INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 23

RAVI VAKIL

Contents

PS10 due today. PS11 out today. PS9 and PS10 back on Tuesday.

PS11 due Thurs. Dec. 9. PS12 due Monday December 13.

Invertible sheaves are manipulated in a really formal way, so it's hard to see how geometric it is. The earlier major topics (extension of morphisms to projective varieties over nonsingular points of curves; identification of nonsingular projective curves with finitely generated field extensions of transcendence degree 1) involved proving some technically difficult results with relatively few moving parts. Invertible sheaves and line bundles involves much less technical sophistication, but there are many many more ideas involved, so it's tricky juggling them all.

I really encourage you to play around with invertible sheaves / line bundles in explicit examples. Choose some nice variety, such as $\mathbb{P}^1$ or $\mathbb{P}^2$ or $\mathbb{P}^2$ minus some curve, and choose some nice invertible sheaf like $\mathcal{O}(3)$, and work out spaces of global sections.

Remark. An $\mathcal{O}_X$-module is an invertible sheaf if there is an open cover $U_1, \ldots,$ U_n where $\mathcal{L}_{U_i} \cong \mathcal{O}_{U_i}$. Reason: all we need to find are the transition functions f_{ij}. Let 1_i be the section of $\mathcal{L}$ over U_i corresponding to 1 in $\mathcal{O}_{U_i}$. Then $f_{ij} = 1_i/1_j$ (although I might have this backwards). *Perhaps it is best to say this when first introducing invertible sheaves.*

You can use a variant of this idea to show that if there is a global section that vanishes nowhere, then $\mathcal{L}$ is triival (i.e. $\cong \mathcal{O}_X$). I'll give the proof later today, but some of you might already see how to show it.

Date: Thursday, December 2, 1999.

1. More background on invertible sheaves

1.1. **Operations on invertible sheaves.** Last time I described some basic things you can do with invertible sheaves.

i) Pullback. You can pull back invertible sheaves (or line bundles).

Using that, we proved:

Proposition. $\mathcal{O}_{\mathbb{P}^n}(m_1)$ is not isomorphic to $\mathcal{O}_{\mathbb{P}^n}(m_2)$ if $m_1 \neq m_2$.

Fact. Roya mentioned this last time. (We probably won't prove this for all n, but the proof isn't difficult to follow.) These are *all* the line bundles on $\mathbb{P}^m$. Hence $\operatorname{Pic} \mathbb{P}^n \cong \mathbb{Z}$, with $\mathcal{O}(1)$ the generator.

ii) Tensor product of two invertible sheaves. Take 1: Suppose you have two invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ on X, given by the *same* open cover U_i and (possibly different) transition functions l_{ij} and m_{ij}. Then define the tensor product invertible sheaves $\mathcal{L} \otimes \mathcal{M}$ by the same open cover, and the transition function $l_{ij}m_{ij}$. (You can immediately check that this satisfies the cocyle condition.)

Take 2: If $\mathcal{L}$ and $\mathcal{M}$ have *possibly different* trivializing open covers, you can "refine" both covers to get a common trivializing open cover.

JP asked why this was called tensor product. Answer: this is tensor product in the category of $\mathcal{O}_X$-modules.

Things you might want to check: that this construction is independent of the "representation" of the invertible sheaf. Also, $\mathcal{L} \otimes \mathcal{O}_X$ is isomorphic to $\mathcal{L}$.

Remark. You can see how this works with $\mathcal{O}_{\mathbb{P}^1}(m)$. Immediately, we have $\mathcal{O}_{\mathbb{P}^1}(m+n) \cong \mathcal{O}_{\mathbb{P}^1}(m) \otimes \mathcal{O}_{\mathbb{P}^1}(n)$.

Remark. If a is a section of $\mathcal{L}$ (over some open set) and b is a section of $\mathcal{M}$ (over the same open set), then ab is naturally a section of $\mathcal{L} \otimes \mathcal{M}$. For example, x^2 is a global section of $\mathcal{O}_{\mathbb{P}^1}(2)$, and $(x+y)^3$ is a global section of $\mathcal{O}_{\mathbb{P}^1}(3)$. What is their product as a global section of $\mathcal{O}_{\mathbb{P}^1}(5)$? Answer: $x^2(x+y)^3$.

iii) Inverse invertible sheaves. Suppose you have an invertible sheaf $\mathcal{L}$ defined by the open cover U_i and transition functions f_{ij}. Then define the *inverse*, denoted $\mathcal{L}^{-1}$, by the same open cover, and the transition functions f_{ij}^{-1}. Note that the cocycle condition is satisfied, and also that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.

