INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 22

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Contents

Problem Set 8 back at end of lecture. PS9 due today.

PS10 due Thursday December 2. PS11 due Thurs. Dec. 9. PS12 due Monday December 13. If you don't get a chance to do PS12 by December 13, but still want to think about it, I'll be around, so you can drop by and we can talk.

1. Example: The invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(m)$.

Caution. I don't I think I was consistent with transition functions last time. I think everything is correct in the lecture notes. If g_i is a section over U_i and g_j is a section over U_j, then $g_i = f_{ij} g_j$.

I'll now describe some invertible sheaves on projective space. I think this description will help you see how to deal with invertible sheaves in general.

Let $U_0, \dots, U_n$ be the standard opens of $\mathbb{P}^n$. Consider the invertible sheaf (known as $\mathcal{O}_{\mathbb{P}^n}(m)$) given by the transition functions $f_{ij} = (x_j/x_i)^m$. Recall for this sheaf, the sections corresponding to an open set U are given by the data of sections $g_i \in \mathcal{O}(U \cap U_i)$ (i.e functions on $U \cap U_i$), satisfying the transition information $g_i = f_{ij} g_j$ on the open set $U \cap U_i \cap U_j$.

Note that $f_{ii} = 1$, and $f_{ik} = f_{ij} f_{jk}$.

We can talk about *rational* (or *meromorphic*) sections as well.

Date: Tuesday, November 30, 1999.

1.1. **Example: $\mathbb{P}^1$.** Let's do an example. To keep with last day's example, let's let $n = 1$, and let's first take $m = -1$. Here are the variables we'll use.

On $\mathbb{P}^1$, use projective coordinates $[x_0; x_1]$.

On open set U_0, choose coordinate $y_1 = x_1/x_0$, so U_0 are the points $[1; y_1]$. (Note that this is equal to $[x_0; x_1]$!)

On open set U_1, choose coordinate $z_0 = x_0/x_1$, so U_0 are the points $[z_0; 1]$.

Here's a section over U_0: $g_0(y_1) = y_1 - 1$. Let me ask you some stupid questions. Where does it have zeroes or poles? Answer: a zero of order 1 at $y_1 = 1$, i.e. at $[x_0; x_1] = [1; 1]$.

Let's consider this as a *rational* section over all of $\mathbb{P}^1$. What does this section look like over U_1? It is given by a function $g_1(z_0) = f_{10} f_0(y_0) = x_1/x_0(1/z_0 - 1) = (z_0 - 1)/z_0^2$. Zero of order 1 at [1;1], and pole of order 2 at [0;1].

Remark. Note that we calculated that this section had a zero of order 1 at $[1; 1]$ in both open neighbourhoods. But the functions whose order of vanishing we checked were different! Why were we guaranteed to get the same answer? Answer: they differed by a factor that was invertible (i.e. had 0 valuation).

Proposition. $\mathcal{O}_{\mathbb{P}^1}(-1)$ is not isomorphic to the structure sheaf.

Proof 1. The sums of orders of vanishing of this meromorphic section is -1. The sums of orders of vanishing of a meromorphic structure sheaf is 0. (This is the proof I gave last time. □

Remark. Notice that In general, the sums of orders of vanishing of a meromorphic section of $\mathcal{O}_{\mathbb{P}^1}(m)$ is m. So they are all non-isomorphic invertible sheaves.

Proof 2. You can check that $\mathcal{O}_{\mathbb{P}^1}(-1)$ has no non-zero global sections. But the structure sheaf has non-zero global sections (the non-zero constants). □

1.2. **Identification of global sections of $\mathcal{O}_{\mathbb{P}^n}(m)$ with the degree m polynomials in x_0, ..., x_n ($m \geq 0$).** As another example, I'll let you calculate the space of global sections of $\mathcal{O}_{\mathbb{P}^1}(2)$.

Using the same variables as before, $g_0(y_1)$ is a polynomial in y_1. $g_1(z_0)$ is a polynomial in z_0, and $g_1(z_0) = z_0^2 g_0(1/z_0)$. Hence g_0 can have degree at most 2. Conversely, any polynomial of degree 2 will work. Hence the space of global sections of $\mathcal{O}_{\mathbb{P}^1}(2)$ has dimension 3. (The same argument will show that the space of global sections of $\mathcal{O}_{\mathbb{P}^1}(n)$ has dimension $n + 1$ if $n \geq 0$.)

