INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 21

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Contents

Problem set info! PS8 back Tuesday November 30. PS9 due Tuesday November 30. PS10 due Thursday December 2. (PS11 due Thurs. Dec. 9. PS12 due Monday Dec. 13.)

Brand new topic:

1. Line bundles and invertible sheaves

Invertible sheaves are fundamental objects. They are essentially the same thing as line bundles, which are geometrically easy to visualize. So I'll tell you intuitively about line bundles first, and then introduce invertible sheaves. (I won't discuss motivation just yet, so I hope you believe me when I say how important this concept is!)

1.1. **Line bundles, classically.** Line bundle on a complex manifold (with classical topology). Draw a picture. Precisely, given a complex manifold X, a line bundle is given by the data of a cover U_i ($i \in I$ some index set), and trivializations $U_i \times \mathbb{C}$, and transition functions f_{ij} on $U_i \cap U_j$ that is an analytic function, nowhere zero, satisfying $f_{ij} f_{jk} = f_{ik}$ (the *cocyle condition*), from which $f_{ii} = 1$ and $f_{ij} = f_{ji}^{-1}$. What this means: if $u \in U_i \cap U_j$, then the point $(u, z) \in U_j \times \mathbb{C}$ is identified with the point $(u, f_{ij} z) \in U_i \times \mathbb{C}$. *(I think I said this incorrectly in class.)* If you've seen Cech cohomology before, you can check that the data of a line bundle is the same as an element of $H^1(X, \mathcal{O}^*)$, where $\mathcal{O}^*$ is the sheaf of invertible (=nowhere zero) analytic functions.

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In particular, the total space L of a line bundle is also a complex manifold (of dimension one higher than that of X), with a morphism $L \to X$.

A *section* of a line bundle is the data of maps $g_i : U_i \to \mathbb{C}$ (or if you prefer, $U_i \to U_i \times \mathbb{C}$), satisfying $g_i(p) = f_{ij}(p)g_j(p)$ for points $p \in U_i \cap U_j$. (Draw a picture of a section of $L \to X$.)

Note that there is always a *zero-section* given by $g_i(p) = 0$ for all i, $p \in U_i$.

Two line bundles $L_1 \to X$, $L_2 \to X$ are isomorphic if there is an isomorphism $L_1 \to L_2$ commuting with the morphisms to X.

Example. The trivial line bundle $X \times \mathbb{C} \to X$. Global sections are just analytic functions.

At this point you might think that all line bundles are trivial (which isn't true). Indeed there are line bundles that seem nontrivial that are trivial. *Exercise.* Suppose U_i is a finite open cover of X, and for each i, h_i is an invertible function on U_i. Prove that the data of the open cover U_i and the transition functions $f_{ij} = h_i/h_j$ defines a line bundle, and prove that this is isomorphic to the trivial line bundle. (In fancy language, this shows that a zero-element of $H^1(X, \mathbb{C}^*)$ indeed induces a trivial line bundle.)

Example. Inside $\mathbb{C}^{n+1}$, we know that $\mathbb{P}^n$ parametrizes lines through the origin. This will give a line bundle on $\mathbb{P}^n$ (as we'll soon see rigorously): "above" each point of $\mathbb{P}^n$, we have the line corresponding to that point. (We will later see that this is not trivial, if $n > 0$.)

Example. Let X be a one-dimensional complex manifold. Then there is a *tangent line bundle* (and similarly a *cotangent line bundle*). (Describe it pictorially.) This will be tricky to describe precisely.

1.2. **Line bundles on varieties (and schemes).** We can use the same definition for varieties, except this time we use the Zariski topology, and the functions should be invertible regular functions. (Experts in Cech cohomology can later check that line bundles are parametrized be $H^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the *sheaf of invertible functions* and the topology on X is of course the Zariski topology; in fact, if you've seen Cech cohomology of sheaves in any other setting, you know a whole lot about cohomology of sheaves in algebraic geometry.)

1.3. **Sheaf of sections of a line bundle, and the correspondence with line bundles.** Note that the sections actually form a *sheaf.* (Sketch intuitively first.) Here, explicitly, is how. Suppose the line bundle L on our variety X is defined by an open cover U_i and transition functions f_{ij} (on $U_{ij} = U_i \cap U_j$). Then the sheaf of sections $\mathcal{L}$ is defined by:

Elements of $\mathcal{L}(U)$ are functions h_i on $U \cap U_i$ such that $h_i = f_{ij}h_j$ on $U \cap U_i \cap U_j$.

Notice that this sheaf $\mathcal{L}$ is an $\mathcal{O}_X$-module. What this means is: for each open U, $\mathcal{L}(U)$ is an $\mathcal{O}_X(U)$-module (and this action commutes with restriction: if $U \subset V$, and $a \in \mathcal{L}(V)$ and $f \in \mathcal{O}_X(V)$, then the restriction of (fa) to U is the same as the restriction of f to U acting on the restriction of a to U.

We can now define an invertible sheaf.

