INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 7

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Contents

Problem set 3 is now due one week from today; future problem sets will be due on Thursdays, and returned the following Tuesday. I'll give out copies of parts of the red book.

Where we are.

Last day, we played around with the structure sheaf of the plane $\mathbb{A}^2$, and observed that the sections of the structure sheaf over some random open set U was a subring of $\overline{k}(x,y)$.

We now formally know what a prevariety is: it is a topological space with a structure sheaf, that "locally" looks like an irreducible algebraic set. We'll next make this concrete by working out a fair number of examples.

Some of you are wondering about that prefix "pre"; we'll begin to answer that today.

I was initially going to tell you about the *function field* of a prevariety beforehand, that generalizes the role of $\overline{k}(x,y)$ in the plane, but instead, I'll say a few words now, and talk about it at length after the examples.

Function fields, quickly.

To find the function field of a prevariety, take any affine open set, find its ring of regular functions R, and take its field of fractions $K(R)$. We'll see later that this is well-defined.

Quick comment about sheaves: gluability and identity are not just for finite covers.

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Also, at some point during the lecture, we discussed why two non-empty open subsets of a prevariety must intersect. Here's the summary: (i) This is true for any irreducible topological space. (ii) Affine varieties are irreducible (by definition). (iii) If you glue two irreducibles along non-empty (homeomorphic) open sets, you get an irreducible topological space. (iv) A prevariety is a connected union of a finite number of irreducible spaces, so you can use (iii) inductively to show that a prevariety is irreducible.

1. Examples of prevarieties

Let's start with a prevariety that isn't affine.

Last time we showed that an open subset of a prevariety (with the induced topology and structure sheaf) is also a prevariety. This is called an *open subprevariety.*

Consider $\mathbb{A}^2 \setminus (0,0)$.

Exercise. Prove that $\mathbb{A}^2 \setminus (0,0)$ is not an affine variety. (Hint: If it were, then what would its ring of functions be?)

Hence not all open subsets of affine varieties are affine varieties.

The function field of this prevariety is $\overline{k}(x,y)$.

Example: $\mathbb{P}^1$. Take two copies of $\mathbb{A}^1$; call them X and Y, and say their coordinates are a and b. Glue them together along the open set $X \setminus 0$ and $Y \setminus \{0\}$, where the two open sets are identified via $a = 1/b$.

You can see these patches in terms of coordinates on $\mathbb{P}^1$.

As mentioned earlier, if $\overline{k}$ is the complex numbers, you can picture this quite explicitly. You're gluing two complex planes together, and you end up with a sphere.

The function field of $k(X)$ is $\overline{k}(a) \cong \overline{k}(b)$, and the isomorphism is via $a \mapsto 1/b$. (Discuss a bit.) To see what happens if we chose the isomorphism $a \mapsto b$, see the line with the doubled origin below.

Example: Global sections of $\mathcal{O}_{\mathbb{P}^1}$.

Notice that $\mathbb{P}^1$ isn't affine: the affine algebraic set associated to the global sections of $\mathcal{O}_{\mathbb{P}^1}$ is a point.

A pathological example: the line with the doubled origin. Now let's glue our two copies X and Y of $\mathbb{A}^1$ together, along the same open sets, except this time identify the open sets via $x = y$. (Explain.) This is something nasty that we'll want to get rid of when defining varieties. (For the same reason, a *real manifold* isn't just defined as a topological space that is locally isomorphic to a neighborhood of the origin in $\mathbb{R}^n$; an additional condition (the *Hausdorff* condition) is necessary.)

Again, the function field of $k(X)$ is $\overline{k}(x)$.

Example: $\mathbb{P}^n$. Recall that projective space (over the field $\overline{k}$) is usually defined as the set of $(n+1)$-tuples $(x_0; \dots; x_n)$ where the x_i are not all zero, under the equivalence $(x_0; \dots; x_n) \sim a(y_0; \dots; y_n)$ where $a \in \overline{k}^*$. We'll define $\mathbb{P}^n$ as an algebraic variety. We have $n+1$ open sets, each isomorphic to $\mathbb{A}^n$, and we glue them together just as we did $\mathbb{P}^1$. For convenience, I'll just describe the first two, and show how they glue together. The affine open U_0 has coordinates $(a_1, \dots, a_n)$, and the affine open U_1 has coordinates $(b_0, b_2, \dots, b_n)$; the open subset of U_0 corresponding to $a_1 \neq 0$ is glued to the open subset of U_1 corresponding to $b_0 \neq 0$, and the identification is via:

$$(1; a_1; a_2; \dots; a_n) \sim (b_0; 1; b_2; \dots; b_n),$$

i.e. $a_2/a_1 = b_2/b_0$, $a_3/a_1 = b_3/b_0$, $\dots$, $a_n/a_1 = b_n/b_0$.

In an analogous manner, this tells you how to glue together any U_i and U_j. We still have something further to check: that these gluings agree. You can think through yourself that it suffices to prove that the gluing data of U_i and U_j, U_j and U_k, and U_k and U_i all "agree".

This also tells you how to think of points in $\mathbb{P}^n$ with projective coordinates. For example, consider $\mathbb{P}^3$, which is covered by U_0, $\dots$, U_3. Then (4;3;0;1) is a point which corresponds to $(3/4, 0, 1/4)$ on U_0, $(4/3, 0, 1/3)$ on U_1, and $(4, 3, 0)$ on U_3; it doesn't lie in the open U_2.