You can identify the space of global sections with the vector space of degree 2 polynomials in x_0 and x_1. Here's the identification. Given a polynomial $ax_0^2 + bx_0x_1+cx_1^2$, to get the corresponding section $g_0(y_1)$ over U_0, just divide by x_0^2. Hence get $a + b(x_1/x_0) + c(x_1/x_0)^2 = a + by_1 + cy_1^2$. To get the corresponding section $g_1(z_0)$ over U_1 just divide by x_1^2, and get $a(x_0/x_1)^2 + b(x_0/x_1) + c = az_0^2 + bz_0 + c$. Note that g_0 and g_1 satisfy the desired transition data: $g_0 = (x_0/x_1)^2 g_1$.

In the same way you can prove:

Proposition. Let $m \geq 0$. Then the vector space of global sections of $\mathcal{O}_{\mathbb{P}^n}(m)$ can be identified with the vector space of polynomials $h(x_0, \ldots, x_n)$ of degree m. The identification is this: $g_i = h/x_i^m$.

Probably an exercise. You've actually done this in the case when $m = 0$, when you proved that the only regular functions on projective space are the constants.

This is really concrete, and to prove to you how concrete it is, let me ask you some questions.

Here's a section of $\mathcal{O}_{\mathbb{P}^2}(3)$: $x^3 + y^3 + z^3$. Where does it have zeroes? How about xy^2?

We can extend this to rational sections. Once again, they have to be homogeneous degree m. How about $(x^3 + y^3 + z^3)/(x^2yz)$, which is a meromorphic section of $\mathcal{O}_{\mathbb{P}^2}(-1)$. Where are the poles and zeroes?

2. More background on invertible sheaves

2.1. **Operations on invertible sheaves.** Here are some basic things you can do with invertible sheaves.

i) Pullback. You can pull back invertible sheaves (or line bundles). (Give picture first.) Here's how. If you have a morphism $\pi : X \to Y$, and you have an invertible sheaf $\mathcal{L}$ on Y defined by open sets U_i and transition functions f_{ij} on $U_i \cap U_j$, then define the *inverse invertible sheaf* $\pi^*\mathcal{L}$ by the open sets $\pi^{-1}(U_i)$, and the functions $\pi^* f_{ij}$, which are functions on $\pi^{-1}(U_i \cap U_j) = \pi^{-1}(U_i) \cap \pi^{-1}(U_j)$.

You can pull back sections as well: if g is a section of $\mathcal{L}$ over some open subset $U \subset Y$, then you can define a section $\pi^{-1}g$ of $\pi^{-1}\mathcal{L}$ over $\pi^{-1}(U)$ (which is also open). For convenience of notation, I'll show you how this works for global sections. If g is a global section given by regular functions g_i on U_i related by $g_i = f_{ij}g_j$ on $U_i \cap U_j$, when consider the regular functions $\pi^* g_i$ on $\pi^{-1}U_i$. (Remember that you can pull back regular functions via morhpisms!) Then because $g_i = f_{ij}g_j$ on $U_i \cap U_j$, we have $\pi^{-1}g_i = \pi^{-1}f_{ij}\pi^{-1}g_j$ on $\pi^{-1}(U_i \cap U_j) = \pi^{-1}(U_i) \cap \pi^{-1}(U_j)$. In other words, these regular functions define a section of $\pi^*\mathcal{L}$.

As an example of how you can use this, consider:

Proposition. $\mathcal{O}_{\mathbb{P}^n}(m_1)$ is not isomorphic to $\mathcal{O}_{\mathbb{P}^n}(m_2)$ if $m_1 \neq m_2$.

Proof. Recall that we've proved this for $n = 1$: any non-zero meromorphic section has "total order of vanishing" m.

