth manifolds, this is quite reasonable.) If $\phi$ is the vertical projection of the parabola $x = y^2$ onto the $x$-axis, show that the induced map of tangent spaces at the origin is the zero map.

Coming next: Examples. Checking for nonsingularity in projective space. The singular points form a closed subset.