# INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 24

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Contents



PS9 back on today; PS12 out.

PS10 back, PS11 due Thurs. Dec. 9. PS12 due Monday December 13.

Show them a moebius strip, with sections.

## 1. Degree of a line bundle / invertible sheaf

1.1. **Last time.** Last time, I defined the Picard group of a variety $X$, denoted $\mathrm{Pic}(X)$, as the group of invertible sheaves on $X$. In the case when $X$ was a nonsingular curve, I defined the Weil divisor class group of $X$, denoted $\mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{Lin}(X)$, and sketched why $\mathrm{Pic}(X) \cong \mathrm{Cl}(X)$.

Let me remind you of the two morphisms. For $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$, take your invertible sheaf $\mathcal{L}$, and take any non-zero rational section $s$. Then take the class of $(s)$ in $\mathrm{Cl}(X)$.

For $\mathrm{Cl}(X) \to \mathrm{Pic}(X)$, take any Weil divisor $D$ representing your element of the class group. Then $\mathcal{O}_C(D)$ defined last time is an invertible sheaf.

Here's how $\mathcal{O}_C(\sum n_p p)$ was defined. Basically, sections over any open set are rational sections over that open set, where you are allowed to have certain poles and required to have certain zeros, depending on the $n_p$ for $p$ in your open set. Precisely:

$$\mathcal{O}_C(\sum n_p p)(U) = \{x \in k(C) | v_p(x) + n_p \geq 0 \text{ for all } p \in U\}.$$

You will essentially check on PS11 that the order of vanishing at $p$ of the section corresponding to $x$ is $v_p(x) + n_p$.

*Date*: Tuesday, December 7, 1999.

1.2. **New material.** Assume from now on that $C$ is *projective* (and still nonsingular). In this case, we can define the *degree* of an invertible sheaf. The key technical result is:

**Theorem.** Suppose $s$ is a non-zero rational function on $C$. Let $(s) = \sum_{p \in C} n_p p$. Then $\sum n_p = 0$.

Notice that we earlier proved this result when $C = \mathbb{P}^1$. This theorem is not too tricky, but for the sake of time, I'll omit the proof. Let's look at its consequences.

**Definition / Easy Proposition.** Define the *degree* function $\deg : \mathrm{Div} \to \mathbb{Z}$, by $\deg(\sum n_p p) = \sum n_p$. Then for $D \in \mathrm{Lin}$, $\deg D = 0$ (the theorem). Hence deg descends to give a function $\deg : \mathrm{Cl}(C) \to \mathbb{Z}$ and hence $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$.

If you unwind the identification of the Picard group and the Class group, you get the following recipe to calculate the degree of an invertible sheaf $\mathcal{L}$: find any non-zero rational section, and calculate find the sums of the orders of its zeros (and poles).

*Example (not on problem set).* Use this recipe to check that $\mathcal{O}_{\mathbb{P}^1}(2)$ has degree 2.

Suppose $\mathcal{L}$ were an invertible sheaf with a non-zero section $s$. Then this section $s$ would have no poles, and possibly some zeros, so its degree would be non-negative. Moreover, if $s$ had *no* zeros, then it would be invertible, so $\mathcal{L}$ would be the trivial sheaf. Hence:

**Proposition.** An invertible sheaf of negative degree has no non-zero sections. An invertible sheaf of degree 0 has no non-zero sections unless it is the trivial sheaf, in which case it has a one-dimensional family of sections.

*Proof.* All that's left to prove is that the trivial sheaf has precisely a one-dimensional family of sections. In other words, $C$ has a one-dimensional vector space of regular functions, the constants. This will follow from the next result. $\square$

**Proposition.** Let $C$ be a nonsingular projective curve. Then there are no nonconstant regular functions on $C$.

(You proved this when $C = \mathbb{P}^1$.)

*Proof.* Suppose $s$ were a nonconstant regular function on $C$. Let $p$ be a point on $C$, and suppose $s(p) = a \in \overline{k}$. Then $s(p) - a$ is another nonconstant regular function, and it has a zero at $p$. Hence its divisor $(s - a)$ has at least one zero and no poles, which has positive degree. But this is a rational function with positive degree, contradicting our Theorem. $\square$

## 2. The sheaf of differentials of a nonsingular curve

I'm going to describe this sheaf in a down-to-earth way. The best way to understand this sheaf is very explicitly. The best way to prove things is by describing it very abstractly. So I'll prove fewer things in the hope that you'll still understand what's going on. What I'd most like is for you to be able to calculate things.