Now we'll deal with the general n. Take the closed immersion $\mathbb{P}^1 \hookrightarrow \mathbb{P}^m$, sending $\mathbb{P}^1$ to a line, e.g. $(x_0; x_1) \mapsto (x_0; x_1; 0; \dots; 0)$. Check that $\mathcal{O}_{\mathbb{P}^n}(m_i)$ pulls back to $\mathcal{O}_{\mathbb{P}^1}(m_i)$. But $\mathcal{O}_{\mathbb{P}^1}(m_1)$ is not isomorphic to $\mathcal{O}_{\mathbb{P}^1}(m_2)$. $\square$

Fact. (We probably won't prove this, but the proof isn't difficult to follow.) These are *all* the line bundles on $\mathbb{P}^m$. Hence $\operatorname{Pic} \mathbb{P}^n \cong \mathbb{Z}$, with $\mathcal{O}(1)$ the generator.

Example: Line bundles on curves. This example is somewhat lame. Consider the curve C $y^2 z = x^3 + xz^2$. We can pull back the line bundle $\mathcal{O}_{\mathbb{P}^2}(2)$ to C. We already can see that it has at least 6 linearly-independent sections. Where on the curve does the section $x^2 + y^2$ vanish? Answer: where $x^2 + y^2 = 0$. We can pull back the line bundle $\mathcal{O}_{\mathbb{P}^2}(3)$ to C. We can already see that it has at least 9 linearly independent sections. $\mathcal{O}_{\mathbb{P}^2}(3)$ has 10 sections. But one of them vanishes on C: the section $y^2 z - x^3 - xz^2$.

ii) Tensor product of two invertible sheaves. Take 1: Suppose you have two invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ on X, given by the *same* open cover U_i and (possibly different) transition functions l_{ij} and m_{ij}. Then define the tensor product invertible sheaves $\mathcal{L} \otimes \mathcal{M}$ by the same open cover, and the transition function $l_{ij} m_{ij}$. (You can immediately check that this satisfies the cocyle condition.)

Take 2: Suppose now that you have two invertible sheaves $\mathcal{L}$ and $\mathcal{M}$, given with the data of *possibly different* open covers U_i and V_j respectively, with transition functions $l_{ii'}$ and $m_{jj'}$ respectively. Then what you want to do is to take a different cover *finer* than the U_i and the V_j so you can express $\mathcal{L}$ and $\mathcal{M}$ using this new cover; then use the construction above. The following does the trick: $\mathcal{L} \otimes \mathcal{M}$ is defined using the open sets $U_i \cap V_j$, and the transition functions are given by $f_{ij,i'j'} = l_{ii'} m_{jj'}$.

Things you might want to check: that this construction is independent of the "representation" of the invertible sheaf. Also, $\mathcal{L} \otimes \mathcal{O}_X$ is isomorphic to $\mathcal{L}$.

Remark. If a is a global section of $\mathcal{L}$ (or more generally, a section over some open set), and b is a section of $\mathcal{M}$, then you can interpret ab as a section of $\mathcal{L} \otimes \mathcal{M}$.

Remark. You can see how this works with $\mathcal{O}_{\mathbb{P}^1}(m)$. Immediately, we have $\mathcal{O}_{\mathbb{P}^1}(m+n) \cong \mathcal{O}_{\mathbb{P}^1}(m) \otimes \mathcal{O}_{\mathbb{P}^1}(n)$.

Remark. If a is a section of $\mathcal{L}$ (over some open set) and b is a section of $\mathcal{M}$ (over the same open set), then ab is naturally a section of $\mathcal{L} \otimes \mathcal{M}$. For example, x^2 is a global section of $\mathcal{O}_{\mathbb{P}^1}(2)$, and $(x+y)^3$ is a global section of $\mathcal{O}_{\mathbb{P}^1}(3)$. What is their product as a global section of $\mathcal{O}_{\mathbb{P}^1}(5)$? Answer: $x^2(x+y)^3$.

iii) Inverse invertible sheaves. Suppose you have an invertible sheaf $\mathcal{L}$ defined by the open cover U_i and transition functions f_{ij}. Then define the *inverse*, denoted $\mathcal{L}^{-1}$, by the same open cover, and the transition functions f_{ij}^{-1}. Note that the cocycle condition is satisfied, and also that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.

Coming next: The Picard group. Maps to projective space correspond to a vector space of sections of an invertible sheaf. The class group. The canonical invertible sheaf (= the sheaf of differentials). Genus. Riemann-Roch Theorem: statement (no proof) and applications. The Riemann-Hurwitz formula.