Remark. You can even divide by non-zero sections, to get meromorphic sections. The quotient in the above example is $x^2/(x+y)^3$, which is a meromorphic section of $\mathcal{O}_{\mathbb{P}^1}(-1)$. Important note: if you have two non-zero rational sections f, g of an invertible sheaf $\mathcal{L}$, then f/g is a non-zero rational section of the trivial sheaf $\mathcal{O}$, i.e. is a rational function. Let me say that again: the ratio of any two (non-zero) rational sections of a line bundle is a rational function.

You can now check that invertible sheaves form an abelian group:

Definition. The (abelian) group of invertible sheaves on X is called the *Picard group* of X, and is denoted $\operatorname{Pic} X$.

In general, people often write $\mathcal{L}^{\otimes m}$ for $\mathcal{L} \otimes \cdots \otimes \mathcal{L}$ (m times, if $m > 0$), $\mathcal{O}_X$ (if $m = 0$), or $\mathcal{L} \otimes \cdots \otimes \mathcal{L}$ ($-m$ times, if $m < 0$); you can check that $\mathcal{L}^{\otimes m} \otimes \mathcal{L}^{\otimes n} \cong \mathcal{L}^{\otimes (m+n)}$.

Remark for the arithmetically minded. If you do this construction in the language of schemes, for Spec of the ring of integers in a number field (a number field is a finite extension of $\mathbb{Q}$, and the ring of integers are the integral closure of $\mathbb{Z}$ in the field), then you get the *class group* of the field. If you're interested, and know what the class group is, ask me about it.

We'll see later that $\operatorname{Pic} \mathbb{P}^1$ is $\mathbb{Z}$, and I'd mentioned that $\operatorname{Pic} \mathbb{P}^n$ is $\mathbb{Z}$ as well. Here's some other behavior: Picard groups can be torsion.

Exercise. Let X be the variety $\mathbb{P}^2$ minus an irreducible conic. Let $\mathcal{O}_X(m)$ be the restriction of $\mathcal{O}_{\mathbb{P}^2}(m)$ to X. Show that $\mathcal{O}_X(2)$ is trivial, but that $\mathcal{O}_X(1)$ isn't. Hence $\mathcal{O}_X(1)$ is a 2-torsion element of $\operatorname{Pic} X$. (Hint: Show that $\mathcal{O}_X(2)$ has a global section vanishing nowhere, and that every global section of $\mathcal{O}_X(1)$ vanishes somewhere. Use the following proposition.)

Proposition. Suppose $\mathcal{L}$ is an invertible sheaf on a variety X, and that there is a global section s of $\mathcal{L}$ vanishing nowhere. Then $\mathcal{L}$ is isomorphic to $\mathcal{O}_X$, the "trivial sheaf".

Proof. We give an identification of $\mathcal{L}(U)$ with $\mathcal{O}_X(U)$ (that commutes with restriction maps); this will do the trick. (Do you see why?) To the section a of $\mathcal{O}_X(U)$ associate the section as of $\mathcal{L}(U)$. (More precisely, the section $a \operatorname{res}_{X,U} s$.) Conversely, to the section b of $\mathcal{L}(U)$, associate the section bs^{-1} of $\mathcal{L}(U)$. □

1.2. **Maps to projective space correspond to a vector space of sections of a invertible sheaf.** Suppose you have a variety X, and an invertible sheaf $\mathcal{L}$, and $n+1$ sections $s_0, \dots, s_n$ that don't have a common zero. Then this induces a map to projective space:

$$(s_0, \dots, s_n) : X \to \mathbb{P}^n.$$

Here's how.

