# INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 8

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*Correction.* Brian pointed out that I goofed when defining gluing data for $\mathbb{P}^n$ last class. I'll eventually correct them in the notes.

Suppose we're thinking about $\mathbb{P}^n$ with projective coordinates $(x_0; x_1; \dots; x_n)$, and we have coordinate patches $U_0$ and $U_1$, where $U_0$ has coordinates $a_1, \dots, a_n$, and corresponds to the locus where $x_0 \neq 0$, so

$$(x_0; x_1; \dots; x_n) = (1; a_1; \dots; a_n),$$

and $U_1$ has coordinates $b_0, b_2, \dots, b_n$, and corresponds to the locus where $x_1 \neq 0$, so

$$(x_0; x_1; \dots; x_n) = (b_0; 1; b_2; \dots; b_n).$$

Then the patching data is $a_1 = 1/b_0$, $a_2 = b_2/b_0$, … , $a_n = b_n/b_0$ (where you are gluing together the open set $D(a_1)$ on $U_0$ and the open set $D(b_0)$ on $U_1$.

**Perspective.** Everything is defined on 3 levels: (i) on the level of sets, (ii) on the level of topological spaces, and (iii) on the level of structure sheaves.

(I also went over what the sheaf $F|_U$ means, if $F$ is a sheaf on a topological space $X$, and $U$ is an open subset of $X$.)

## 1. Morphisms of prevarieties

First, some philosophy. We know what morphisms from one affine open to another looks like: it's the same as the maps of rings of regular functions (as rings over $\overline{k}$) in the opposite direction.

So here is how we might like to think of morphisms of one prevariety to another. (Explain.) This is actually correct, but what gets confusing is showing that this description is independent of the choice of cover. Reason useful: Start off with map

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of sets, level (i) above. Then levels (ii) and (iii), i.e. continuity and structure sheaf, come out in the wash.

Thus we'll define morphisms in another way first, that doesn't involving covers, then show that this is the same as the "naive" description I've just given. Then in real life, we'll use the "naive" description.

**Linking "Morphisms for algebraic sets" and structure sheaves.**

The key result:

**Theorem.** Let $X \subset \mathbb{A}^m$, $Y \subset \mathbb{A}^n$ be irreducible algebraic sets, and let $\pi : X \to Y$ be a continuous map. Then the following are equivalent. (Draw pictures!)

i) $\pi$ is a morphism of algebraic sets

ii) for all $g \in \mathcal{O}_Y(Y) = A(Y)$, $g \circ \pi \in \mathcal{O}_X(X) = A(X)$.

iii) for all open $U \subset Y$ and $g \in \mathcal{O}_Y(U)$, $g \circ \pi \in \mathcal{O}_X(\pi^{-1}(U))$

iv) For all $x \in X$ and $g \in \mathcal{O}_{Y,\pi(x)}$, $g \circ \pi \in \mathcal{O}_{X,x}$.

I talked about what iv) means a bit. Central is the idea that if $F^X$ (resp. $F^Y$) is the sheaf of functions on $X$ (resp. $Y$), then

$$\begin{array}{ccc} \mathcal{O}_{Y,\pi(x)} & \subset & F^Y_{\pi(x)} \\ \downarrow & & \downarrow \\ \mathcal{O}_{X,x} & \subset & F^X_x \end{array}$$

*Note added later: part of this essentially follows from $\cap \mathcal{O}_Y(U) = \cap_{y \in U} \mathcal{O}_{Y,y}$.*

*Proof.* We've already shown that i) $\Leftrightarrow$ ii). (Explain.) Note that we've described morphisms in a new way: they are continuous maps, where pullback takes regular functions to regular functions.

Immediately iii) $\Rightarrow$ ii) (just take $U = Y$).

Also, iv) $\Rightarrow$ iii); that's how the structure sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$ were defined.

We'll finish off the proof by showing that ii) $\Rightarrow$ iv).

Let $g \in \mathcal{O}_{Y,\pi(x)}$ (draw picture). We write $g = a/b$ where $a$, $b \in \mathcal{O}_Y(Y)$, where $b(\pi(x)) \neq 0$. By ii), $a \circ \pi$, $b \circ \pi \in \mathcal{O}_X(X)$. Thus $g \circ \pi = (a \circ \pi)/(b \circ \pi) \in \mathcal{O}_{X,x}$, since we have $b \circ \pi(x) \neq 0$. $\square$

**Definition.** A morphism $f : X \to Y$ of prevarieties is a (continuous) morphism of topological spaces such that, for all open sets $V$ in $Y$, $g \in \mathcal{O}_Y(V) \Rightarrow g \circ f \in \mathcal{O}_X(f^{-1}V)$. (We're using iii) from the Theorem.)

Note that by that Theorem, our tentative definition of morphisms of irreducible affine algebraic sets is actually a special case of our new definition.

Immediately we have:

**Proposition.** The composition of two morphisms is a morphism.

*Proof.* Continuous is clear. Regular functions on an open pull back to regular functions on the preimage. □

Let's link this to our naive definition of morphisms described earlier.