Definition. An invertible sheaf on a variety X is an $\mathcal{O}_X$-module $\mathcal{L}$ such that there is an open cover U_i of X satisfying:

- there are isomorphisms $\phi_i : \mathcal{L}|_{U_i} \to \mathcal{O}_{U_i}$,
- there are invertible functions f_{ij} on $U_i \cap U_j$ such that for $z \in \mathcal{L}(U)$ for any $U \subset U_i \cap U_j$, $\phi_i(z)$ (which is a function on U) equals $f_{ij}\phi_j(z)$ (the product of two functions on U).

Notice that the cocycle condition $f_{ik} = f_{ij}f_{jk}$ on $U_i \cap U_j \cap U_k$ follows from this definition.

Notice that if you have a line bundle, its sheaf of sections is an invertible sheaf. If you have an invertible sheaf, you can cook up a line bundle. And these constructions commute.

Philosophy: Invertible sheaves are easier to deal with (if less intuitive to picture) than line bundles, so we'll deal with them in the future. Here's why this fits into the the philosophy of algebraic geometry. Classically, people would study varieties by looking at them as topological spaces. In algebraic geometry, we study varieties by looking at *functions* on them (in some sense a dual point of view). In the same way, we will study line bundles by instead looking at their *sheaf of sections*.

1.4. **Examples.** (I'm going to discuss this example again later on.) Let's bring this back down to earth. I'll now give some examples. Try to think of them analytically at the same time as algebraically.

Define a *meromorphic section* (or "rational section") to be a section of a line bundle over some non-empty open set.

Example. The trivial line bundle $X \times \mathbb{A}^1 \to X$. Global sections are just regular functions on X.

A specific case: let $X = \mathbb{P}^1$. Then the meromorphic sections of the trivial line bundle are just rational functions. They are (in affine coordinates) of the form $(t - a_1)^{n_1} \dots (t - a_r)^{n_r}$, where the n_r are integers, possibly negative. Define the order of vanishing of a meromorphic function (at a_i to be n_i). What's the order of vanishing of this meromorphic function at ∞? Answer: $-\sum n_i$. Hence *the sum of the order of vanishings of any non-zero meromorphic section of the trivial line bundle on* $\mathbb{P}^1$ *is 0.* (Remark: whenever you are talking about orders of vanishing, keep in mind that there is DVR theory lurking in the background.)

Example: The "tautological" line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ on $\mathbb{P}^n$. Let $\mathbb{P}^n$ have coordinates $(x_0; \dots; x_n)$. This will be the line bundle that assigns to a point of $\mathbb{P}^n$ the corresponding line through the origin in $\mathbb{A}^{n+1}$. Perhaps ask about section of $\mathcal{O}_{\mathbb{P}^1}(-1)$ corresponding to $x_0 = 1$ here. We'll define this on patches. On the patch U_0 (i.e. the points $(1; y_1; \dots; y_n)$, so we can take coordinates y_i on U_0), the line is of the form $(s, sy_1, \dots, sy_n)$ in $\mathbb{A}^{n+1}$, with coordinate s.

On U_1 (i.e. the points $(z_0; 1; z_2; \dots; z_n)$), the line is of the form $(tz_0; t; tz_2; \dots; tz_n)$. How do we change coordinates from s to t on $U_0 \cap U_1$? $s = (x_0/x_1)t$. The best way to see this is by example. Consider the point $(1, 2, 3, ..., n+1) \in \mathbb{A}^{n+1}$. This is a point on the line above $(1; 2; \dots; n+1)$. Its s-coordinate is 1, and its t-coordinate is 2. Indeed $s/t = x_0/x_1$. Precisely: $f_{10} = x_0/x_1 = 1/y_0 = z_0$. $f_{01} = x_1/x_0 = y_0 = 1/z_0$.

In general, you can define coordinates s_0 on U_0, $\dots$, s_n on U_n, and we have transition functions $f_{ij} = x_j/x_i$. Note that f_{ij} is indeed an invertible function on $U_i \cap U_j$.

Proposition. $\mathcal{O}_{\mathbb{P}^1}(-1)$ on $\mathbb{P}^1$ is not isomorphic to the trivial sheaf.

(I'll restate this proof, and give a second proof next day.)

Proof. (Advantages: foreshadows idea of "degree of a line bundle".) We construct a meromorphic section corresponding to the plane $x_0 = 1$ in $\mathbb{A}^n$ (draw picture). It you think about it, you might convince yourself that it blows up at the point $x_0 = 0$ in $\mathbb{P}^n$, and nowhere else (and it has no zeroes).

Cover $\mathbb{P}^1$ with two standard opens, U_0 and U_1. On the first standard open, we have the section $s_0 = 1$; it has no zeroes. On the second standard open, with coordinates $(z_0; 1)$, the coordinates on the line is s_1, and points on the line are of the form $(z_0 s_1, s_1)$. Thus the section $(1, 0)$ gives (in this coordinate system) $s_1 = 1/z_0$. This has a simple pole at $z_0 = 0$.

In short, this nonzero meromorphic section of $\mathcal{O}_{\mathbb{P}^1}(-1)$ has a single simple pole, so the sum of its orders of vanishing is -1. But we earlier checked that any nonzero meromorphic section of the trivial sheaf has sum of orders of vanishing 0. So they are not the same. $\square$

Coming next: The line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$. Maps to projective space correspond to a vector space of sections of a line bundle. The canonical invertible sheaf, genus. Riemann-Roch Theorem: statement (no proof) and applications. Riemann-Hurwitz.