INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 4

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Contents

This is an initial draft of the notes. I hope to clarify some of the argument in light of some questions that were asked, and re-post these then. But there won't be major substantive changes.

Homework was collected at end. Ana-Maria Castravet (the grader) was introduced. I handed out the second problem set, and talked about it.

Where we are.

By the last part of Thursday's lecture, we had the following definitions.

Algebraic sets in $\mathbb{A}^n$. Zariski topology on an algebraic set — the closed subsets are algebraic subsets. Irreducible algebraic subsets. Ring of regular functions on an algebraic set $V(I)$ is $R = \overline{k}[x_1, \dots, x_n]/I$.

Caution: Last day I called these **varieties** — I'd prefer to use that word for something more precise, so please forget I said it!

Noetherian algebraic space. Irreducible components. Every algebraic set (by definition in $\mathbb{A}^n(\overline{k})$) has a finite number of irreducible components. Dimension. $\dim \mathbb{A}^n(\overline{k}) = n$.

The plan for the next little while (2 lectures): give a tentative definition *morphism* between affine algebraic sets (in a classical way), and we'll get some consequences. This will have some problems, which will motivate us to introduce the *structure sheaf* on an irreducible algebraic set. To do this, I'll first discuss *sheaves*. Then we'll be ready to define *affine varieties*, *varieties*, and *morphisms*. At some point, as an aside, I'll define a *scheme*.

Let me first tell you a bit about the history of the subject. (Three periods: classical; ferment; post-Grothendieck.)

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1. Morphisms between algebraic sets, classical definition

Suppose we have two algebraic sets $X \subset \mathbb{A}^m$, $Y \subset \mathbb{A}^n$. Suppose for convenience, the co-ordinates on $\mathbb{A}^m$ are $x_1, \ldots, x_n$ and the co-ordinates on $\mathbb{A}^n$ are $y_1, \ldots, y_n$. We'd like to define morphisms in this restrictive world, where we can only think in terms of polynomials.

Well, naively, you'd imagine that:

Definition (Naive and tentative).

A morphism is a map given by

$$\pi : (x_1, \ldots, x_m) \mapsto (f_1(x_1, \ldots, x_m), \ldots f_n(x_1, \ldots, x_m)) = (y_1, \ldots, y_m).$$

where the f_i are polynomials, and such that each point on X maps to a point on Y.

Let's analyze this more closely. It induces a morphism in the opposite direction on rings of functions:

$$\pi^* : A(Y) \to A(X).$$

Let me describe it in two ways.

First, as functions (explain that we get functions, and they are polynomials). Hence this is a "pullback" of functions.

Second, more explicitly: the functions above give us a map

$$\begin{array}{rcl} \overline{k}[y_1, \ldots, y_n] & \to & \overline{k}[x_1, \ldots, x_m] \\ y_i & \mapsto & f_i(x_1, \ldots, x_m) \end{array}$$

But any polynomial $g(y_1, \ldots, y_n)$ which vanishes on Y pulls back to a polynomial which vanishes on X, so we get a map $I(Y) \to I(X)$, so we get a map

$$\overline{k}[y_1, \ldots, y_n]/V(Y) \to \overline{k}[x_1, \ldots, x_m]/V(X).$$

Conversely, any such morphism of rings of functions defines a morphism! This is explicit: given such a morphism of rings, then one can represent it by $y_i \to f_i(x_1, \ldots, x_m)$, so let's check that

$$(x_1, \ldots, x_m) \to (f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m))$$

really sends points of X to points of Y.

In the course of doing this, we will see how to interpret maps of points in terms of the algebra of this ring homomorphism $A(Y) \to A(X)$.

Consider a point p of X.

This corresponds to a maximal ideal $\mathfrak{m}_p$ of $A(X)$; the quotient is $\overline{k}$. Hence a point of X is the same as the data of a surjective map

$$A(X) \to \overline{k}.$$

(Recall, this is the evaluation map at p.) We have a map $A(Y) \to A(X)$, so we get a map $A(Y) \to \overline{k}$. We even know it is surjective: it's a map of rings, which means 1 goes to 1. (Any ring-map from a ring to a field is surjective.) *(Those last two sentences aren't right!)*

Hence this corresponds to the data of a maximal ideal of $A(Y)$, which by the Nullstellensatz corresponds to a point of Y.

Let's check that this really corresponds to the image of $\pi(p)$, i.e. that this map really is the evaluation map at $\pi(p)$. *If you have a function $f \in A(Y)$, then to work out its evaluation at $\pi(p)$, you could also pull it back to $A(X)$, and evaluate it at p.* Algebraically, this explicitly describes evaluation at $\pi(p)$ as the composition

$$A(Y) \to A(X) \to A(X)/\mathfrak{m}_p \cong \overline{k}.$$

So with this definition of morphism, we've proved something extremely strong.

