INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 15

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Contents

Hand in problem sets at end.

Scheme aside: All varieties can be considered as schemes.

1. Non-singularity

Last day:

Definition. Let Y be a dimension d affine variety in $\mathbb{A}^n$ (with coordinates x_1, $\dots$, x_n. Suppose Y is defined by equations f_1, $\dots$, f_t (i.e. $I(Y)$ is generated by the f_i; recall that any ideal in $\overline{k}[x_1, \dots, x_n]$ is finitely generated!). Warning: We know that t is at least the codimension $n-d$, but they two aren't necessarily equal! Then $Y \subset \mathbb{A}^n$ is *nonsingular at a point* $p \in Y$ if the rank of the *Jacobian matrix* $(\partial f_i / \partial x_j(p))_{i,j}$ is $n-d$.

Remark. Derivatives are just "formal", i.e. $dx^n/dx = nx^{n-1}$, even in characteristic p; there are no limits here.

1.1. A more algebraic definition of nonsingularity; hence nonsingularity is intrinsic.

Algebraic Definition. Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and algebraically residue field $\overline{k}$. Then A is a *regular local ring* if $\dim_{\overline{k}} \mathfrak{m}/\mathfrak{m}^2 = \dim A$. *I should have made clear that the dimension here is just the dimension of* $\mathfrak{m}/\mathfrak{m}^2$ *as a* $\overline{k}$*-vector space.*

The reason this definition will be relevant is:

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Theorem (*). Let $Y \subset \mathbb{A}^n$ be an affine variety. Let $p \in Y$ be a point. Then Y is nonsingular at p if and only if the local ring $\mathcal{O}_{Y,p}$ is a regular local ring. *Leave on board until proof is complete.*

I promised you a proof, and I'll delay the proof for a few minutes after I summarize last day some more.

Hence the concept of nonsingularity is intrinsic, so we can make the following definitions:

Definition. Let Y be any prevariety. Then Y is *nonsingular at a point* $p \in Y$ if the local ring $\mathcal{O}_{Y,p}$ is a regular local ring; otherwise it is *singular at* p. Y is *nonsingular* if it is nonsingular at any point. Otherwise it is *singular*.

Remark. To check that something is singular, it is still easier to look at an affine cover and use the Jacobian definition.

Theorem (**). Let A be the localization of $\overline{k}[x_1, \ldots, x_n]$ at the origin, so A has dimension n. Then $\mathfrak{m}/\mathfrak{m}^2$ is naturally isomorphic to the vector space $(\alpha_1, \ldots, \alpha_n) \in \overline{k}^n$ (call it V), where points of the vector space can be associated with linear forms $\alpha_1 x_1 + \ldots \alpha_n + x_n$. Hence A is a regular local ring.

Remark. $\mathfrak{m}/\mathfrak{m}^2$ is clearly basis-free.

Recap of proof of Theorem (**): The proof identified the vector space V of "hyperplanes through the origin" $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ with $\mathfrak{m}/\mathfrak{m}^2$. The map from left to right is easy; the map from right to left is given by

$$f \in A \mapsto (\partial f/\partial x_1(0, ..., 0), \ldots, \partial f/\partial x_n(0, ..., 0))$$

(where f vanishes at the origin). We checked that this is well-defined, i.e. if $f \in \mathfrak{m}^2$, then this is the 0-map; this was the chain rule.

Proof of Theorem (*). You'll notice the similarity to the above proof.

Let p be a point $(a_1, \ldots, a_n)$ in $\mathbb{A}^n$, and let $\mathfrak{a} = (x_1 - a_1, \ldots, x_n - a_n)$ be the maximal ideal corresponding to p in $A = \overline{k}[x_1, \ldots, x_n]$. We define a linear map $\theta : A \to \overline{k}^n$ by

$$\theta(f) = \left(\frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_n}(p) \right)$$

for any $f \in A$. Notice that this is exactly the same map as in the proof about $\mathbb{A}^n$. (You should really think of the $\overline{k}^n$ as being the space of linear forms vanishing at the origin.) As we said before, θ induces an isomorphism $\theta' : \mathfrak{a}/\mathfrak{a}^2 \to \overline{k}^n$.

Now let's bring Y into the picture. Let $\mathfrak{b}$ be the ideal of Y in A, and let $f_1, \ldots, f_t$ be a set of generators of $\mathfrak{b}$. Then the rank of the Jacobian matrix $J = (\partial f_i/\partial x_j)(p)_{i,j}$ is just the dimension of $\theta(\mathfrak{b})$ as a subspace of $\overline{k}^n$. Using the isomorphism θ', this is the same as the dimension of the subspace $(\mathfrak{b} + \mathfrak{a}^2)/(\mathfrak{a}^2)$ of $\mathfrak{a}/\mathfrak{a}^2$.