**Proposition.** Suppose $X$ and $Y$ are prevarieties. Let $f : X \to Y$ be any map of sets (not even assumed a priori to be continuous). Let $\{V_i\}$ be a collection of open affine subsets covering $Y$. Suppose $\{U_i\}$ is an open affine covering of $X$ such that $f(U_i) \subset V_i$, and $f^*$ maps $\mathcal{O}_Y(V_i)$ into $\mathcal{O}_X(U_i)$ (i.e. regular function on $V_i$ pull back to regular functions on $U_i$). Then $f$ is a morphism.

*Remark.* We don't even have to assume that the $U_i$ are affine. If $U_i$ isn't affine, then cover it with affines $W_1, \ldots, W_n$ (picture). If the continuous map induces a map $\mathcal{O}_Y(V_i)$ to $\mathcal{O}_X(U_i)$, then it induces a map from $\mathcal{O}_Y(V_i)$ to $\mathcal{O}_X(W_j)$. And conversely, if you have the latter maps, you can reconstruct the original map $\mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$ by gluability and identity.

Intuitively, we have a bunch of morphisms of affine things that we are gluing together.

*Proof.* First of all, the restriction $f_i$ of $f$ to a map from $U_i$ to $V_i$ is a morphism (previous proposition). As $f_i$ is continuous, $f$ is continuous too.

So it remains to check that for any open set $V$, $f^*$ always takes a section $g \in \mathcal{O}_Y(V)$ to some section of $\mathcal{O}_X(f^{-1}(V))$. Now $g \circ f$ is at least a section of $\mathcal{O}_X$ over the sets $f^{-1}(V \cap V_i)$, hence on the sets $f^{-1}(V \cap U_i)$. As $\mathcal{O}_X$ is a sheaf, these glue together to give a section on all of $f^{-1}(V)$. □

*Post mortem.* The proof went through the 3 steps: sets, topological spaces, structure sheaf.

## 2. Examples of morphisms

**Example 1: open immersions; open subprevarieties.** Here's an easy one, but an important definition.

Suppose $(Y, \mathcal{O}_Y)$ is a prevariety, and $U$ is an open subset of $Y$, so $(U, \mathcal{O}_U := \mathcal{O}_Y|_U)$ is a prevariety.

Then the inclusion morphism $U \to Y$ induces a morphism $(U, \mathcal{O}_U) \to (Y, \mathcal{O}_Y)$.

This is called an *open immersion.* One often says that $U$ is an *open subprevariety* of $Y$.

*Example:* There is an open immersion from the affine line to the line with the doubled origin.

**Example 2.** $\mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{P}^1$.

Send $(a, b) \to (x; y)$.

**Example 3: Projection.** The morphism $\pi : \mathbb{P}^2 \setminus (0; 0; 1) \to \mathbb{P}^1$ given by $(x; y; z) \mapsto (x; y)$. Note that the left side is a prevariety. Note also why we had to throw out the point.

I'm showing it to you for two reasons: this is an important example, and it will show you how to prove something is a morphism.

First, draw a picture.

Recall that to show something is a morphism, you have to do three things (not necessarily independently): show there is a map on points, show it is continuous, and show the pullback property on structure sheaves (usually using affine opens).

First, the map above is a map on points. So we just need to show that it is *continuous*, and then the pullback property.

Let $U_x$ be the standard open of $\mathbb{P}^2$ where $x \neq 0$, and $U_y$ similarly. Note that $U_x \cup U_y$ is the left side.

Let $V_x$ be the standard open of $\mathbb{P}^1$ where $x \neq 0$, and $V_y$ similarly. Note that $V_x \cup V_y = \mathbb{P}^1$, the right side.

Note that $U_x$ maps to $V_x$, and $U_y$ maps to $V_y$. We'll show that both maps are morphisms of affine varieties; then we'll have shown that $\pi$ is a morphism.

On $U_x$, let the coordinates be $a$ and $b$, so we think of points of $U_x$ as $(x; y; z) = (1; a; b)$. On $V_x$, let the coordinate be $c$, so we think of points of $V_x$ as $(x; y) = (1; c)$. Then the map is $(1; a; b) \mapsto (1; a) = (1; c)$, so on the level of rings, it is given by $c = a$, Hence $U_x \to V_x$ is a morphism of affine varieties, and we're done (once we do the same argument with $x$ replaced by $y$)!

You might not realize we're done yet, so let me be more explicit. In the definition of morphism, we required that regular functions on $V_x$ (i.e. polynomials in $c$) pull back to regular functions on $U_x$ (i.e. polynomials in $a$ and $b$). But that map just sends a polynomial $f(c)$ to the polynomial $f(a)$.

You should think about this at length on your own to convince yourself about what happened.

More generally, we can define the projection $\mathbb{P}^n \setminus \{(0;0;\ldots;0;0;a;b;\ldots;z)\} \to \mathbb{P}^k$ where there are $k+1$ zeros on the left side.