Theorem. Morphisms from one affine algebraic set $X \subset \mathbb{A}^m$ to another $Y \subset \mathbb{A}^n$ canonically correspond to morphisms of rings $A(Y) \to A(X)$:

$$\mathrm{Hom}_{\text{alg sets}}(X, Y) \cong \mathrm{Hom}_{\text{rings over } \overline{k}}(A(Y), A(X)).$$

Corollary. $A(X)$ is canonically isomorphism to the set of morphisms from X to $\mathbb{A}^1(\overline{k})$.

Proof. Note that $A(\mathbb{A}^1(\overline{k})) \cong \overline{k}[t]$. □

Let me repeat the theorem in category-language. You should take my categorical statements in one of three ways. If you've never seen this language before, ignore it. If you've vaguely seen this language before, then this will tell you something about categories and functors. And if you're happy with categories, this will tell you something about algebraic sets and rings.

Theorem take 2. The assignment $X \mapsto A(X)$ extends to a contravariant functor: { Category of algebraic sets and morphisms } $\to$ { Category of nilpotent-free finitely generated algebras over $\overline{k}$ and $\overline{k}$-homomorphisms } which is an equivalence of categories.

In particular:

Corollary. Every nilpotent-free finitely generated algebra over $\overline{k}$ is the ring of regular functions of some algebraic set.

Remark. If X is an irreducible algebraic set, then $A(X)$ is not just a nilpotent-free finitely generated algebra over $\overline{k}$, but it is a finitely generated integral domain over $\overline{k}$. Then we get a contravariant functor { Category of algebraic sets and morphisms } $\rightarrow$ { Category of nilpotent-free finitely generated algebras over $\overline{k}$ and $\overline{k}$-homomorphisms } which is an equivalence of categories.

Here's another important fact:

Proposition. Morphisms are continuous.

Proof. Suppose we have a morphism $\pi : X \rightarrow Y$, where $X \subset \mathbb{A}^m$, $Y \subset \mathbb{A}^n$, and π is given by $y_i = f_i(x_1, \ldots, x_m)$.

We need to show that the preimage of every closed set $Z \subset Y$ is closed. Now Z is defined as the zero-set of some finite number of functions $g_j(y_1, \ldots, y_n) = 0$ $(1 \leq j \leq p)$. Then $\pi^{-1}(Z) \subset X$, the preimage of Z, is given by $g_j(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)) = 0$ $(1 \leq j \leq p)$. But anything defined as the zero-locus of a bunch of polynomials is closed, so we're done. $\square$

Consequences of this definition.

We can now talk about isomorphisms!

Example. For example, the line L described by $3x + 4y = 7$ in $\mathbb{A}^2$ is isomorphic to $\mathbb{A}^1$. We can see this in a couple of ways. This is perhaps the cleanest, although not the most naive:

$$A(L) \cong \overline{k}[x, y]/(3x + 4y - 7) \cong \overline{k}[x] = A(\mathbb{A}^1).$$

That choice of congruence means that we've chosen the isomorphism $\mathbb{A}^1 \rightarrow L$ given by $(x) \mapsto (x, (-3x+7)/4)$, or the reverse morphism $L \rightarrow \mathbb{A}^1$ given by $(x, y) \mapsto x$.

Example. There is a morphism from the affine line $\mathbb{A}^1$ to the curve C $y^2 = x^3+x^2$ (draw it; nodal cubic) in the plane, given by $t \mapsto (t^2 - 1, t(t^2 - 1))$. This is not an isomorphism. One reason why is that the curve C has a node, but we're not yet able to talk about that. Another reason is because the points $t = 0$ and $t = 1$ map to the same point of the plane.

Example. There is a morphism from the affine line $\mathbb{A}^1$ to the curve C $y^2 = x^3$ given by (t^2, t^3). It is a homeomorphism: there is a correspondence of points, and the topologies are the same (open sets are the empty set, or all but a finite number of points). (But: *Exercise.* This is also not an isomorphism.)

Example of a weird (but incredibly important) morphism: Frobenius. Suppose $\overline{k}$ has characteristic $p > 0$, so we have a ring morphism $F : \overline{k} \rightarrow \overline{k}$ given by $t \mapsto t^p$ (called the Frobenius morphism). Then this gives us a ring homomorphism $\overline{k}[t] \rightarrow \overline{k}[t]$ given by

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \mapsto a_0 + a_1(t^p) + a_2(t^p)^2 + \cdots + a_n(t^p)^n$$

This corresponds to a morphism $\mathbb{A}^1 \to \mathbb{A}^1$ You can check that the fixed points of this morphism are the points on $\mathbb{A}^1$ whose coordinates lie in $\mathbb{F}_p$, the finite field with p elements. Hence when people want to study equations over finite fields, which they do a lot, they look at the fixed points of this morphism.

