INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 25

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Contents

PS10 back today; PS11 due today. PS12 due Monday December 13.

1. The genus of a nonsingular projective curve

The definition I'm going to give you isn't the one people would typically start with. I prefer to introduce this one here, because it is more easily computable.

Definition. The *tentative genus* of a nonsingular projective curve C is given by $\deg \Omega^1_C = 2g - 2$.

Fact (from Riemann-Roch, later). g is always a nonnegative integer, i.e. $\deg K = -2, 0, 2, \dots$.

Complex picture: Riemann-surface with g "holes".

Examples. Hence $\mathbb{P}^1$ has genus 0, smooth plane cubics have genus 1, etc.

Exercise: Hyperelliptic curves. Suppose $f(x_0, x_1)$ is a polynomial of homogeneous degree n where n is even. Let C_0 be the affine plane curve given by $y^2 = f(1, x_1)$, with the generically 2-to-1 cover $C_0 \to U_0$. Let C_1 be the affine plane curve given by $z^2 = f(x_0, 1)$, with the generically 2-to-1 cover $C_1 \to U_1$. Check that C_0 and C_1 are nonsingular. Show that you can glue together C_0 and C_1 (and the double covers) so as to give a double cover $C \to \mathbb{P}^1$. (For computational convenience, you may assume that neither $[0; 1]$ nor $[1; 0]$ are zeros of f.) What goes wrong if n is odd? Show that the tentative genus of C is $n/2 - 1$. (This is a special case of the *Riemann-Hurwitz formula*.) This provides examples of curves of any genus.

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Exercise. Suppose C is a nonsingular degree d projective plane curve. Show that the tentative genus of C is $\binom{d-1}{2}$.

2. The Riemann-Roch Theorem with Applications but No Proof

Definition. If X is a topological space, and S is a sheaf on X, define $H^0(X, S)$ to be the set of global sections of S over X (what we've called $S(X)$). If X is a variety, and S is an $\mathcal{O}_X$-module, then $H^0(X, S)$ has the structure of a $\overline{k}$-vector space. In this situation, define $h^0(X, S)$ to be the dimension of $H^0(X, S)$ as a $\overline{k}$-vector space.

From here on in, C is a nonsingular projective curve, and we'll deal with the case when S is an invertible sheaf. In this case, we have shown that h^0 is finite.

The Celebrated Riemann-Roch Formula. $\mathcal{L}$ a line bundle of degree d on a nonsingular projective curve C of genus g. Then $h^0(C, \mathcal{L}) - h^0(C, \Omega^1 \otimes \mathcal{L}^\vee) = d - g + 1$.

Proof. Omitted!

I think you know enough to understand a proof, or at least a proof under mild assumptions. If there are 2 or 3 of you interested, I could try to come up with a proof that you'd understand in half an hour, and present it during IAP.

Example. Let's check this out for $\mathbb{P}^1$. (Make a table of degree, $h^0(C, \mathcal{L})$, $h^0(C, \Omega^1 \otimes \mathcal{L}^\vee)$, $d - g + 1$.)

Here come lots of appliations.

Corollary. Plug in $\mathcal{L} = \mathcal{O}$. Then we see that $h^0(C, \Omega^1) = g$. Hence the tentative genus is always a nonnegative integer.

Definition. Define the genus of a nonsingular projective curve C to be $h^0(C, \Omega^1_C)$.

Corollary. If $d > 2g - 2$ (i.e. the degree of $\mathcal{L}$ is greater than the sheaf of differentials), then $h^0(C, \Omega^1 \otimes \mathcal{L}^\vee) = 0$. Hence $h^0(C, \mathcal{L}) = d - g + 1$.

Reason: the invertible sheaf $\Omega^1 \otimes \mathcal{L}^\vee$ has negative degree.

Corollary. Given $p \in C$. Then there is a rational function with a pole only at p.

Proof. Consider the divisor $(2g+1)p$. Then $\mathcal{L} := \mathcal{O}((2g+1)p)$ is a degree $2g$ line bundle, and it has $h^0(C, \mathcal{L}) = d - g + 1 = 2g + 1 - g + 1 = g + 2 > 1$. Hence there are two linearly independent sections of $\mathcal{L}$. But the global sections of $\mathcal{L}$ corresponded to rational functions whose valuation at every point distinct from p was nonnegative, and whose valuation at p was at most $-(2g+1)$. One such function is the constant function; thus there is a function that is nonconstant, which must therefore have a pole somewhere; the pole must be at p. (We also see that the order of pole can be made to be at most $2g+1$.)

2.1. **A criterion for closed immersions.** Given a nonsingular projective curve C, and invertible sheaf $\mathcal{L}$. Let $\{s_0, \ldots, s_n\}$ be a basis for the (finite-dimensional) vector space $H^0(C, \mathcal{L})$ of global sections. If, for each $p \in C$, there is a section s of $\mathcal{L}$ such that s doesn't vanish at p, then this basis gives a morphism

$$(s_0, \ldots, s_n) : C \mapsto \mathbb{P}^n.$$

This morphism is denoted $|\mathcal{L}|$, and is called the *complete linear system* associated with $\mathcal{L}$.