Let U_i be a cover of X trivializing the invertible sheaf, and let f_{ij} be the transition functions of $\mathcal{L}$ with respect to this trivialization. I'll i) define the morphism from $U_i \to \mathbb{P}^n$, and ii) show that they agree (as maps of sets) on the $U_i \cap U_j$.

i) For the morphism from $U_i \to U_j$, let $(g_0, \dots, g_n)$ be functions on U_i corresponding to sections $(s_0, \dots, s_n)$. Then $g_0, \dots, g_n$ don't have a common zero in U_i, and hence define morphism to $\mathbb{P}^n$.

ii) Now consider the two morphisms from $U_i \cap U_j$ to $\mathbb{P}^n$, one given by considering it as a subset of U_i, and the other by considering it as a subset of U_j. As in i), let $(g_0, \dots, g_n)$ be the functions on U_i that correspond to our sections $(s_0, \dots, s_n)$. Similarly, let $(h_0, \dots, h_n)$ be the functions on U_j. Then under the first morphism, the point $p \in U_i \cap U_j$ is sent to

$$(g_0(p); \dots; g_n(p)) = (f_{ij}(p)h_0(p); \dots; f_{ij}(p)h_n(p)) = (h_0(p); \dots; h_n(p))$$

which is precisely where it is sent to under the second morphism.

Remark. This generalizes the fact stated earlier, that if you have a projective variety $X \subset \mathbb{P}^l$, and $n+1$ polynomials of the same degree m: s_0, ..., s_n, such that they don't have a common zero on X, then they define a morphism $X \to \mathbb{P}^n$. Just apply this construction with the line bundle $\mathcal{O}_{\mathbb{P}^l}(m)$, pulled back to X.

Proposition. Under this morphism π, there is a natural isomorphism $\mathcal{L} \cong \pi^{-1}\mathcal{O}(1)$.

This isn't too hard to prove. It's harder than an exercise, but certainly something I could explain to you reasonably quickly (using the remark before teh first section of today's lecture). But we won't be using it later in the course, so I'll omit the proof in the interests of time.

2. The class group

Let C be a nonsingular curve. (What I say will work, with relatively simple modification, for any nonsingular variety, so experts might want to make these modifications as I explain things.) Our goal is to descibe the *Picard group* (the group of line bundles or invertible sheaves) in a different, hopefully more tractable way.

Definition. The *Weil divisor group* $\mathrm{Div}(C)$ is the free group generated by the points of C. Elements of $\mathrm{Div}(C)$ are called *Weil divisors* (or informally, just *divisors*) of C.

In other words, elements of $\mathrm{Div}(C)$ can be written in the form

$$\sum_{p \in C} n_p p$$

where the n_p are integers, and only finitely many of them are zero. It's clear how these objects form a group.

Definition. Elements of $\mathrm{Div}(C)$ where all n_p are non-negative are called *effective (Weil) divisors* on C.

Definition. Suppose s is a non-zero rational section of an invertible sheaf $\mathcal{L}$ on C, so $s \in \mathcal{L}(C)$. Let (s) be the "divisors of poles and zeros of s".

In other words, if $(s) = \sum_{p \in C} n_p p$, then n_p is zero if p is neither a zero nor a pole; if p is a zero, then n_p is the order of the zero of s at p; if p is a pole, then n_p is the negative of the order of the pole.

For example, let $C = \mathbb{P}^1$, $\mathcal{L}$ be the structure sheaf, and s the rational function $x_0(x_0 - x_1)/x_1^2$. Then $(s) = [0;1] + [1;1] - 2[1;0]$.

Definition/Proposition. The divisors associated to non-zero rational functions (i.e. the case when $\mathcal{L}$ is the trivial bundle) is a subgroup of the divisor group, called the *subgroup of divisors linearly equivalent to 0*, denoted $\mathrm{Div}_0(C)$.

Proof. We just need to check that this forms a group. If f and g are rational functions, then $(f) + (g) = (fg)$ (explain), and $-(f) = (1/f)$. $\square$

Definition. Define the *divisor class group* of C $\mathrm{Cl}(C)$ as the quotient $\mathrm{Div}(C)/\operatorname{Lin}(C)$.

Proposition. $\operatorname{Lin} \mathbb{P}^1 = \{\sum n_p p \text{ where } \sum n_p = 0\}$.

Here's why. Choose some standard affine $\mathbb{A}^1$ with coordinate t, so that no p is at ∞. Then $\left(\prod_{p \in \mathbb{A}^1} (t-p)^{n_p}\right) = \sum n_p p$ (explain). Thus the right side is contained in the left side. Conversely, we've seen that the sums of the orders of vanishing of any rational function on $\mathbb{P}^1$ is 0, showing that the left side is contained in the right side.

