INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 20

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Contents

New problem set out.

1. Recap of where we are

We are in the midst of proving the following.

Theorem. Finitely generated fields over $\overline{k}$ of transcendence degree 1 correspond to nonsingular projective curves (over $\overline{k}$).

Corollary. The following 3 categories are equivalent:

(i) nonsingular projective curves, and dominant morphisms;
(ii) curves, and dominant rational maps *(in the problem set, I added quasiprojective, but that isn't necessary)*;
(iii) function fields of dimension 1 over $\overline{k}$, and $\overline{k}$-homomorphisms.

The details are left to you in an exercise. This is important to understand. (Sketch what the objects and morphisms are, and how the links go.)

With these insights, you can prove things such as: any nonsingular rational curve is an open subset of $\mathbb{P}^1$.

Proof. To go from a nonsingular projective curve to a finitely generated field, we just take the function field. So we're dealing with the opposite direction: given K, we're constructing a projective nonsingular curve C with this function field. We've already shown that there is at most one such curve, as last day we saw that any two nonsingular projective curves that are birational (i.e. have the same function field) are in fact isomorphic.

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At this point, we have constructed *a nonsingular curve C (with function field K) such that all the valuations of K correspond to points of C (and vice versa).*

We will be done once we prove:

Proposition. C is projective.

Proof. Suppose it isn't projective. By the previous section, C is quasiprojective, so embed it in some $\mathbb{P}^n$, and let p be any point in its closure $\overline{C}$ but not in C. Using p, we'll construct a *new* DVR of $K/\overline{k}$, contradicting the fact that C is supposed to have points corresponding to *all* the DVRs of $K/\overline{k}$.

We have shown that: *p can't be a nonsingular point of $\overline{C}$.* (Leo has pointed out that this doesn't really matter.) Our proof used the following

Earlier Lemma. Let Y be a prevariety, and suppose p and q are two points contained in a single affine open U, and $\mathcal{O}_{Y,q} \subset \mathcal{O}_{Y,p}$ (as subrings of $k(Y)$). Then $p = q$.

(I also gave a geometric argument that could replace this.)

In our situation

In our situation, $\mathcal{O}_{Y,q} = \mathcal{O}_{Y,p}$. Hence we've shown that p can't be a nonsingular point of $\overline{C}$, so we have a picture like this (draw it).

Take any affine open $U = U_i \cap C$ as before, containing p, so U is an affine curve. Take the normalization in its function field. (Draw a picture.) Let $\tilde{U}$ be the variety corresponding to the normalization of $A(U)$; note that we have a normalization morphism $\nu : \tilde{U} \to U$. If we could prove that ν is surjective (which we will in a moment), then we'd be essentially done. Let $\tilde{p}$ be a point mapping to p; it is a nonsingular point of $\tilde{U}$. It corresponds to a DVR of $K/\overline{k}$. By our *earlier lemma*, it is a different DVR from any of the points of U. But we could have chosen U to include any given point of of C, so this DVR isn't the same as the DVR corresponding to any point of C, giving a contradiction. □

All that's left is the following commutative algebra lemma.

Commutative Algebra Lemma. Let U be an affine variety, with coordinate ring $A(U)$. Let $\tilde{U}$ be the variety corresponding to the normalization of $A(U)$ (in its function field), and let $\nu : \tilde{U} \to U$ be the *normalization morphism* corresponding to the inclusion $A(U) \hookrightarrow A(\tilde{U})$. Then ν is surjective.

Before proving that, let's prove the famous Nakayama's lemma. There are many other forms of this. (Variation on version on Shafarevich p. 283.)

Nakayama's Lemma. Let M be a non-zero finitely-generated module over a ring A and $\mathfrak{a} \subset A$ an ideal. Suppose that for any element $a \in 1+\mathfrak{a}$, $aM \neq 0$. Then $\mathfrak{a}M \neq M$.