Remark. The sections of $\mathcal{L} \otimes \mathcal{O}_C(-p)$ (often written $\mathcal{L}(-p)$) over U can be identified with the sections of $\mathcal{L}$ vanishing at p. (You've seen this before if $\mathcal{L}$ is $\mathcal{O}_C$; the proof is the same.) Hence there is a natural inclusion

$$H^0(C, \mathcal{L}(-p)) \hookrightarrow H^0(C, \mathcal{L}).$$

The cokernel is dimension 0 or 1; it is 1 if there is a global section of $\mathcal{L}$ that does not vanish at p. Hence we've proved:

Proposition. $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L} \otimes \mathcal{O}_C(-p)) \leq 1$, and if equality holds then then there is a section of $\mathcal{L}$ that does not vanish at p.

Proposition. Suppose p and q are distinct points of C. If $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L} \otimes \mathcal{O}_C(-p-q)) = 2$, then p and q are mapped to distinct points of $\mathbb{P}^n$ via $|\mathcal{L}|$. In other words, $|\mathcal{L}|$ is injective.

Proof. (You can check that $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) \leq 2$, and that the hypotheses of this proposition imply the hypotheses of the previous one, so that $\mathcal{L}$ really *does* induce a morphism into projective space.)

It suffices to show that there is a section of $\mathcal{L}$ that vanishes at p but doesn't vanish at q. (Explain.) □

When the degree of $\mathcal{L}$ is big, the hypotheses of these propositions are automatically satisfied.

Proposition. If $\deg \mathcal{L} > 2g-1$, then for each $p \in C$, there is a section s of $\mathcal{L}$ such that s doesn't vanish at p. If $\deg \mathcal{L} > 2g$, then the morphism $|\mathcal{L}|$ is injective.

Proof. In this range, $h^0(C, \mathcal{L}) = d - g + 1$, $h^0(C, \mathcal{L}(-p)) = (d-1) - g + 1$, and (in the second case) $h^0(C, \mathcal{L}(-p-q)) = (d-1) - g + 1$. □

Under the conditions of the second proposition, $|\mathcal{L}|$ is an injection of sets. But it still might get knotted up at some point (explain). It would be great to know criteria for when $|\mathcal{L}|$ gives a closed immersion. Here's one:

Proposition. Suppose for any pair of points p and q of C (*not necessarily distinct*), $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L} \otimes \mathcal{O}_C(-p-q)) = 2$. Then $|\mathcal{L}|$ is a closed immersion.

Proof. Omitted. Shafarevich p. 217; Hartshorne.

Remark. By the same argument as above, if $\deg \mathcal{L} > 2g$, then the hypotheses are satisfied.

This is powerful. Here are applications.

Proposition. A projective nonsingular curve C of genus 0 must be isomorphic to $\mathbb{P}^1$.

Proof. Choose any point $p \in C$. Then $\mathcal{L} := \mathcal{O}(p)$ has degree $1 > 2g$, so $|\mathcal{L}|$ is a closed immersion. But $\mathcal{L}$ has 2 sections by Riemann-Roch: $h^0(C, \mathcal{L}) = \deg \mathcal{L} - g + 1 = 2$. Thus $|\mathcal{L}|$ describes a closed immersion of C into $\mathbb{P}^1$, so C must be isomorphic to $\mathbb{P}^1$. □

Thus there is only one projective nonsingular genus 0 curve.

Let's try a similar trick in genus 1. Let C be a genus 1 (projective nonsingular) curve, and let $\mathcal{L}$ be any degree 3 invertible sheaf (e.g. $\mathcal{O}(3p)$). Then by Riemann-Roch, $h^0(C, \mathcal{L}) = 3 - 1 + 1 = 3$.

Then by the Proposition, $|\mathcal{L}|$ gives a closed immersion of C into $\mathbb{P}^2$, and it must be as a cubic curve. (In fact the degree of the image is precisely the degree of the line bundle. We haven't analyzed this carefully yet, so I'll resort to another argument: from an exercise in PS12, you can calculate that the genus of a degree d nonsingular plane curve is $\binom{d-1}{2}$. Hence as the image of $|\mathcal{L}|$ has genus 1, we know $d = 3$.)

Hence we have proved:

Proposition. Every genus 1 curve can be written as a cubic plane curve.

Thus we've "seen" all genus 1 curves. By special request, we proved that:

Proposition. There is a natural bijection between the degree 1 line bundles on a genus 1 curve C and the points of the curve.

Proof. The link is as follows. If p is a point on the curve, then $\mathcal{O}_C(p)$ is a degree 1 invertible sheaf. If $\mathcal{L}$ is a degree 1 invertible sheaf, then by Riemann-Roch, $h^0(C, \mathcal{L}) = 1$, so there is (up to multiple) precisely one section of $\mathcal{L}$. It's divisor is necessarily a point. These two constructions clearly commute. □

In the problem set, you'll see that all genus 2 curves can be written as double covers of $\mathbb{P}^1$ branched at six points. So now we've "seen" all genus 2 curves.