Cotangent line bundle (= the sheaf of differentials).

Definition of this sheaf. On an affine variety, say what it is: $\Omega^1(U)$ is the $A(U)$-module generated by $ds$ where $r, s \in A(U)$. Rules: $ds = 0$ if $s \in \overline{k}$. $d(r+s) = dr + ds$. $d(rs) = rds + sdr$.

*Exercise.* $r \in A(U)$. Show $d(r^n) = nr^{n-1}dr$ if $n \geq 0$. If $r \in A(U)^*$, i.e. $r$ is invertible, $dr^n = nr^{n-1}dr$.

**Proposition.** This is a finitely-generated module generated by $dx_1, \ldots, dx_n$, where $x_i$ are generators of $A(U)$.

*Proof.* $ds(x_1, \ldots, x_n) = \sum_{i=1}^n \frac{\partial s}{\partial x_i} dx_i$. □

*Example.* If $U = \mathbb{A}^1$ with coordinate $x$, then the global differentials on $U$ are of the form $f(x)dx$ where $f(x) \in A(U) = \overline{k}[x]$. Clearly it is an $A(U)$-module. $dx$ is the generator of this module. It doesn't vanish anywhere.

If the variety is a nonsingular curve, this describes an invertible sheaf. I'll make this precise soon; assume it for now. But I first want to make some calculations, to show you that they are reasonable and straightforward.

I won't explicitly explain why it is a sheaf, but I'll wave my hands in that direction. The following proposition will show how restriction maps from affines to distinguished opens work.

**Proposition.** $\Omega^1(D(f)) = \Omega^1(U)_f$.

More precisely: there is a canonical isomorphism.

*Proof.* Suppose $A(U)$ is generated by $x_1, \ldots, x_n$ as an algebra. Then $\Omega^1(U)$ is generated by $dx_1, \ldots, dx_n$ as an $A(U)$-algebra.

Now $A(D(f)) = A(U)_f = A(U)[1/f]$ is generated by $x_1, \ldots, x_n, 1/f$ as an algebra.

$\Omega^1(D(f))$ is generated by $dx_1, \ldots, dx_n$, $d(1/f) = -(1/f^2)df \in \frac{1}{f^2}\Omega^1(U)$ as an $A(D(f))$-algebra. This is isomorphic to $\Omega^1(U)_f$ (explain). □

The restriction map $\Omega^1(U) \to \Omega^1(D(f))$ is just the inclusion $\Omega^1(U) \hookrightarrow \Omega^1(U)_f$.

*Remark.* This description is the cotangent sheaf $\Omega_{X/\overline{k}}$ for any scheme $X$. Not invertible sheaf in general; it is a *coherent sheaf*, whatever that is. Note that it is an $\mathcal{O}_X$-module. Vector bundle if $X$ is nonsingular. Dual to tangent bundle. The *dualizing* or *canonical sheaf* in that case is the top wedge power of the cotangent sheaf. *Exercise.* Calculate $\Omega^1_{\mathbb{P}^n}$.

In the case we're interested in, that of a nonsingular curve, this sheaf is actually an invertible sheaf. I'll wave my hands about that in a moment. But first, calculation:

**Proposition.** $\Omega^1_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$.

*Proof.* Assuming that $\Omega^1_{\mathbb{P}^1}$ is an invertible sheaf, we need only find a non-zero rational section and calculate its degree.

Once again, let's set out our coordinates. Let $[x_0; x_1]$ be projective coordinates on $\mathbb{P}^1$. Let $y_1$ be the coordinate on standard affine $U_0$, so $[x_0; x_1] = [1; y_1]$. Let $z_0$ be the coordinate on standard affine $U_1$, so $[x_0; x_1] = [z_0; 1]$. Hence $y_1 = x_1/x_0$, $z_0 = x_0/x_1 = 1/y_1$.