Proof. Suppose that $M = (u_1, \ldots, u_n)$. Suppose $\mathfrak{a}M = M$. Then there are equalities $u_i = \sum_{j=1}^n a_{ij} u_j$ with $a_{ij} \in \mathfrak{a}$. Thus $\sum_{j=1}^n (\delta_{ij} - a_{ij}) u_j = 0$ for $i = 1, \ldots,$ n, so by Cramer's rule $du_i = 0$ for $d = \det(\delta_{ij} - a_{ij})$. Thus $dM = 0$. As $d \in 1+\mathfrak{a}$, it follows by assumption that $M = 0$, giving a contradiction. □

Proof of Commutative algebra lemma. This comes from the theorem on *finiteness of integral closure* (Class 17): $A(\tilde{U})$ is a finite $A(U)$-module. Let p be a point of U, corresponding to maximal ideal $\mathfrak{m}$. Then the preimages of p in $\tilde{U}$ correspond to maximal ideals of $\tilde{A}$ containing $\mathfrak{m}A$. So there *exists* a preimage of p if there is a maximal ideal of $\tilde{A}$ containing $\mathfrak{m}A$, or equivalently $\mathfrak{m}\tilde{A}$. Equivalently, we want to show that $\mathfrak{m}\tilde{A}$, which is an ideal, isn't the unit ideal (as then there must be a maximal ideal containing it).

Since $\tilde{A}$ contains the unit element of A, $a\tilde{A} = 0$ if and only if $a = 0$, and as $\mathfrak{m} \neq (1)$, then $0 \notin 1+\mathfrak{m}$. This verifies the assumptions of Nakayama's Lemma, so $\mathfrak{m}\tilde{A} \neq \tilde{A}$. So we're done. □

Nakayama's lemma, although short, is very powerful. To show you how powerful, here are some exercises. As a hint, each one of these has a proof that is at most two lines long.

Exercise. If $\mathfrak{a} \subset A$ is an ideal such that every element of $1+\mathfrak{a}$ is invertible (which holds for example if $A/\mathfrak{a}$ is a local ring), M a finitely generated A-module and $M' \subset M$ any submodule, then $M' + \mathfrak{a}M = M$ implies that $M' = M$.

Exercise. If $\mathfrak{a} \subset A$ is an ideal such that every element of $1+\mathfrak{a}$ is invertible, M a finitely generated A-module, show that the elements $u_1, \ldots, u_n \in M$ generate M if and only if their images generate $M/\mathfrak{a}M$.

Exercise. Let A be a Noetherian ring, and $\mathfrak{a} \subset A$ an ideal such that every element of $1+\mathfrak{a}$ is invertible in A. Then $\cap_{n>0} \mathfrak{a}^n = (0)$.

Exercise. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Suppose $a_1, \ldots, a_n \in \mathfrak{m}$ generate $\mathfrak{m}/\mathfrak{m}^2$ (as a vector space). Show that $a_1, \ldots, a_n$ generate $\mathfrak{m}$ (as an ideal).

Exercise. In class 16, I invoked a result from commutative algebra. If R s a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\overline{k}$, then $\dim_{\overline{k}} \mathfrak{m}/\mathfrak{m}^2 \geq \dim R$. Prove this.

2. Normalization, and desingularization of curves

Note to myself: perhaps normalization should be done earlier, as soon as integral closure is introduced. That would head off the subtle question about the above proof: how do you know that the point constructed over U_1 also is a point over U_2.

Note to the reader: I think in class I presented this material in different order.

You can get some interesting mileage out of normalization. Given an affine variety X, you can define the normalization $\tilde{X}$. Notice that if $D(f)$ is a distinguished open of the affine variety, so $D(f) \hookrightarrow X$, then $\tilde{D(f)} \hookrightarrow \tilde{X}$. Using this observation, you construct a normalization $\tilde{X}$ for *any* variety X, along with the normalization morphism $\tilde{X} \to X$. *(Exercise.)* For the construction, you construct this for each affine open, and show that they glue together. (You can even describe normalization in a function field. If X is a variety, and L is a finite extension of $k(X)$, then you can define Y, the normalization of X in L.)