In the problem set, you'll see that all genus 3 curves can be written either as double covers of $\mathbb{P}^1$ branched at eight points, or as nonsingular plane quartics. So now we've "seen" all genus 3 curves.

Proposition. A projective nonsingular curve minus a finite number of points is affine.

Proof. Consider the invertible sheaf $\mathcal{O}(kp)$ where k is huge. (Exercise: how big does k have to be for this to work?) Then this gives a morphism into projective space

$$|\mathcal{O}(kp)| : C \to \mathbb{P}^n$$

expressing C as a closed subvariety. Now there is a section s_0 of $\mathcal{O}_C(kp)$ that has only one zero, of order k, at the point p. Take a basis for the sections of $\mathcal{O}_C(kp)$: $s_0, \dots, s_n$. So we have a map

$$(s_0, \dots, s_n) : C \to \mathbb{P}^n$$

$$q \mapsto (s_0(q); \dots ; s_n(q)).$$

Now where does $x_0 = 0$ intersect C? Answer: where $s_0 = 0$, i.e. only at the point p. (Geometry: the curve meets the hyperplane $x_0 = 0$ with multiplicity k there.) Hence the affine open U_0 of C is $C \setminus \{p\}$, so $C \setminus \{p\}$ is affine! □

Corollary. Any nonsingular curve that is *not* projective is affine.

Let's calculate more things.

Suppose C is a genus 1 curve. Let $\mathcal{L}$ be any degree 4 invertible sheaf on C, e.g. $4p$ for some point p. Then $|\mathcal{L}|$ expresses C as a closed subvariety of $\mathbb{P}^3$ (as $\mathcal{L}$ has 4 sections, and the degree satisfies the criterion for closed immersions above). I claim that it is the intersection of 2 quadrics. Here is a pretty good sketch of why.

Let π be the inclusion. Let $w, \dots, z$ be the coordinates on $\mathbb{P}^3$. First, note that the image isn't contained in any hyperplane.

Next, compute the number of sections of $\pi^*\mathcal{O}(1)$, which has degree 4: by Riemann-Roch, it has 4 sections, which are precisely the 4 sections we used to map it to projective space. They correspond to the linear functions w, x, y, z.

Compute the number of sections of $\pi^*\mathcal{O}(2)$, which has degree 8: by Riemann-Roch, it has 8 sections. But here are 10 sections: the sections $w^2, \dots, yz, z^2$. Hence there is a two-dimensional vector space of polynomials that pullback to zero. Let $p(w, x, y, z)$ and $q(w, x, y, z)$ be a general basis of this two-dimensional vector space. Then the image of C lies on $p = 0$ and $q = 0$ (explain).

We know that $p = 0$ and $q = 0$ must intersect in a curve. Here's why: if their intersection included a surface S, the surface must be of degree 2 or degree 1. If S

were degree 2, then it must be the surface $p = 0$ and $q = 0$; but then p is a scalar multiple of q, contradicting the fact that p and q formed a basis. If S were degree 1, then p and q both consist of 2 planes (explain, and explain the contradiction).

Hence $p = 0$ and $q = 0$ intersect in a curve C' (including C). By the exercises on the Hilbert polynomial, the curve C' has degree 4, which means that it meets a general hyperplane at 4 points.

But a general hyperplane corresponds to the zeros of a general section of $\mathcal{L}$. Any nonzero section of $\mathcal{L}$ has 4 zeros, so C is all of C'. Hence C is the intersection of 2 quadrics in $\mathbb{P}^3$.

(Conversely, if you take 2 general quadrics in $\mathbb{P}^3$, it turns out that you get an elliptic curve.)

Remark. You can use this method to understand curves of genus 4 and 5.

For a genus 4 curve C, the sheaf of differentials has degree 6, and has 4 sections. If C is not hyperlliptic, then $|\Omega^1|$ expresses C as a closed subvariety of $\mathbb{P}^3$, (after an application of the crition above). The methods we just used show that C is a complete intersection of a quadric and a cubic. Hence this "describes all genus 4 curves": they are either hyperelliptic, or the intersection of a quadric and a cubic. (There is a uniform way of describing them, that doesn't involve these cases.)

For a genus 5 curve C, something similar happens. C is either hyperelliptic, or it sits in $\mathbb{P}^4$ as the intersection of 3 quadrics. *Not quite! Could be trigonal, in which case the intersection of the 3 quadrics is, not surprisingly, a rational scroll. Johan showed me a paper of Rutger Noot.*

3. Recap of course

Generalities: Algebraic sets. Zariski topology. Nullstellensatz. Hilbert basis theorem. Sheaves and stalks; structure sheaf. Prevarieties and morphisms of prevarieties. Function fields and birational maps. Projective prevarieties. Schemes. Products. Dimension. Nonsingularity.

Curves: Valuation rings. Completitions. Integral closure, Dedekind domains, DVRs. Extending rational maps of nonsingular curves to projective varieties. Finitely generated fields over $\overline{k}$ of tr. deg. 1 correspond to nonsingular projective curves.

Line bundles and invertible sheaves. $\mathcal{O}(m)$. Picard group. Class group. Degree. Riemann-Roch.