Corollary. $\mathrm{Cl}(\mathbb{P}^1) \cong \mathbb{Z}$.

Theorem. $\mathrm{Cl}(C)$ is naturally isomorphic to $\mathrm{Pic}(C)$.

In the course of the proof, we'll see the identification quite clearly.

Corollary. $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$.

The map $\mathrm{Pic}(C) \to \mathrm{Cl}(C)$. First of all, I mentioned earlier that we have a map as sets from data $(\mathcal{L}, s)$ (where s is a non-zero rational section of $\mathcal{L}$) to $\mathrm{Div}(C)$. Note that those ordered pairs form a group: $(\mathcal{L}, s)(\mathcal{M}, t) = (\mathcal{L} \otimes \mathcal{M}, st)$, $(\mathcal{L}, s)^{-1} = (\mathcal{L}^\vee, s^{-1})$. Note also that this is a group homomorphism; call it ϕ.

Next, if s and t are two rational sections of $\mathcal{L}$. Then $(\mathcal{L}, s)(\mathcal{L}, t)^{-1} = (\mathcal{O}_X, st^{-1})$. Hence $\phi(\mathcal{L}, s) \equiv \phi(\mathcal{L}, t) \pmod{\operatorname{Lin}(C)}$. Hence we get a map $\mathrm{Pic}(C) \to \mathrm{Cl}(C)$.

The map $\mathrm{Cl}(C) \to \mathrm{Pic}(C)$. Similarly, we'll construct a map $\mathrm{Div}(C) \to \mathrm{Pic}(C)$, and show that $\operatorname{Lin}(C)$ goes to the identity in $\mathrm{Pic}(C)$, i.e. the invertible sheaf $\mathcal{O}_C$. Given a divisor D, we construct an invertible sheaf $\mathcal{O}_C(D)$.

Define a sheaf as follows. Recall that the sections of the structure sheaf were defined as

$$\mathcal{O}_C(U) = \cap_{p\in U}\mathcal{O}_{C,p} = \{x \in k(C) | v_p(x) \geq 0 \text{ for all } p \in U\}.$$

Remember why it was easy to check that this was a sheaf. Define

$$\mathcal{O}_C(\sum n_p p)(U) = \cap_{p\in U}\mathcal{O}_{C,p} = \{x \in k(C) | v_p(x) + n_p \geq 0 \text{ for all } p \in U\}.$$

Example. Suppose $C = \mathbb{P}^1$, with coordinates $[x_0; x_1]$, and $D = [1; 0]$. Then let's find the global sections of $\mathcal{O}_C(D)$. These are rational functions, whose valuations are non-negative everywhere, except possibly -1 at $[1; 0]$. One such function is 1. Another is x_0/x_1, which has a simple pole at $[1; 0]$. Hence $\mathcal{O}_C([1; 0])$ has at least 2 global sections.

Exercise. $\mathcal{O}(p)$ on $\mathbb{P}^1$ is $\mathcal{O}(1)$. (Hence we've found *all* the global sections; any global section is a linear combination of these 2.) Another consequence: *all* invertible sheaves on $\mathbb{P}^1$ are of the form $\mathcal{O}(m)$. (For the experts: you can jack of this proof to show the same result for $\mathbb{P}^n$. I can give you the details in a minute or two at some other time if you like.) Addition to problem on problem set: the section corresponding to 1 in the previous paragraph actually has a zero (despite what you think) — find it!

Now we've gotten this sheaf; I'll now show that it is *invertible*, by giving a trivializing cover.

Suppose $p_1, \ldots, p_n$ are the points appearing in D (i.e. where $n_p \neq 0$).

Let U_0 be the open set $C \setminus \{p_1, \ldots, p_n\}$. Then the restriction of the sheaf $\mathcal{O}_C(D)$ to this open U_0 is (isomorphic to) the trivial sheaf visibly: For $U \subset U_0$, the sections of $\mathcal{O}_C(D)|_{U_0}$ over U are $\cap_{p\in U}\mathcal{O}_{C,p} = \{x \in k(C) | v_p(x) \geq 0 \text{ for all } p \in U\}$, which is precisely $\mathcal{O}_C|_{U_0}$.