I'll give the pieces of the construction, and let you put them together in the problem set. The key algebraic piece is the following.

Lemma. Let R be a ring, and $f \in R$, $f \neq 0$. Let $\tilde{R}$ be the normalization of R, so $R \to \tilde{R}$, and let $\widetilde{R_f}$ be the normalization of R_f. Then $\tilde{R}_f$ is naturally isomorphic to $\widetilde{R_f}$, so the following diagram commutes:

$$\begin{array}{ccc} \tilde{R} & \rightarrow & \tilde{R}_f = \widetilde{R_f} \\ \uparrow & & \uparrow \\ R & \rightarrow & R_f. \end{array}$$

When R is a domain, which is the case we are concerned with here, all morphisms above are injections. When doing this exercise, you will have to prove this lemma in the case of domains.

What does this tell us geometrically? It means that if you have an affine variety X, and you construct the normalization $\tilde{X}$ mapping to it, and then you look at the preimage of an open set $D(f)$, you see the normalization $\widetilde{D(f)}$ of the open set. This means that you can construct normalizations of arbitrary variety X: over any affine open set U of X, construct the normalization $\tilde{U}$ mapping to it. This lemma tells you that you can glue these $\tilde{U}$'s together.

Here's one reason why we care: a normal curve is a nonsingular curve (and vice versa). So this gives a way of desingularizing curves. In other words, given a singular curve C, it constructs a nonsingular $\tilde{C}$ mapping to it.

Explicit example: a node. $y^2 = x^3 + x^2$. Draw a picture. You can see that this isn't normal, as normal curves are nonsingular. Let's normalize the ring $R = \overline{k}[x, y]/(y^2 - x^3 - x^2)$. Notice that $t = y/x$ is in the function field, and it satisfies an integral equation (ask them): $t^2 - (x + 1) = 0$.

So now we have the variety in 3-space (with coordinates x, y, t) which satisfies equations $y = tx$, $y^2 = x^3 + x^2$, $t^2 = (x + 1)$. It maps to our original curve C by projection to the xy-plane. Note that there are 2 branches mapping to $(x, y) = (0, 0)$: $(x, y, t) = (0, 0, 1)$, $(x, y, t) = (0, 0, -1)$.

So this isn't surprisingly nonsingular. Let's check. Jacobian matrix is

$$\begin{pmatrix} -t & 1 & -x \\ -3x^2 - 2x & 2y & 0 \\ -1 & 0 & 2t \end{pmatrix}$$

We want this to be rank 2. (As a check: this shouldn't be rank 3. I haven't checked that the determinant is always 0.) So we want to show that there are always 2 linearly independent columns. The first and second columns work.

Explain why this is the normalization (because the ring is integrally closed, as it is a Dedekind domain).

Example: a cusp. $y^2 = x^3$. (Draw a picture.) What should we add (in the function field, that satisfies an integral equation)? *Exercise.* Finish this.

Example: a triple point. $y^3 = x^3 + x^4$. (Draw a picture.) What should we add? *Exercise.* Finish this.

Example: a tacnode. $y^2 = x^4 + x^5$. What should we add? Careful: $t = y/x$ is not quite clever enough.

If X is higher-dimensional, then $\tilde{X}$ isn't necessarily nonsingular. For example, the cone $x^2 + y^2 = z^2$ in $\mathbb{A}^3$ is normal, but singular. (*Exercise*: H II.6.4 p. 147)

You can identify normal varieties geometrically. Fact: normal is equivalent to "regular (nonsingular) in codimension 1", and "Serre's S2 condition". Regular in codimension 1 just means that the singular locus is codimension at least 2. Serre's S2 condition is more mysterious, involving a commutative algebra concept called *depth*; I'm not sure how best to picture it.