(You should check: R is a ring, $\mathfrak{m}$ is a maximal ideal, $R_\mathfrak{m}$ localization. Then $R/\mathfrak{m} \cong R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} \cong \overline{k}$, $R/\mathfrak{m}^2 \cong R_\mathfrak{m}/\mathfrak{m}^2 R_\mathfrak{m}$, and $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}R_\mathfrak{m}/\mathfrak{m}^2 R_\mathfrak{m}$. Exercise on Problem Set 8.)

Now the local ring $\mathcal{O}_{Y,p}$ of p on Y is obtained by modding out by $\mathfrak{b}$ and localizing at the maximal ideal $\mathfrak{a}$. Thus if $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{Y,p}$, then $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{a}/(\mathfrak{b}+\mathfrak{a}^2)$. Counting dimensions of vector spaces, $\dim \mathfrak{m}/\mathfrak{m}^2 + rkJ = n$.

Now the dimension of the local ring $\mathcal{O}_{Y,p}$ (as a ring) is the dimension of Y (as a variety), so $\mathcal{O}_{Y,p}$ is regular if and only if $\dim \mathfrak{m}/\mathfrak{m}^2 = r$. But this is equivalent to $rkJ = n - r$, which says that p is a nonsingular point of Y. $\square$

Important observation from Theorem (**). Notice that the elements of $\mathfrak{m}/\mathfrak{m}^2$ are naturally identified with linear functions on $\mathbb{A}^n$. Now $\mathbb{A}^n$ can canonically be identified with the tangent space of $\mathbb{A}^n$ at the origin. So we've made an identification of $\mathfrak{m}/\mathfrak{m}^2$ with the *cotangent space* of $\mathbb{A}^n$ at the origin.

Based on this observation we made the following definition:

Definition. Let $(A, \mathfrak{m})$ be the local ring of a point $p \in Y$. Call $\mathfrak{m}/\mathfrak{m}^2$ the *Zariski co-tangent space* to Y at p, and $(\mathfrak{m}/\mathfrak{m}^2)^*$ the *Zariski tangent space.*

Exercise. Suppose $f : X \to Y$ is a morphism of varieties, with $f(p) = q$. Show that there are natural morphisms $f^* : \mathfrak{m}_q/\mathfrak{m}_q^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2$ (the induced map on cotangent spaces) and $f^* : (\mathfrak{m}_p/\mathfrak{m}_p^2)^* \to (\mathfrak{m}_q/\mathfrak{m}_q^2)^*$ (the induced map on tangent spaces). (If you imagine what is happening on the level of tangent spaces and cotangent spaces of smooth manifolds, this is quite reasonable.) If ϕ is the vertical projection of the parabola $x = y^2$ onto the x-axis, show that the induced map of tangent spaces at the origin is the zero map.

1.2. **Examples.** It isn't hard to check for singular points, especially on hypersurfaces in $\mathbb{A}^n$. For example, consider the plane curve $y^2 = x^3 - x^2$. (Do it.)

Exercise. Hartshorne Ex. I.5.1. Find the singular points, and sketch the following plane curves: (a) $x^2 = x^4 + y^4$, (b) $xy = x^6 + y^6$, (c) $x^3 = y^2 + x^4 + y^4$, (d) $x^2y + xy^2 = x^4 + y^4$. Hartshorne Ex. I.5.2, some two-dimensional examples.

Hypersurfaces in projective space are also easy, because there is a trick.

Exercise. Suppose the characteristic of $\overline{k}$ is 0. Suppose a hypersurface $Y \subset \mathbb{P}^n$ is given by $f(x_0, \ldots, x_n) = 0$. Show that the locus of points $p \in \mathbb{P}^n$ where $\partial f/\partial x_i(p) = 0$ for all i are precisely the singular points of Y. (In particular, if $\partial f/\partial x_i(p) = 0$ for all i, then $f(p) = 0$, i.e. $p \in Y$! To see why, calculate $\sum_i \partial f/\partial x_i$.)

As an example, consider $y^2z - x^3 = 0$. The only singular point is $(x; y; z) = (0; 0; 1)$.

Scheme examples.

Exercise. (a) Show that both $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec}\mathbb{Z}[i]$ are nonsingular curves. (b) Let $\mathfrak{m} = (1+i)$ in $\mathbb{Z}[i]$. Then under the map $f : \operatorname{Spec}\mathbb{Z}[i] \to \mathbb{Z}$, $f(\mathfrak{m}) = (2)$. Check that the map on cotangent spaces (or equivalently, that the dual map on tangent spaces) is the zero-map. For all other primes of $\mathbb{Z}[i]$, calculate the map on cotangent spaces.