Now define U_i ($1 \leq i \leq n$) as follows. Choose any rational function $f_i \in k(C)$ such that $v_{p_i}(f_i) = n_i$. Then $(f_i) = n_i p_i +$ other stuff. Let U_i be C minus the other p_j minus the other stuff. Hence the "part of the divisor (f_i) on U_i" is just $n_i p_i$. Note that the U_i's and U_0 cover all of C; so we just need to show that the sheaf $\mathcal{O}_C(D)$ is trivial on U_i ($1 \leq i \leq n$). For $U \subset U_i$, the sections of $\mathcal{O}_C(D)|_{U_i}$ over U are $\cap_{p\in U}\mathcal{O}_{C,p} = \{x \in k(C) | v_p(x/f) \geq 0 \text{ for all } p \in U\}$. This is isomorphic to $\cap_{p\in U}\mathcal{O}_{C,p} = \{y \in k(C) | v_p(y) \geq 0 \text{ for all } p \in U\}$ (just take $y = x/f$), which in turn is precisely $\mathcal{O}_C|_{U_i}$.

So at this point we have a map $\mathrm{Div}(C) \to \mathrm{Pic}(C)$. To construct the desired map $\mathrm{Cl}(C) \to \mathrm{Pic}(C)$, we just need to show that the image of $\mathrm{Lin}(C) \to \mathrm{Pic}(C)$ is 0. (I didn't do this in class.) Suppose $D \in \mathrm{Lin}(C)$. Let s be a rational function such that $(s) = D$. Then

$$\begin{gathered}\mathcal{O}_C(D)(U) = \cap_{p\in U}\mathcal{O}_{C,p} = \{x \in k(C) | v_p(x) + n_p \geq 0 \text{ for all } p \in U\} \\ = \cap_{p\in U}\mathcal{O}_{C,p} = \{x \in k(C) | v_p(sx) \geq 0 \text{ for all } p \in U\} \\ = \cap_{p\in U}\mathcal{O}_{C,p} = \{y \in k(C) | v_p(y) \geq 0 \text{ for all } p \in U\} = \mathcal{O}_C(U).\end{gathered}$$

Notice that this identification commutes with restriction maps. Hence $\mathcal{O}_C(D) \cong \mathcal{O}_C$.

Both of these maps $\operatorname{Pic}(C) \to \operatorname{Cl}(C)$ *and* $\operatorname{Cl}(C) \to \operatorname{Pic}(C)$ *commute.*

I didn't explain this in class. I'll just sketch both directions. There's a lot of content packed into a few sentences; ask me about any steps you're suspicious about.

Start with a divisor D on C. Consider $\mathcal{O}_C(D)$ as just defined. Choose the section s corresponding to the rational function $1 \in k(C)$. You can check that the divisor of zeros of s is precisely D. Hence the composition $\operatorname{Cl}(C) \to \operatorname{Pic}(C) \to \operatorname{Cl}(C)$ is the identity.

For the other direction, suppose you have an invertible sheaf $\mathcal{L}$. Let s' be any rational section, so we get a divisor $(s') = D$. We wish to show that $\mathcal{L} \cong \mathcal{O}_C(D)$. Construct the section s of $\mathcal{O}_C(D)$ as in the previous paragraph (corresponding to the rational function $1 \in k(C)$). You can check that $(s') = D$ as well. Hence s/s' is a rational section of $\mathcal{L} \otimes \mathcal{O}_C(D)^\vee$; this rational section has associated divisor $D - D = 0$, hence has *no* zeros or poles. Hence this is a regular section (as it has no poles) that is invertible (as it has no zeros). But we've seen earlier that any invertible sheaf with a nowhere vanishing section is the trivial sheaf, so $\mathcal{L} \otimes \mathcal{O}_C(D)^\vee \cong \mathcal{O}_C$, so $\mathcal{L} \cong \mathcal{O}_C(D)$ as desired.

This completes the proof of the theorem. $\square$

Coming up in the last two classes: Degree of a line bundle/invertible sheaf. The sheaf of differentials (= the canonical sheaf) of a nonsingular curve. The celebrated Riemann-Roch theorem, with applications but no proof.