Consider the differential $dy_1$ on $U_0$, that doesn't vanish anywhere. This will give a (rational) differential $f(z_0)dz_0$ on $U_1$; we must figure out $f$.

What to do: $dy_1 = d(1/z_0) = (-1/z_0^2)dz_0$.

Hence this has a double pole at $z_0 = 0$; the degree of this invertible sheaf is -2. □

**Proposition.** $\Omega^1$ is invertible sheaf.

*"Proof."* I'll only prove this for plane curves (i.e. lying in $\mathbb{A}^2$), but you'll see how the proof generalizes.

Consider first the curve $C$ given by $f(x, y) = y^2 - x^3 + x = 0$ in $\mathbb{A}^2$; suppose we're in characteristic 0. (Draw it.) We know that $df = 2ydy + (1 - 3x^2)dx = 0$. Hence "on the locus where $2y \neq 0$, $dy$ is a multiple of $dx$". (We've thrown out the locus where the tangent space is vertical.

Precisely: Consider the distinguished open $D(y)$. Then $A(D(y)) = A(C)[1/y]$. Also, $\Omega^1(D(y))$ is generated (as an $A(D(y))$-module) by $dx$ and $dy$, and $dy$ is in $A(D(y))dx$, so $\Omega^1(D(y))$ is generated by $dx$.

Similarly, "on the locus where $1 - 3x^2 \neq 0$, $dx$ is a multiple of $dy$" These two open sets cover our curve $C$.

More generally, suppose $C$ were a plane curve given by $f(x,y) = 0$. We know that $df = f_x dx + f_y dy$. Hence "on the locus where $f_x \neq 0$", $dx$ is a multiple of $dy$, and "on the locus where $f_y \neq 0$", $dy$ is a multiple of $dx$. Ask them: how do we know we've hit all points, i.e. how do we know that for all points $p$ of $C$, either $f_x \neq 0$ or $f_y \neq 0$? Answer: $C$ is nonsingular, so this is just the Jacobian condition! $\square$

*Remark.* Let's go back to the plane curve example $y^2 - x^3 + x = 0$. (Draw picture.) Notice that $dx$ is a section of $\Omega^1_X$. It is a uniformizer everywhere but where $2y = 0$, and there we can take $dy$ as the uniformizer. Then $dx$ vanishes only where $2y = 0$, and there it vanishes to order 1. Hence $(dx) = (-1,0) + (0,0) + (1,0)$.

**Proposition.** Suppose $C$ is a nonsingular plane cubic. Then $\Omega^1_C \cong \mathcal{O}_C$.

Immediately, we have a third or fourth proof of the corollary:

**Corollary.** A nonsingular plane cubic is not isomorphic to $\mathbb{P}^1$.

(This is because $\deg \Omega^1_C = 0$, $\deg \Omega^1_{\mathbb{P}^1} = -2$.)

By the same method, you can prove that:

*Exercise.* Suppose $C$ is a nonsingular degree $d$ curve in $\mathbb{P}^2$. Then $\Omega^1_C \cong \mathcal{O}_C(d-3)$.

*Proof of Proposition.* I'm going to make some reductions. They won't make the proof much easier, but they will make the proof of the above exercise more tractable. First of all, assume there is some line meeting the plane cubic at 3 distinct points (transversely, i.e. with multiplicity 1). This isn't hard; feel free to make this assumption in the exercise.

Then choose coordinates $[x_0; x_1; x_2]$, so this line is the line at infinity $x_2 = 0$, and none of the 3 points are $[0;1;0]$.

I'll do the rest of the proof in a particular case: $f(x,y,z) = x_2^2 x_0 - x_1^3 + x_1 x_0^2 = 0$. This is the projectivization of the curve $a^2 = b^3 - b$, with appropriate choice of coordinates. Notice where it meets the line at infinity $x_2 = 0$: precisely where $-x_1^3 + x_1 x_0^2 = 0$, i.e. at the points $[1;1;0]$, $[1;-1;0]$, $[1;0;0]$.

(The conditions correspond to the fact that $x_1^3$ has a nonzero coefficient in the expression, and the cubic you get by setting $x_2 = 0$ has distinct roots.)