The U_i are often called the standard affine cover of $\mathbb{P}^n$. We will talk about projective varieties at great length soon.

The function field is $\overline{k}(a_1, \dots, a_n)$.

Exercise. Prove that the global sections of $\mathcal{O}_{\mathbb{P}^n}$ are constant.

Remark. You don't have to know much about the structure sheaf to glue together prevarieties, thanks to the fact that we understand affine prevarieties well, and distinguished opens of affines are affines too.

2. Function fields of prevarieties

Definition. Given a prevariety X, define the *function field* $k(X)$ as follows. (Caution: that k isn't the field k.) Elements of $k(X)$ are called *rational functions* on X.

Intuition: Meromorphic functions on $\mathbb{C}$ (i.e. functions defined almost everywhere). There's no point where all such functions are defined, yet they definitely form a ring: you can add two and get another such, or multiply, etc.

Elements of $k(X)$ is the data (U, f) where U is an open subset of X and $f \in \mathcal{O}_X(U)$ is a section of the structure sheaf over U, modulo an equivalence relation: $(U, f) \sim (V, g)$ if there is an open W in U and V such that f and g agree on W, or precisely:

$$\operatorname{res}_{U,W} f = \operatorname{res}_{V,W} g.$$

This might remind you of the definition of the stalk of a sheaf. This is no coincidence; in the land of schemes, this actually *is* a stalk, albeit at a weird kind of point (called the *generic point* of X).

It is a complicated way of saying something that turns out to be quite simple:

Claim. Let U be an affine open set of X. Then $k(X) = K(A(U))$; the function field is just the ring of fractions of $A(U)$.

Hence to find $k(X)$, you just have to find *any* affine open, take the ring of regular functions on that affine open, and take its quotient field.

Example. $k(\mathbb{A}^n) = \overline{k}(x_1, \dots, x_n)$. In fact, if X is any open subset of $\mathbb{A}^n$, then $k(X) = k(\mathbb{A}^n)$.

Proof. First, we show the result when X is affine; let its ring of regular functions be R. Then an element of $k(X)$ is of the form $(U, f \in \mathcal{O}_X(U))$. Using the equivalence relation, make U smaller so that it is a distinguished open (possible as distinguished opens form a base for the topology); this element is of the form $(D(g), f \in \mathcal{O}_X(D(g)) = R_{(g)})$. Thus f can be written as a/b where $a, b \in R$ (in fact b is a power of g).

Conversely, given an element of $K(R)$, write it as a/b. Then this gives a section of $\mathcal{O}_X$ over $D(b)$: $a/b \in R_{(b)} = \mathcal{O}_X(D(b))$. Then you just have to check that these two constructions commute. □

Consequences (proofs omitted).

1. For any open $U \subset X$, $\mathcal{O}_X(U)$ is a subring of $k(X)$.
2. The restriction maps are inclusions, i.e. $U \subset V$ implies $\mathcal{O}_X(V) \subset \mathcal{O}_X(U) \subset k(X)$.
3. $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \cap \mathcal{O}_X(V)$ inside $k(X)$. (The right side wouldn't even make sense unless $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ were subrings of the same thing.)
4. $\mathcal{O}_X(U \cap V)$ is the subring of $k(X)$ generated by $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$. *(False! See problem on Problem Set 4. If X is separated, I suspect this is true.)*

3. Morphisms of prevarieties: first thoughts

First, some philosophy. We know what morphisms from one affine open to another looks like: it's the same as the maps of rings of regular functions (as rings over $\overline{k}$) in the opposite direction.

So here is how we might like to think of morphisms of one prevariety to another. (Explain: to describe a morphism X to Y, we might write X as a union of affine opens U_i, and Y as a union of affine opens V_i, where U_i maps to V_i, and everything glues nicely.) This is actually correct, but what gets confusing is showing that this description is independent of the choice of cover; plus it is kind of annoying to check, so it would be good to have a (slightly) more usable definition.

So we'll define morphisms in another way first, that doesn't involving covers, then show that this is the same as the "naive" description I've just given. Then in real life, we'll use (a variant of) the "naive" description.

We begin by recasting morphisms of affines in a different light.

Theorem. Let $X \subset \mathbb{A}^m$, $Y \subset \mathbb{A}^n$ be irreducible algebraic sets, and let $\pi : X \to Y$ be a continuous map. Then the following are equivalent. (Draw pictures!)

i) π is a morphism of algebraic sets

ii) for all $g \in \mathcal{O}_Y(Y) = A(Y)$, $g \circ \pi \in \mathcal{O}_X(X) = A(X)$.

We've already proved this Theorem.

But note that we've described morphisms in a new way: they are (i) continuous maps, such that (ii) pullback takes regular functions to regular functions.

Coming in the next few lectures (in some order):

1. Morphisms of prevarieties.
2. Examples of morphisms, including open immersions, closed immersions.
3. The isomorphism of $x^2 + y^2 = z^2$ in $\mathbb{P}^2$ with $\mathbb{P}^1$.
4. Projective varieties. Rational maps (and maps of function fields?). Degree of rational maps. Birationality.