**Example 4: Closed immersions.**

First: *closed immersions of affine varieties.* Suppose $X \subset Y \subset \mathbb{A}^n$, $X$, $Y$ affine varieties. Then we have a morphism of affine varieties $X \to Y$ given by $x_1 \mapsto x_1$, etc. This is called a *closed immersion of affine varieties.*

On the level of rings: $I(Y) \subset I(X)$, so $R/I(Y) \to R/I(X)$, i.e. $A(Y) \twoheadrightarrow A(X)$. Conversely, given a surjective morphism of nilpotent-free finitely generated $\overline{k}$-algebras $A(Y) \to A(X)$, then this describes a closed immersion of affine varieties: let $x_1$, $\ldots$, $x_n$ be generators of $A(Y)$, so $Y$ sits inside $\mathbb{A}^n$ (with coordinates $x_1$, $\ldots$, $x_n$). Then the image of the $x_i$'s in $A(X)$, which we'll also call $x_i$ if it isn't too confusing, also generate $A(X)$, and this reproduces the picture of closed immersions in the previous paragraph.

Hence: a morphism of affine varieties $X \to Y$ is a *closed immersion* if it induces a surjective map of rings $A(Y) \to A(X)$. In particular, you might have worried that the definition earlier was dependent on how $Y$ was chosen to be affine, i.e. the choice of how to put $Y$ in an affine space; this shows that such fears are groundless.

**Definition.** A morphism of prevarieties $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a *closed immersion* (of prevarieties) if

(i) $X \to Y$ is a homeomorphism between $X$ and a closed subset of $Y$ (with the induced topology), and
(ii) $Y = \cup_\alpha U_\alpha$ of affine opens where

$$(X \cap U_\alpha, \mathcal{O}_X|_{X\cap U_\alpha}) \to (U_\alpha, \mathcal{O}_Y|_{U_\alpha})$$

is a closed immersion of affine varieties.

One often says that $X$ is a *closed subprevariety* of $Y$.

Now to the example: Suppose char $\overline{k} \neq 2$ (for technical reasons). In $\mathbb{P}^2$, $x^2+y^2 = z^2$ is a closed subset. It even describes a closed subprevariety. Let $U_x$ be the open where $x \neq 0$, and similarly for $U_y$ and $U_z$.

On $U_x$, the coordinates are $(1;a;b)$, so $x^2+y^2 = z^2$ translates to $1+a^2-b^2 = 0$ in $\mathbb{A}^2$ (coordinates $a, b$). Irreducible (as polynomial is irreducible). (If the characteristic were 2, it wouldn't be irreducible: $(1+a+b)^2 = 0$.

On $U_y$, the coordinates are $(c;1;d)$, so $x^2+y^2 = z^2$ translates to $c^2+1 = d^2$ in $\mathbb{A}^2$ (coordinates $c, d$).

On $U_z$, the coordinates are $(e; f; 1)$, so $x^2 + y^2 = z^2$ translates to $e^2 + f^2 = 1$ in $\mathbb{A}^2$ (coordinates $e, f$).

Hence this "describes" a closed subprevariety (a conic in the projective plane).

Note that this argument is quite general; suppose you have an irreducible polynomial of degree $d$ in $n + 1$ variables $x_0, \ldots, x_n$. Then this describes a closed subprevariety of $\mathbb{P}^n$; it is called a *hypersurface of degree d*. We will soon see that this is a special case of *projective (pre)varieties*.

**Example 5: Composing morphisms.**

Sometimes it is easier to show things are morphisms by composing with simpler morphisms.

Let $C$ be the plane curve $x^2 + y^2 = z^2$ described above. Then map of sets $\pi : (x; y; z) \in C \rightarrow (x; y)$ is a morphism.

Draw a picture.

First note that this morphism makes sense because $x$ and $y$ on the left side can't both be zero.

It is a morphism because it is a composition of the closed immersion of Example 4 with the projection of Example 3.

*Exercise.* Prove that $C$ is actually isomorphic to $\mathbb{P}^1$. (Caution: The map $\pi$ isn't an isomorphism. Steps will be given; as a consequence, you will find all solutions to $x^2 + y^2 = z^2$ over an arbitrary field of characteristic not 2.)

Let me give you the geometric idea behind this. You might have seen this before if you've seen the formula that gives you all pythagorean triples.

First of all, in the affine open $U_z$, with coordinates $a$, $b$ (so $(a; b; 1) = (x; y; z)$), we can view the formula as $a^2 + b^2 = 1$, which we might as well draw as a circle.

Consider the $\mathbb{P}^1$ of lines through the point $(1, 0)$. A random such line meets the circle at two points, one of which is $(1, 0)$. For example, if the line has slope $-m$, then you can do the algebra and check that the line $y = -mx + 1$ meets the circle at $(1, 0)$ and another point

$$\left( \frac{2m}{1+m^2}, \frac{1-m^2}{1+m^2} \right).$$

(This works even if $m = 0$; you need a special case if the line is vertical and the slope is thus infinite.) This basically gives an identification between the lines through $(1, 0)$, parameterized by $\mathbb{P}^1$, and the points of the circle. You'll make this much more precise in the exercise.

**Coming in the next few lectures (in some order):**

1. Perhaps define schemes.
2. Projective varieties.
3. Rational maps (why equivalent to maps of function fields?). Degree of rational maps. Birationality.