Consider the standard affine $U_2$, with coordinates $(x;y;1) = (x_0;x_1;x_2)$, and the standard affine $U_0$, with coordinates $(1;w;z) = (x_0;x_1;x_2)$. (Note that they cover the curve; the only point of $\mathbb{P}^2$ missed by these standard affines is $(0;1;0)$, which we assumed wasn't on the curve.)

Then consider the differential $dx$ on $U_2$. It has no poles (on $U_2$). Let's find its zeros. We know that $x - y^3 + yx^2 = 0$, so $(1 - 2xy)dx + (-3y^2 + x^2)dy = 0$. The zeros of $dx$ on $U_2$ are precisely where $-3y^2 + x^2 = 0$. We could find their coordinates, but we don't have to: note that this is precisely where $f_{x_1}(x, y, 1) = 0$: $f_{x_1} = -3x_1^2 + x_0^2$. Equivalently, this is precisely where $f_{x_1}(x_0, x_1, x_2) = 0$ (just check that there are no places where this vanishes at infinity). Thus $\mathcal{O}(dx) = \mathcal{O}(2)$; it vanishes at 6 points (with appropriate multiplicity).

Next, let's extend this to a *rational* differential over all of $C$. Let's see how to extend $dx$ over $U_0$. Note that $x = 1/z$, so $dx = -1/z^2 dz$. Where does $dz$ vanish? The same argument shows that $dz$ vanishes precisely where $f_{x_1}(x_0, x_1, x_2) = 0$ vanishes; we've already counted these points. $z = 0$ vanishes at our 3 points on the line at infinity, so $\mathcal{O}((1/z^2)) = \mathcal{O}(-2)$ (explain that statement!). Hence this rational section vanishes at the 6 points in the plane, and has a pole to order 2 at each of the 3 points at infinity, so it has total degree 0. More generally, this line bundle is trivial: those 6 points are in the class of $\mathcal{O}(2)$, and those poles are in the class of $\mathcal{O}(-2)$. $\square$

*Exercise.* Suppose $X$ is a nonsingular curve, and $p$ a point on $X$. Let $U$ be an affine neighborhood of $p$, and let $\mathfrak{m}$ be the maximal ideal of $A(U)$ corresponding to $p$. Then we earlier identified the cotangent space to $X$ at $p$ with $\mathfrak{m}/\mathfrak{m}^2$. As $\Omega^1$ is the "cotangent sheaf", there should be a natural map (of $A(U)$-modules) from $\Omega^1(U) \to \mathfrak{m}/\mathfrak{m}^2$. Describe this map explicitly. (Hence there is a natural map from the stalk of $\Omega^1_p$ to $\mathfrak{m}/\mathfrak{m}^2$.)

*Exercise: Pullback of differentials.* Suppose $\pi : X \to Y$ is a morphism of nonsingular curves. Show that there is a natural map from global differentials on $Y$ (i.e. global sections of $\Omega^1_Y$) to global differentials on $X$. The naturality of your construction will show that there is a natural morphism of invertible sheaves on $X$, $\pi^{-1}\Omega^1_Y \to \Omega^1_X$. (Equivalently, there is a natural morphism of sheaves on $Y$, $\Omega^1_Y \to \pi_*\Omega^1_X$; but $\pi_*\Omega^1_X$ isn't an invertible sheaf, so we don't yet have the language to effectively discuss it.)

*Exercise: Residues.* In complex analysis, a differential on a one-dimensional complex manifold with a simple pole at a point $p$ has a naturally-defined residue at that point: in any analytic coordinate $z$ at $p$ (i.e. $p$ corresponds to $z = 0$), if the differential is of the form $(a_{-1}/z + f(z))dz$ where $f(z)$ is analytic, then the residue is $a_{-1}$, and this is *independent of the choice of analytic coordinate*. Show that the same is true in algebraic geometry: suppose $X$ is a nonsingular curve, $p$ a point of $X$, and $s$ a differential on $X$ with simple pole at $p$; define the residue of $s$ at $p$ (and show that it is well-defined, i.e. that it doesn't depend on any choice).

**Coming soon:** The genus of a nonsingular projective curve. The Riemann-Roch theorem with applications but no proof.