1.3. **The singular points form a closed subset. Theorem.** Let Y be a prevariety. Then the set of singular points $\operatorname{Sing} Y$ is a closed subset of Y.

In the proof, we'll use the following (which will come up in the Commutative Algebra course).

Commutative Algebra Fact. If A is a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\overline{k}$, then $\dim_{\overline{k}} \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$.

Remark. It is also true that $\operatorname{Sing} Y \neq Y$, but I won't prove this fact.

Proof of Theorem. First step: reduce to the affine case. It suffices to show that for some open covering $Y = \cup Y_i$ of Y, that $\operatorname{Sing} Y_i$ is closed for each i. So assume that Y is affine.

By the above C.A. fact, we know that the rank of the Jacobian matrix is at most $n-r$, so the set of singular points is the set of points where the rank is *less than* $n-r$. Thus $\operatorname{Sing} Y$ is the set defined by the ideal $I(Y)$ together with all determinants of $(n-r)\times(n-r)$ submatrices of $(\partial f_i/\partial x_j)_{i,j}$, which is a closed set. $\square$

2. Curves: Valuation rings and nonsingular points, take 1

Dimension 1 varieties, or curves, are particularly simple, and most of the rest of the course will concentrate on them.

We saw that nonsingularity has to do with local rings, so we'll discuss *one-dimensional local rings.*

First we'll recall some facts about discrete valuation rings and Dedekind domains.

Definition. Let K be a field. A *discrete valuation* of K is a map $v : K \setminus \{0\} \to \mathbb{Z}$ such that for all x, y non-zero in K, we have: $v(xy) = v(x) + v(y)$, $v(x+y) \geq \min(v(x), v(y))$. Notice that the set $R = \{x \in K | v(x) \geq 0\} \cup \{0\}$ is a subring of K; call this the *discrete valuation ring*, or *DVR*, of K. The subset $\mathfrak{m} = \{x \in K | v(x) > 0\} \cup \{0\}$ is an ideal in R, and $(R, \mathfrak{m})$ is a local ring. A *discrete valuation ring* is an integral domain which is the discrete valuation ring of some valuation of its quotient field. If $\overline{k}$ is a subfield of K such that $v(x) = 0$ for all

$x \in \overline{k} \setminus \{0\}$, then we say v is a *discrete valuation of $K/\overline{k}$*, and R is a *discrete valuation ring of $K/\overline{k}$.*

I'll need to patch this to prevent pathologies such as the "zero valuation". More on this next day.

Example. Let $K = \overline{k}(t)$, and for $f \in K$, let $v(f)$ be the order of the zero of f at $t = 0$ (negative if f has a pole). Check all properties. Notice that *discrete valuation ring* of of v are those quotients of polynomials whose denominator doesn't vanish at 0, i.e. $\overline{k}[t]_{(t)}$. In geometric language, it is the stalk of the structure sheaf of $\mathbb{A}^1$ at the origin.

Also, $\overline{k}[t]_{(t)}$ is a discrete valuation ring: it is indeed an integral domain, and it is the valuation ring of some valuation in its quotient field $\overline{k}(t)$.

Similarly, we could get other valuations by replacing 0 with any other element of $\overline{k}$. Have we found all the valuations? No:

Example. Let $K = \overline{k}(t)$ as before. For $f \in K$, write f in terms of $u = 1/t$, and let $v(f)$ be the order of zero of f at $u = 0$. Again, it is indeed a valuation, and it has geometric meaning. (Ask them.) It corresponds to the point of $\mathbb{P}^1$ "at ∞" (when looking at it with respect to the t-coordinate).

Fact (that we'll later prove). These are all the valuations of $\overline{k}(t)$, the function field of $\mathbb{P}^1$. They naturally correspond to the points of $\mathbb{P}^1$.

Example. Let $K = \mathbb{Q}$. (Ask for valuations.) If $f \in \mathbb{Q}$, let $v(x)$ be the highest power of 2 dividing x, so $v(14) = 2$, $v(3) = 0$, $v(13/12) = -2$. Check all properties. What's the discrete valuation ring? Those fractions with no 2's in the denominators. Geometrically, $\mathbb{Q}$ is the function field of $\operatorname{Spec} \mathbb{Z}$, and the valuations turn out to correspond to the maximal prime ideals of $\mathbb{Z}$, i.e. the "closed points" of $\operatorname{Spec} \mathbb{Z}$.

Coming soon. Describing discrete valuation rings in many ways. Integral closure. Dedekind domains.