When we define a variety properly, we want to describe it in abstract terms, and not have the data of an affine space lying around. Technically it makes life harder in the short run, but much easier in the long run. The previous example shows that we need more than the minimal amount of information of the set of points, and the Zariski topology: we saw two algebraic sets that aren't the "same", and yet there was a homeomorphism between them.

Another example is this: Consider the map $\pi : \mathbb{A}^1 \to \mathbb{A}^1$ that is the identity, except that the points 0 and 1 are swapped (draw picture). This is most definitely not a morphism, as it isn't given by a polynomial. Then this is a bijection on sets, and it's even a homeomorphism in the Zariski topology: let's check that the preimage of an open set is an open set. (Do it.)

Hence we need some more information. And this information is contained in a certain sheaf, the structure sheaf. Then we'll define varieties as a topological space, along with the data of a structure sheaf, which encapsulates algebraic data.

Before saying what that is, let me tell you what a sheaf is.

2. Sheaves

The fundamental idea of a sheaf on a topological space isn't actually complicated, although proving things with sheaves is. Let me give you a prototypical example of a sheaf, and then give you the definition, so you can see how this example obviously satisfies all the properties.

The example is: the sheaf of continuous functions on $\mathbb{R}^2$. Other examples are differentiable functions, all functions, C^1-functions, polynomial functions, etc.

Definition. A *presheaf* F of rings on a topological space X is the following data.

- To every open subset U of X there is associated a ring denoted $F(X)$ or $\Gamma(U, F)$, called "sections over U".
- If $U \subset V$ (both open) then there is a restriction map $\operatorname{res}_{V,U} F(V) \to F(U)$.
- Restriction maps "commute": if $U \subset V \subset W$ (all open), then

$$\begin{tikzcd} F(W) \arrow[rr, "\operatorname{res}_{W,V}"] \arrow[dr, "\operatorname{res}_{W,U}"'] & & F(V) \arrow[dl, "\operatorname{res} V, U"] \\ & F(U) & \end{tikzcd}$$

commutes.

Remark. A presheaf of rings on X is a contravariant functor from the category of open subsets of X (where the morphisms are inclusions) to the category of rings, and vice versa.

Definition. A *morphism* $F \to G$ of presheaves on a topological space is the same as the data of ring-morphisms $F(U) \to G(U)$ compatible with restriction maps, i.e. if $U \subset V$, then the following diagram commutes:

$$\begin{array}{ccc} F(V) & \to & G(V) \\ \mathrm{res}_{V,U} \downarrow & & \downarrow \mathrm{res}_{V,U} \\ F(U) & \to & G(U). \end{array}$$

As an example, let X be $\mathbb{R}^2$, and let F be the presheaf of differentiable functions, and G the presheaf of continuous functions; $F(V) \to G(V)$ is inclusion.

Caution: Words like *injective* and *surjective* can be defined for presheaves and sheaves, but one has to be cautious.

Definition. A presheaf F is a *sheaf* if it satisfies two further conditions:

Gluability. If $U_1 \cup \cdots \cup U_n = V$ and you are given sections $f_i \in F(U_i)$ such that "f_i and f_j" agree on $U_i \cap U_j$", i.e.

$$\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$$

in $F(U_i \cap U_j)$, then there is an $f \in F(V)$ such that $f_i = \mathrm{res}_{V,U_i}$ in $F(U_i)$.

Identity. If $U_1 \cup \cdots \cup U_n = V$ and you have a sections $f, g \in F(V)$ such that "f and g agree on each U_i", i.e. $\mathrm{res}_{V,U_i} f = \mathrm{res}_{V,U_j} g$ in $F(U_i)$ for all i, then $f = g$ in $F(V)$.

(One can also have a presheaf of sets, or modules, or groups, etc.; the definition doesn't change.)

Examples. a) Rings of functions on a topological space.

b) Rings of continuous functions (to $\overline{k}$) on a topological space. Rings of differentiable functions on a manifold.

c) (Sheaf of sets) X and Y are topological spaces. If U is an open subset of X, let $F(U)$ be the set of continuous morphisms from U to Y. Then this is a sheaf on X.

d) If F is a sheaf on a topological space X, and U is an open subset of X, then the restriction of F to U is also a sheaf (on the topological space U).

Non-example. If X is a disconnected topological space, let F be the presheaf of constant integer-valued functions on X. In other words, for each open $U \subset X$, let $F(U)$ be the ring $\mathbb{Z}$, and the restriction maps are just the identity. Then we don't have gluability. (I didn't explain this example to my satisfaction; I'll talk more about it on Thursday.)

Coming up:

(1) Defining the stalk of a (pre)sheaf of rings at a point.

(2) The structure sheaf $\mathcal{O}_X$ of an irreducible algebraic set $X \subset \mathbb{A}^n(\overline{k})$

(3) Linking "Morphisms for algebraic sets" and structure sheaves

(4) Defining Affine varieties and prevarieties

(Homework collected.)